Age-Induced Acceleration of Time: Implications for Intertemporal Choice*

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University of Arizona Working Paper 15-02

April 2016

First version: March 2015

Our perception of time is both nonlinear and nonstationary, which makes preference reversals possible. I decompose the sources of dynamic inconsistency into a time acceleration effect and a time compression effect. Standard economic models focus only on the second effect. I show that when the perception of time accelerates with age, the two effects can offset each other for hyperbolic discounters but not for exponential discounters. In line with the results of recent economic experiments, such hyperbolic discounters would report discount rates that seem to imply dynamic inconsistency but would nonetheless manifest dynamic consistency in actual choices over time.

JEL: A12, C91, D91

Keywords: age, discounting, dynamic consistency, hyperbolic, intertemporal choice, preference reversals, time

*I thank George Ainslie, Martin Dufwenberg, Asaf Plan, and participants at the 2016 Southwest Economic Theory Conference for helpful comments. An old version circulated under the title "Age and Perspective: Dynamic Consistency With Time-Varying Discount Rates".

When as a child I laughed and wept,
Time crept.
When as a youth I waxed more bold,
Time strolled.
When I became a full grown man,
Time ran.
When older still I daily grew,
Time flew.
Soon I shall find, in passing on,
Time gone.
O Christ! wilt Thou have saved me then?
Amen.

"Time's Paces", Henry Twells (1901)

1 Introduction

People dislike waiting. This preference makes them discount future utility payoffs. The intensity of the distaste for waiting depends on how long the wait feels. Waiting three months for a vacation feels like an eternity to a child but trivial to an adult. We therefore may expect the child to discount the pleasure of the vacation more strongly. The intensity of the distaste for an additional period of waiting also changes with the duration of the wait. Waiting three additional months for a vacation feels less painful when one is already planning to wait a year rather than a week. We therefore may expect our self a year from now to discount the pleasure of the vacation more strongly over those same three additional months. Psychologists discuss both of these effects, but economists have primarily focused on the latter, as "hyperbolic" discounting.[1] In particular, hyperbolic discounters are said to be marked by dynamic inconsistency: they may be more willing to defer the vacation by three months if you ask them a year ahead of time than if you ask them a week ahead of time.[2] I extend economic analyses of discounting to allow for both aspects of time perception. I show that accounting for the effects of aging can reverse standard intuition about which types of agents are dynamically inconsistent.

I introduce psychological models of the subjective acceleration of time with age to the economics literature and formally explore their implications for discounting and dynamic

[1]For reviews of the evidence in favor of hyperbolic discounting, see Ainslie and Haslam (1992) and Frederick et al. (2002). Much additional work has occurred since then. For contrary perspectives, see Rubinstein (2003) and Andreoni and Sprenger (2012). I will focus on discount rates that decline at all horizons rather than on present-biased preferences as in Laibson (1997) and O'Donoghue and Rabin (1999).

[2]The implication of dynamic inconsistency has long been a focal point of the economics literature (Strotz, 1955–1956; Pollak, 1968; Laibson, 1997).

consistency. Ever since the foundations of the modern field (James, 1890), psychologists have been interested in how the subjective perception of time changes with age.[3] Folk wisdom, reflected in the opening verse, holds that time passes more quickly as we age. Psychologists have proposed that the subjective length of an interval of actual (or clock) time is inversely proportional either to the amount of actual time one has lived (Janet, 1877; Doob, 1971) or to the accumulated subjective time one has lived, which works out to being inversely proportional to the square root of actual time lived (Lemlich, 1975). Some have proposed a biological basis for the effect of aging: for example, dopamine appears to speed up the body's internal clock, and dopamine production falls with age, so it is conceivable that our internal clock slows with age and thereby makes time appear to pass faster (Tien and Burnes, 2002). Other channels include age-related reductions in basal metabolism or brain temperature (Block et al., 1998).[4] A body of experimental evidence using survey measures such as recall of past intervals and using physical measures such as response times has supported the notion that time speeds as we age (e.g., Walker, 1977; Joubert, 1990; Craik and Hay, 1999; Crawley and Pring, 2000; Carrasco et al., 2001; Wittmann and Lehnhoff, 2005).[5]

While economists have devoted many experiments and much analysis to how time discounting changes with delay, the discipline has mostly ignored how the perception of time changes with age.[6] Yet both should be important for discounting. I adapt standard psychological models for mapping clock time into subjective time, which an agent may then discount according to her distaste for delay. First, I show that psychological models of aging imply that agents become more patient as they age.[7] This result is consistent with much empirical evidence (Green et al., 1994; Rogers, 1994; Read and Read, 2004; Chu et al., 2010; Tanaka et al., 2010), though Meier and Sprenger (2015) report contrary results. Second, I explore the implications for intertemporal choice when agents are either naive or fully sophisticated about how aging affects their perception of time. Naive agents assume that time

[3] "The same space of time seems shorter as we grow older—that is, the days, the months, and the years do so; whether the hours do so is doubtful, and the minutes and seconds to all appearance remain about the same." James (1890, Chapter XV)

[4] Another channel for the effect of aging is that we become more integrated with our future selves as we age. People with a stronger sense of self-continuity do tend to have smaller discount rates and greater savings (Ersner-Hershfield et al., 2009). Mitchell et al. (2011) provide a neurological basis for the relationship between self-reference and discounting, and Parfit (1984) provides a philosophical basis.

[5] Some suggest that these studies may actually be capturing the effects of time pressure (Friedman and Janssen, 2010; Janssen et al., 2013) or intensity of experience (Fraisse, 1984) rather than age per se. Block et al. (1998) suggest that time might actually slow for the more elderly. They interpret this effect in terms of the attentional resources devoted to the assigned task of estimating time intervals.

[6] Only a few economists have recognized the potential importance of accounting for the evolution of our subjective perception of time (Wolf, 1970; Samuelson, 1976; McFadden, 2013; Aadland and Shaffer, 2015). Heal (1998, p. 106) briefly makes a point related to this paper's when he discusses logarithmic discounting.

[7] I abstract from mortality risks, which can make agents less patient as they age (e.g., Yaari, 1965). Introducing mortality risks into the present setting would generate a U-shaped time profile for the mortality-adjusted discount rate. This type of profile is consistent with evidence in Read and Read (2004).

will always pass as rapidly as it does at the present moment, whereas sophisticated agents recognize that time will pass more rapidly in the future.[8]

I show that fully sophisticated agents are dynamically consistent if and only if they are exponential discounters, but naive agents are dynamically consistent only if they discount hyperbolically or do not discount at all. Approaching closer to a future interval of time increases the discount rate that a hyperbolic discounter applies to the interval, which I call a "time compression effect". This change in the discount rate makes the hyperbolic discounter less willing to defer that interval's utility as she draws closer to it, which generates dynamic inconsistency in the standard model. However, as the agent ages, any given interval of time feels shorter, which I call a "time acceleration effect". This effect tends to reduce the discount rate applied to a future interval of time as the agent approaches it. Surprisingly, these two effects offset each other under benchmark specifications for hyperbolic discounting from the economics literature and benchmark specifications for the effect of aging from the psychology literature. In these cases, a hyperbolic discounter does not change her discount rate as she approaches a given interval of clock time, whereas an exponential discounter who does not correctly anticipate the effect of aging uses a progressively smaller discount rate as she approaches a given interval of clock time.[9] Further, I show that even if a hyperbolic discounter is dynamically inconsistent, she may be less severely dynamically inconsistent than an exponential discounter, especially if the latter uses a high discount rate. Preference reversals may occur for both hyperbolic and exponential discounters, but they may be easier to detect for exponential discounters.

Intriguingly, the psychology literature has demonstrated a second aspect of time perception that can generate hyperbolic discounting. Several psychology studies support the notion that people apply a constant discount rate to each perceived unit of time but that their perception of time compresses intervals of objective time more strongly as they look farther into the future (Ekman and Lundberg, 1971; Kim and Zauberman, 2009; Zauberman et al., 2009; Bradford et al., 2014). These studies support the use of hyperbolic discount rates at a given age (Takahashi, 2005).[10] Now consider an agent who experiences time flowing faster as she ages and who compresses intervals of time more strongly as she looks farther into the future. Our results imply that if this agent discounts each perceived unit of time at a constant rate and is naive about the psychology of time perception, then her behavior can

[8]This distinction between naivete and sophistication mirrors a distinction common in the literature on hyperbolic and present-biased preferences (Laibson, 1997; O'Donoghue and Rabin, 1999).

[9]Ray and Bossaerts (2011) previously observed that the combination of exponential discounting and a biological clock that does not match objective time can generate dynamic inconsistency.

[10]Ainslie and Haslam (1992) and Karp (2015) similarly argue that declining, hyperbolic discount rates reflect how our faculty for perceiving time mimics our faculty for perceiving space. From the perspective of the present, a given interval of time appears to shrink as we look farther into the future, as a given length of railroad track appears to shrink as we look farther into the distance. The economics literature has also explored how uncertainty about the probability (Sozou, 1998; Azfar, 1999) or timing (Dasgupta and Maskin, 2005) of future payoffs can generate hyperbolic discounting.

be approximately dynamically consistent. In contrast, a similar agent who either does not experience the effect of aging or does not more strongly compress more distant intervals will tend to be dynamically inconsistent.

These results have important implications for the experimental and life-cycle savings literatures in economics. First, most experiments elicit hyperbolic discount schedules using choice experiments that take place at a given time. Declining discount rates in these conventional cross-sectional experiments are commonly identified with dynamic inconsistency. However, my analysis suggests that if the subjects were to return to the lab as the potential dates of payment drew closer, then the postulated dynamic inconsistency may not materialize (compare Sayman and Öncüler, 2009; Sprenger, 2015). These predictions are in fact borne out in the longitudinal experiment of Read et al. (2012): they find the typical evidence of hyperbolic discounting in a cross-sectional analysis, but a longitudinal analysis then fails to find the pattern of preference reversals predicted by standard models of hyperbolic discounting. They interpret their longitudinal results as evidence against hyperbolic discounting. However, in the present paper's setting, hyperbolic discounters in the cross-section would indeed plausibly demonstrate dynamic consistency in a longitudinal analysis. In another rare example of a longitudinal analysis, Halevy (2015) finds that many subjects appear to have time-varying preferences yet are dynamically consistent in practice.[11] He proposes that this result may be driven by implicit risk or by time-varying liquidity demands. The present paper suggests an additional possibility: acceleration of subjective time. Future experimental work in economics and psychology should measure changes in the subjective perception of the rate of passage time (over both long and short intervals) and compare them to cross-sectional measures of hyperbolic discounting.

Second, the life-cycle literature seeks to explain the age profile of consumption and savings. Most life-cycle models generate the empirically observed hump-shaped consumption profile by introducing borrowing constraints or precautionary savings motives. Yet puzzles remain. For instance, Gourinchas and Parker (2002) underpredict consumption early in the life cycle. Feigenbaum (2008) notes that allowing the discount rate to vary with age offers an alternate means of generating the standard hump-shaped consumption profile. He explores time-varying mortality risk as the source of a time-varying discount rate, but he finds that mortality risk does not explain the overall age profile of consumption very well. It is possible that age-induced declines in the discount rate could be an important and hitherto overlooked component of life-cycle models. For instance, allowing the young to have greater discount rates could resolve the problem of underpredicting their consumption. Future work should

[11] A small handful of further studies have undertaken longitudinal comparisons. Augenblick et al. (2015) find evidence of dynamic consistency when payouts involve money but not when payouts involve effort. In a developing country context, Giné et al. (2012) find that realized preference reversals are correlated with present-biased preferences. Solnick et al. (1980) and Ainslie and Haendel (1983) report evidence of dynamic inconsistency in experiments with nonmonetary rewards and with substance abuse patients, respectively. Sayman and Öncüler (2009) test for the possibility of "reverse time inconsistency" over short timeframes. Read and van Leeuwen (1998) explore the interaction between appetite and dynamic consistency.

explore the potential for age-dependent perception of duration to explain consumption and savings data.

The next section introduces standard psychological models of the subjective acceleration of time and shows that they imply that agents become more patient as they age. Section 3 considers dynamic consistency when agents are naive about the effects of age. Section 4 analyzes agents who are fully sophisticated about the effects of age. The final section concludes.

2 Subjective perception of time

We consider a single agent moving through time. Let s be the clock time of the present moment and t be the clock time of some future moment, measured since some agent-specific initial time. This initial time could be the agent's birth or could be the beginning of the agent's economic life. Using the interpretation of time since birth, I often say that the agent is of age s. We are interested in how the time s agent discounts payoffs to be received at times $t > s$.

The agent discounts future payoffs based not on their distance in actual time but on their distance in subjective (or perceptual) time. Let the function $\tau(t, s)$ describe how long an interval $t - s$ feels to an age s agent. Assume that age affects time perception in a continuous, differentiable manner. Based on the psychology literature, the agent's perception of an increment of clock time is a power function of the agent's age:[12]

$$\frac{\partial \tau(t, s)}{\partial t} = \frac{k}{s^{\alpha}},$$

where $k > 0$ is a constant of proportionality. We consider cases with $\alpha \geq 0$. When $\alpha = 0$, the subjective perception of an interval of time is independent of the agent's age. This case matches the standard discounting model used in economics. When $\alpha > 0$, a given interval of clock time feels shorter as the agent ages. This power form is commonly used in the psychology literature to represent how people perceive physical stimuli. The general form matches that of Stevens' power law, which relates the actual magnitude of a stimulus to its perceived strength. In the special case of $\alpha = 1$, we have an application of the Weber-Fechner law, with the subjective duration of an interval of clock time decreasing in proportion to total clock time lived. In the case of time perception, Janet (1877) and Doob (1971) proposed a model with $\alpha = 1$, and Lemlich (1975) proposed a model with $\alpha = 0.5$.

To start, assume that a time s agent does not anticipate the effects of aging on time perception. Integrating over the interval from s to t, we have:

$$\tau(t, s) = \frac{k}{s^{\alpha}}(t - s). \tag{1}$$

[12] In keeping with the psychology literature, I treat the perception of time as changing continuously. Using a discrete-time setting would not change the main results.

The key property we are seeking to capture is that time flows faster as we age. This means that the subjective length of a given interval of clock time appears to shorten as s increases:

$$\frac{\partial^2 \tau(t,s)}{\partial t \partial s} = k \frac{-\alpha}{s^{\alpha+1}} \leq 0. \tag{2}$$

Consistent with intuition, the subjective length of a given interval of clock time accelerates more rapidly for the young:

$$\frac{\partial^3 \tau(t,s)}{\partial t \partial s^2} = k \frac{\alpha(\alpha+1)}{s^{\alpha+2}} \geq 0.$$

Aging has less of an effect on time perception for agents who are already old.[13]

We are interested in how an agent of age s discounts payoffs that arrive at time t. The agent's time preference is captured by her instantaneous discount rate $\delta(\tau(t,s))$. By analogy to the standard model of discounting in clock time, we assume that $\delta(\cdot) \geq 0$ and that $\delta'(\cdot) \leq 0$. The discount rate is a function of the subjective delay $\tau(t,s)$ from the present; it depends on the delay $t-s$ measured by the clock only insofar as that delay translates into subjective time (see evidence in Brocas et al., 2015). The transformation $\tau(t,s)$ reflects how long an interval $t-s$ feels for an agent of age s, and the discount rate $\delta(\tau(t,s))$ reflects how much an agent dislikes that feeling of waiting. The discount factor applied by an agent of age s to a payoff to be received at time t is then

$$\beta(\tau(t,s)) = e^{-\int_0^{\tau(t,s)} \delta(w)\,\mathrm{d}w}. \tag{3}$$

We say that an agent becomes more *patient* as she ages if, as s increases, she applies a strictly lower discount factor to payoffs that will be received at time $s+n$, for fixed $n>0$. A more patient agent is more willing to delay rewards by n units of clock time. The following proposition establishes that if time speeds up as an agent ages and the agent applies a strictly positive discount rate to subjective time, then the agent becomes more patient over time.

Proposition 1. *An agent with $\alpha > 0$ and $\delta(\cdot) > 0$ becomes more patient as she ages.*

Proof. From equation (3), we have

$$\frac{\mathrm{d}\beta(\tau(s+n,s))}{\mathrm{d}s} = -\delta(\tau(s+n,s)) \left[\left.\frac{\partial \tau(t,s)}{\partial t}\right|_{t=s+n} + \left.\frac{\partial \tau(t,s)}{\partial s}\right|_{t=s+n} \right] \beta(\tau(s+n,s)).$$

From equation (1), we have

$$\begin{aligned} \left.\frac{\partial \tau(t,s)}{\partial t}\right|_{t=s+n} + \left.\frac{\partial \tau(t,s)}{\partial s}\right|_{t=s+n} &= k\frac{1}{s^\alpha} + k\frac{-\alpha}{s^{\alpha+1}}(s+n-s) - k\frac{1}{s^\alpha} \\ &= -\,n\,k\frac{\alpha}{s^{\alpha+1}} \leq 0. \end{aligned}$$

[13] If $\alpha > 0$ and we fix $t = s+n$, then an infinitely-lived agent would see time pass so quickly that every future moment would feel instantaneous: $\lim_{s\to\infty} \tau(s+n,s) = 0$. This notion is similar to philosophical arguments in Stump and Kretzmann (1981) and subsequent literature about how an eternal being relates to time.

The inequality is strict if $\alpha > 0$. Substituting into the previous expression, we see that

$$\frac{\mathrm{d}\beta(\tau(s+n,s))}{\mathrm{d}s} > 0 \text{ if } \alpha > 0 \text{ and } \delta(\tau(s+n,s)) > 0.$$

□

As the agent ages, a given wait "by the clock" feels shorter. The agent therefore becomes more willing to wait a given interval of clock time in order to receive a reward. This effect arises for any form of the discount rate, as long as the discount rate is positive and depends only on the subjective length of an interval of delay. Holding mortality risk and income constant, this result suggests that agents become more willing to invest in education or to save as they age.[14] Empirical evidence indeed suggests that discount rates may decline with age, even after controlling for income (Green et al., 1994; Rogers, 1994; Read and Read, 2004; Chu et al., 2010; Tanaka et al., 2010).[15]

3 Dynamic consistency

A dynamically consistent agent does not revise her preference ordering over future outcomes as she draws closer to them in clock time. Becoming more patient as we age does not directly imply dynamic inconsistency: patience concerns our willingness to wait for an interval of length n in clock time, whereas dynamic consistency concerns our choices for a specific instant t in clock time. In the standard model, a dynamically consistent agent may employ a discount rate that varies with calendar time, but a dynamically consistent agent may not employ a discount rate that varies with the interval of delay between the evaluation moment and the payoff.

In the present setting, dynamic consistency requires that the agent's discount factor for some fixed future time t declines at the same rate regardless of her age.[16] This condition is equivalent to stating that the agent's time t discount rate measured in *clock time* must be constant as she ages. Dynamic consistency thus requires:

$$\frac{\partial\tilde{\delta}(t,s)}{\partial s} = 0 \text{ for all } t > 0 \text{ and all } s < t, \text{ where } \tilde{\delta}(t,s) \triangleq -\frac{\mathrm{d}\beta(\tau(t,s))/\mathrm{d}t}{\beta(\tau(t,s))} = \delta(\tau(t,s))\,\frac{\partial\tau(t,s)}{\partial t}.$$

[14] We may speculate that discount rates vary cross-culturally if different societies have different types or lengths of collective memory.

[15] See Meier and Sprenger (2015) for contrary results.

[16] To see this, consider a time s agent optimizing consumption choices over an interval $\mathrm{d}t$ beginning at some future time $t > s$. She has stationary instantaneous utility $u(c(t))$ over consumption $c(t)$. Via standard first-order conditions, she equates $\beta(\tau(t,s))\,u'(c(t))$ and $\beta(\tau(t+\mathrm{d}t,s))\,u'(c(t+\mathrm{d}t))$. Using a first-order Taylor series expansion of $\beta(\tau(t+\mathrm{d}t,s))$ around $\beta(\tau(t,s))$ and dividing through by $\beta(\tau(t,s))$, we find that she chooses her consumption profile so that $u'(c(t)) = [1-\tilde{\delta}(t,s)\,\mathrm{d}t]\,u'(c(t+\mathrm{d}t))$, where $\tilde{\delta}(t,s)$ is the clock-time discount rate defined immediately below. This condition holds for all ages $s \leq t$ if and only if $\partial\tilde{\delta}(t,s)/\partial s = 0$ for all $s \leq t$.

The function $\delta(\tau(t,s))$ gives the discount rate measured in subjective time $(-[\mathrm{d}\beta(\tau(t,s))/\mathrm{d}s]/\beta(\tau(t,s)))$; the function $\tilde{\delta}(t,s)$ gives the discount rate measured in clock time $(-[\mathrm{d}\beta(\tau(t,s))/\mathrm{d}t]/\beta(\tau(t,s)))$. Dynamic consistency applies to choices revealed as clock time passes and so requires that $\tilde{\delta}(t,s)$ be independent of s.

Substituting in previous definitions, we find the central expression of the paper:

$$\frac{\partial \tilde{\delta}(t,s)}{\partial s} = \underbrace{\frac{\partial^2 \tau(t,s)}{\partial t \partial s}\delta(\tau(t,s))}_{\text{time acceleration effect, } \leq 0} + \underbrace{\frac{\partial \tau(t,s)}{\partial t}\delta'(\tau(t,s))\frac{\partial \tau(t,s)}{\partial s}}_{\text{time compression effect, } \geq 0}. \tag{4}$$

The first set of terms on the right-hand side describes the effect of time accelerating as the agent ages: as s increases, the weakly negative cross-partial reflects that a given interval of clock time feels shorter, which tends to reduce the discount rate as measured in clock time. From equation (2), the time acceleration effect is zero under the standard model ($\alpha = 0$) and is strictly negative under the time-acceleration model ($\alpha > 0$) when $\delta(\cdot) > 0$. The second set of terms on the right-hand side describes the effect of aging changing our perspective on a future interval of time, as moving forward in space changes our perspective on the gap between railroad ties (Karp, 2015). Several psychology studies find that people compress intervals of time more strongly as they look further into the future (Ekman and Lundberg, 1971; Kim and Zauberman, 2009; Zauberman et al., 2009; Bradford et al., 2014). By bringing the agent forward in clock time, aging shortens the subjective time remaining until any given fixed point t in clock time ($\partial\tau/\partial s < 0$, as is easy to verify). If the subjective-time discount rate is constant ($\delta'(\tau) = 0$, as for exponential discounters), the shortening of the subjective interval until clock time t is irrelevant for the clock-time discount rate and this time compression effect is zero. However, if we can more finely distinguish the first intervals of subjective time (as with perspective in space), then the discount rate declines in subjective time ($\delta'(\tau) < 0$, as for hyperbolic discounters). In this case, the shortening of the subjective interval until clock time t increases the discount rate applied to the point in subjective time corresponding to clock time t, which makes the time compression effect strictly positive.

The time acceleration and time compression effects oppose each other. In the standard model of time perception with $\alpha = 0$, subjective time is equivalent to clock time, and we obtain dynamic consistency if and only if the discount rate is constant in subjective time. This is the commonly understood condition for dynamic consistency. The following proposition establishes an additional set of conditions under which we obtain dynamic consistency.

Proposition 2. *An agent with $\alpha > 0$ is dynamically consistent if and only if, for all times t and ages $s < t$, she has a discount rate $\delta(\tau)$ of the hyperbolic form $\frac{C}{1+\rho(s)\tau}$, where $\rho(s) = \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}s}s^{\alpha}$ and C is weakly positive and constant with respect to τ.*

Proof. Assume $\alpha > 0$. From the condition for dynamic consistency and equation (4), we have that the agent is dynamically consistent if and only if

$$\delta'(\tau(t,s)) = -\delta(\tau(t,s))\frac{\partial^2\tau(t,s)}{\partial t\partial s}\left[\frac{\partial\tau(t,s)}{\partial t}\frac{\partial\tau(t,s)}{\partial s}\right]^{-1} < 0.$$

This clearly holds if $\delta(\tau(t,s)) = 0$, which matches the form in the proposition when $C = 0$. Now assume that $\delta(\tau(t,s)) > 0$. Substitute in from previous definitions and expressions to obtain that the agent is dynamically consistent if and only if:

$$\begin{aligned}\frac{\delta'(\tau(t,s))}{\delta(\tau(t,s))} =& \frac{-1}{\tau(t,s) + \frac{k}{\alpha s^{\alpha-1}}}\\ =& \frac{-\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}s}s^{\alpha}}{1+\frac{1}{k}\left[\frac{\mathrm{d}}{\mathrm{d}s}s^{\alpha}\right]\tau(t,s)}\\ =& \frac{-\rho(s)}{1+\rho(s)\,\tau(t,s)},\end{aligned}$$

where $\rho(s) \triangleq \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}s}s^{\alpha}$. Integrating with respect to τ, we have:

$$\ln(\delta(\tau(t,s))) = -\ln(1+\rho(s)\,\tau(t,s)) + D,$$

where D is an arbitrary constant of integration. We thus have:

$$\delta(\tau(t,s)) = \frac{C}{1+\rho(s)\,\tau(t,s)},$$

where $C \triangleq e^D > 0$.

□

An agent who does not discount the future at all ($C = 0$) is dynamically consistent because her decisions are insensitive to the rate at which time passes. Further, we have established that hyperbolic discounters ($C > 0$) can also be dynamically consistent. Herrnstein (1981), Mazur (1987), and Loewenstein and Prelec (1992) each proposed hyperbolic discount factors which imply, for $\alpha = 1$, discount rates of the form given in the statement of the proposition.[17] Under knife-edge parameterizations, such hyperbolic discounting may not induce preference reversals. And experiments in Green et al. (1994) found that $\rho(s)$ decreases in age s, which suggests that $\alpha \leq 1$, as proposed in the psychology literature.

Why might a hyperbolic discounter be dynamically consistent? A hyperbolic form for $\delta(\cdot)$ suggests that the agent sees subjective time "in perspective": the agent can better

[17] Loewenstein and Prelec (1992) proposed the discount factor $(1+\gamma\,\tau)^{-\xi/\gamma}$, for $\gamma, \xi > 0$. Herrnstein (1981) and Mazur (1987) investigated the special case of $\xi = \gamma$. These discount factors generate the discount rates described in Proposition 2 when $C = \xi$ and $\rho(s) = \gamma$. We have $\rho(s) = \gamma$ when $\alpha = 1$ and $k = 1/\gamma$.

discriminate nearby intervals. Because intervals of subjective time closer to $\tau = 0$ loom larger from the agent's viewpoint, the agent applies a greater discount rate $\delta(\tau(t,s))$ to small τ. As the agent advances through "real" clock time, the agent both ages and approaches a given future time t. This has two effects. First, as the agent ages, time seems to pass faster, which makes each interval of subjective time feel shorter and so tends to lower the discount rate as measured in clock time. Second, as the agent draws closer to the subjective time corresponding to the instant t, any given interval of subjective time around clock time t appears larger, tending to increase the discount rate whether measured in subjective time or in clock time. So a given interval of subjective time appears larger as the agent advances closer to the instant t, but each interval of clock time maps into a smaller interval of subjective time as the agent simultaneously ages. For a particular hyperbolic form for $\delta(\cdot)$, these two effects cancel, and the clock-time discount rate applied to instant t remains constant as the agent advances towards t.

Nearly all of the experimental literature elicits each subject's schedule of discount rates at some fixed time s. If this schedule is not constant in the horizon t (as many studies indeed find), then the standard interpretation is that the agent is dynamically inconsistent. However, the present model suggests that a longitudinal study that examined the agent's actual choices over time may nonetheless fail to observe dynamic consistency. This unexpected result is in fact consistent with the results of longitudinal experiments in Read et al. (2012) and Halevy (2015).

In Proposition 2, we obtained dynamic consistency only if the agent does not discount at all or if the parameter ρ in the hyperbolic discount function varies in a precise way with age s. However, there is an important special case of the subjective time function $\tau(t,s)$ for which we obtain dynamic consistency with an age-invariant hyperbolic discount rate.

Corollary 3. *An agent with $\alpha = 1$ and $k = 1$ is dynamically consistent if and only if she has a discount rate $\delta(\tau)$ of the hyperbolic form $\frac{C}{1+\tau}$, where C is weakly positive and constant with respect to τ.*

Proof. Follows directly from Proposition 2. □

The case of $\alpha = 1$ and $k = 1$ is of special interest because it corresponds to the functional form originally proposed by Janet (1877) and Doob (1971). It also corresponds to an application of the Weber-Fechner law to the case of time perception. If we perceive an interval of time by measuring it against our cumulative lived experience, then our revealed choices may be dynamically consistent under a particularly simple form of hyperbolic discounting.[18]

But perhaps the correct model of time perception does not use $\alpha = 1$ and $k = 1$. We still may doubt whether $\rho(s)$ takes the precise form given in Proposition 2. In that case, a hyperbolic discounter will not be dynamically consistent. We now consider degrees of dynamic inconsistency. Assume that aging affects two agents in the same way, so that $\tau(t,s)$

[18] Recall that this measure of cumulative lived experience need not begin with birth.

does not vary by agent. Each agent is currently of age w. Let $\tilde{\delta}_i(t,s)$ be the clock-time discount rate for agent i. We say that agent x is "more dynamically consistent around time t" than is agent y if $|\partial\tilde{\delta}_x(t,s)/\partial s| \leq |\partial\tilde{\delta}_y(t,s)/\partial s|$ for all $s \in [w,t]$.[19] The following proposition describes which subjective-time discount rates generate a greater degree of dynamic consistency than does a constant subjective-time discount rate:

Proposition 4. *Let agent y use a constant subjective-time discount rate of $\delta_y > 0$ and define $\delta^*(\tau(t,s))$ as the nonzero dynamically consistent hyperbolic discount rate from Proposition 2. Agent x is more dynamically consistent around time t than is agent y if $\delta_y \geq \delta_x(0)$ and, for all $s \in [w,t]$, $\delta_x'(\tau(t,s))/\delta_x(\tau(t,s)) \geq \delta^{*\prime}(\tau(t,s))/\delta^*(\tau(t,s))$.*

Proof. From equation (4),

$$\frac{\partial\tilde{\delta}_y(t,s)}{\partial s} = \frac{\partial^2\tau(t,s)}{\partial t\partial s}\delta_y \leq 0.$$

Again using equation (4), agent x is more dynamically consistent around time t than is agent y if and only if

$$\begin{aligned}\left|\frac{\partial^2\tau(t,s)}{\partial t\partial s}\right|\delta_y &\geq \left|\frac{\partial^2\tau(t,s)}{\partial t\partial s}\delta_x(\tau(t,s)) + \frac{\partial\tau(t,s)}{\partial t}\delta_x'(\tau(t,s))\frac{\partial\tau(t,s)}{\partial s}\right| \\ &= \left|\frac{\partial^2\tau(t,s)}{\partial t\partial s}\left[\delta_x(\tau(t,s)) + \delta_x'(\tau(t,s))\tau(t,s)\right] - \left(\frac{\partial\tau(t,s)}{\partial t}\right)^2\delta_x'(\tau(t,s))\right|.\end{aligned}$$

The equality recognizes that $\partial\tau(t,s)/\partial s = [\partial^2\tau(t,s)/\partial t\partial s](t-s) - \partial\tau(t,s)/\partial t$ and that $\tau(t,s) = [\partial\tau(t,s)/\partial t](t-s)$. Explicitly writing out the partial derivatives, we have

$$\begin{aligned}k\frac{\alpha}{s^{\alpha+1}}\delta_y &\geq \left|k\frac{-\alpha}{s^{\alpha+1}}\left[\delta_x(\tau(t,s)) + \delta_x'(\tau(t,s))\tau(t,s)\right] - \left(\frac{k}{s^\alpha}\right)^2\delta_x'(\tau(t,s))\right| \\ \Leftrightarrow \delta_y &\geq \left|-\delta_x(\tau(t,s)) - \delta_x'(\tau(t,s))\tau(t,s) - \frac{1}{\rho(s)}\delta_x'(\tau(t,s))\right| \triangleq |\chi|,\end{aligned}$$

where $\rho(s)$ is as defined in Proposition 2. We have $\chi \leq 0$ if and only if

$$\frac{\delta_x'(\tau(t,s))}{\delta_x(\tau(t,s))} \geq \frac{-\rho(s)}{1+\rho(s)\,\tau(t,s)} = \frac{\delta^{*\prime}(\tau(t,s))}{\delta^*(\tau(t,s))}.$$

[19] Referring to the setting in footnote 16, assume that agents x and y have the same utility function and apply, at age w, the same clock-time discount rate to times $s \leq t$, so that $\tilde{\delta}_x(s,w) = \tilde{\delta}_y(s,w)$. They therefore both trade off consumption in the same way between times s and t. From footnote 16, each agent i chooses her consumption profile so that her marginal utilities satisfy $[u'(c(s))]/[u'(c(t))] = 1 - \tilde{\delta}_i(s,w)\,[t-s]$. The ratio of marginal utilities corresponding to the optimal profile changes by less in age for agent x when $\partial\tilde{\delta}_x(s,w)/\partial w$ is smaller in magnitude than $\partial\tilde{\delta}_y(s,w)/\partial w$. The definition of asymptotic time consistency in Heal (1998, p. 108) has a similar flavor as our definition of greater dynamic consistency.

Assume that this last condition holds, so that $\chi \leq 0$. Assume also that $\delta_y \geq \delta_x(0)$, which implies that $\delta_y \geq \delta_x(\tau(t,s))$. Then we have that agent x is more dynamically consistent around time t than is agent y if and only if, for all $s \in [w,t]$,

$$\begin{aligned}\delta_y \geq& \delta_x(\tau(t,s)) + \delta_x'(\tau(t,s))\tau(t,s) + \frac{1}{\rho(s)}\delta_x'(\tau(t,s))\\ \Leftrightarrow \delta_y - \delta_x(\tau(t,s)) \geq& \delta_x'(\tau(t,s))\tau(t,s) + \frac{1}{\rho(s)}\delta_x'(\tau(t,s)).\end{aligned}$$

This inequality is indeed satisfied, because $\delta_x'(\tau(t,s)) \leq 0$.

□

Dynamic consistency is a knife-edge result, but many types of hyperbolic discounters may nonetheless be less dynamically inconsistent than an exponential discounter. One such type of hyperbolic discounter uses a subjective-time discount rate that starts out below the exponential discounter's subjective-time discount rate and declines less rapidly than does the dynamically consistent subjective-time discount rate $\delta^*(\tau(t,s))$ from Proposition 2. This type of agent's initially lower discount rate ensures that her decisions are no more sensitive to intertemporal tradeoffs than are the exponential discounter's decisions, and the decline in her subjective-time discount rate at least partially offsets the effects of aging. The greater is the discount rate employed by the exponential discounter, the larger is this subset of agents who are more dynamically consistent than that exponential discounter. A high subjective-time discount rate amplifies the importance of time's rate of passage for an exponential discounter and so makes it more likely that a hyperbolic discounter's declining subjective-time discount rate better preserves tradeoffs over time.

Finally, how do our results relate to standard theoretical treatments which identify dynamic consistency with exponential discounting? In particular, Koopmans (1960) shows that exponential discounting is the only type of discount function consistent with particular axioms on intertemporal choice. Brown (2010) and Halevy (2015) have noted that the "stationarity" axiom in Koopmans (1960) actually intertwines dynamic consistency with an assumption that preferences do not change with calendar time. Brown (2010) calls this latter assumption one of "autonomy", and Halevy (2015) calls this assumption one of "time-invariance". The present paper's model of age-dependent perception of time violates the assumption of autonomy or time-invariance and violates the stationarity axiom in Koopmans (1960). Once we allow for non-autonomous or time-varying preferences, dynamic consistency is no longer identified with exponential discounting. Moreover, I show that for plausible models of non-autonomous or time-varying preferences, dynamic consistency is actually identified with types of hyperbolic discounting.

4 Anticipated acceleration of time

Up until now, we have assumed that when the agent discounts future payoffs, she imagines that time will always flow as fast as it does in the present. However, the agent would have experienced time accelerating in previous years, and it is plausible that she projects her experienced acceleration into the future.[20] In this section, we consider the case of correct expectations about the future acceleration of time.[21]

If the time s agent correctly forecasts the future flow of time, then the subjective perception of a future interval $\mathrm{d}t$ of clock time no longer depends on the agent's current age s but instead depends on the agent's future age t. Maintaining our functional form, we now have

$$\frac{\partial \hat{\tau}(t,s)}{\partial t} = \frac{k}{t^{\alpha}},$$

where a hat indicates anticipation of the effect of aging. Integrating over the interval from s to t, we have:

$$\hat{\tau}(t,s) = \frac{k}{1-\alpha}\left(t^{1-\alpha} - s^{1-\alpha}\right) \quad \text{if } \alpha \neq 1,$$
$$\hat{\tau}(t,s) = k\left(\ln(t) - \ln(s)\right) \quad \text{if } \alpha = 1.$$

If $\alpha = 0$ and $k = 1$, we have the standard model, in which $\hat{\tau}(t,s) = t - s$. In this case, it is clearly true that dynamic consistency requires exponential discounting. The following proposition establishes that, in fact, dynamic consistency is identified with exponential discounting of subjective time for any value of α.

Proposition 5. *An agent who anticipates the acceleration of time is dynamically consistent if and only if $\delta'(\tau) = 0$.*

Proof. We have seen that the agent who does not anticipate time accelerating is dynamically consistent if and only if equation (4) equals zero for all times t and ages $s < t$. An agent who anticipates time accelerating is dynamically consistent if and only if a version of equation (4) with $\hat{\tau}(t,s)$ in place of $\tau(t,s)$ equals zero for all times t and ages $s < t$. It is clear that $\partial^2\hat{\tau}(t,s)/\partial t\partial s = 0$ while $\partial\hat{\tau}(t,s)/\partial t \neq 0$ and $\partial\hat{\tau}(t,s)/\partial s \neq 0$. Therefore, from equation (4), the agent who anticipates time accelerating is dynamically consistent if and only if $\delta'(\tau) = 0$. □

[20] Joubert (1990) reports survey evidence of such expectations. However, people who demonstrate projection bias (Loewenstein et al., 2003) will nonetheless underpredict the degree to which time will accelerate.

[21] Much literature discusses the implications for intertemporal decisions of naivete versus sophistication in the context of present-biased preferences (e.g., Laibson, 1997; O'Donoghue and Rabin, 1999). We here focus on the implications of full sophistication about the perception of time for the consistency of the plans formed by an agent who constructs her policy naively. Any type of agent who understands her future selves' tradeoffs could play a Markov perfect equilibrium with her future selves and thereby approximate dynamic consistency.

Why does anticipating the effect of aging restore the traditional result that only exponential discounters are dynamically consistent? The key is that $\partial^2 \hat{\tau}(t,s)/\partial t \partial s = 0$. This zero cross-partial defines sophistication with respect to the acceleration of time: when the agent anticipates the effect of aging, she adjusts her perception of intervals of time around t so that her perception does not change as she approaches t. We are back to the case of autonomous preferences (Brown, 2010) or time-invariant preferences (Halevy, 2015). The present interval of time always passes more quickly for an older person, but a given interval of calendar time passes at exactly the same rate for an old person as for a young person who is sophisticated in the ways of aging. The subjective perception of the rate of passage of time then depends only on calendar time, and both young and old use the same calendar. As is well known (Strotz, 1955–1956), a discount rate that depends on calendar time does not itself challenge dynamic consistency because young agents make plans in full awareness of future discount rates. If the discount rate $\delta(\cdot)$ applied to an instant of subjective time varied with the distance to that instant, the agent would be dynamically inconsistent for exactly the same reason that hyperbolic discounters are dynamically inconsistent in the standard model of intertemporal choice.

5 Conclusions

I have formally modeled the folk wisdom and psychological findings that time accelerates with age. Exponential discounting yields dynamic consistency only if the agent correctly anticipates how time will accelerate as she ages. In the case where the agent does not anticipate this acceleration, we obtain dynamic consistency for a form of hyperbolic discounting. We also see that many variations of hyperbolic discounting may be "more" dynamically consistent than exponential discounting, especially if the latter uses a high discount rate. We thereby see the potential importance of jointly modeling multiple features of the psychology of time perception: modeling a single feature at a time may lead to economically misleading conclusions. For instance, most economic experiments elicit hyperbolic discount rates from a cross-section of agents and infer that these agents will demonstrate dynamic inconsistency, but at least two recent longitudinal studies have failed to find evidence of dynamic consistency in time series of actual choices. The present paper's theoretical framework offers one explanation of this apparent paradox, demonstrating that dynamic inconsistency is not experimentally identified by cross-sectional evidence of hyperbolic discounting. Future experimental work should seek to refine measures of how time perception changes with age, and future life-cycle models should estimate age-dependent discount rates and explore their utility in explaining consumption and savings data.

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