Entropic Discounting of Geometric Means: An Index for Lotteries and Portfolios

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Abstract

This paper models choice under risk, whether such choices are over simple lotteries or over lotteries that arise from a portfolio of endogenously generated shares of assets. Our model satisfies various prudence and risk aversion conditions, is consistent with Allais paradox rankings, and generates various insurance-related results. Within a portfolio framework with compounded reinvestments, our index fits the risks/rewards data from post-war US stock market returns and recent international markets, at least as well as does the standard deviation measures more typically used. It also generates returns that are consistent with the equity premium puzzle.

1 Introduction

Though a portfolio of assets is a type of lottery, the literature on lotteries and portfolio investments remains fractured and self-referentially inconsistent when trying to explain empirical observations. Conventional analysis based on expected utility employs the probability-weighted arithmetic mean of the utility, but is known to be inconsistent with some actual lottery choices (the Allais paradox, for example), and actual portfolio returns to risky assets are much too high under expected utility theory (the equity premium puzzle). Despite these and other difficulties, the expected utility approach to modeling choice under uncertainty is by far the dominant one in this and related literatures. However, this choice is not obviously the correct one.

Many alternatives have been proposed to replace expected utility. Our proposed index, which we call the geometric entropy index, is simple (Ockham’s razor), it is consistent with many of the empirical results that expected utility cannot explain, it provides an easy estimate of the risk/reward trade-off that can be readily employed in empirical research, and it explains lottery and portfolio choices in a unified decision framework. Our index relies on the geometric mean of returns to measure reward and on their entropy to measure risk.

1 The research assistance of Evan Magnusson was most useful. We are grateful for comments from James Faulconer, Kerk Phillips, Grant Gannaway, Mark Machina and discussions with Scott Condie. Wharton Research Data Services (WRDS) was used in preparing the empirical research. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.
The geometric mean of returns is arguably a better measure of reward than the arithmetic. Latané (1959) pointed out that for a stationary process where gains are reinvested proportionately, the portfolio with the greatest geometric mean results asymptotically in the highest terminal wealth.\footnote{Latané called this the maximum chance principle, and asserted that risking the same proportions on the same terms, even in a world experiencing some change, was at least as sensible as applying mean-variance analysis and expected utility theory. If agents adopt strategies that have worked well in the past, there may be a tendency for geometric mean maximization to persist. Samuelson (1971, p. 2496), by contrast, argued that even though maximizing the geometric mean leads to higher terminal wealth, it is still an inferior criterion: “These remarks critical of the criterion of maximum expected average compound growth do not deny that this criterion, arbitrary as it is, still avoids some of the even greater arbitrariness of conventional mean-variance analysis. Its essential defect is that it attempts to replace asymptotically sufficient parameters [E log(X), variance log(X)] by the first of these alone.”}\footnote{Employing a slightly different approach, Jean (1980) showed that geometric mean ranking is a necessary, but not generally sufficient, condition for any degree of stochastic dominance ranking. Often critiques of Latané’s maximum chance principle have simply pointed out that it does not conform to expected utility maximization. (See Elton and Gruber, 1995, p 232). Since expected utility theory has long been criticized as inconsistent with experimental evidence, this is not a compelling critique.}

To include risk in models derived from expected utility, most researchers choose some function of the whole distribution of wealth (such as the lottery variance), or some function of the conditional distribution of wealth (such as value-at-risk measures, based on various quintiles of the distribution). Consistent with Latané’s maximization approach, we develop a “maximum choice” model by generalizing his maximum chance principal. Specifically, we integrate uncertainty by considering optimal betting on wealth outcomes. We assume that the value of such bets is a power function of wealth in state \(i\), \(w_i^{1/\lambda}\), where \(\lambda\) is an unknown parametric constant, and that investors optimize the log of geometric mean value of those bets; such an approach yields a risk measure arising naturally from the Latané geometric mean reinvestment framework.

More specifically, suppose an investor chooses to invest, for each \(i\), a share \(a_i \geq 0\) of his or her wealth in an asset that pays \(w_i\) if the state \(i\) is realized and nothing otherwise, with the objective to maximize

\[
\sum_i p_i \ln (a_i w_i^{1/\lambda}),
\]

or, equivalently, the log of the geometric mean of their value of lottery bets,

\[
(1/\lambda) \sum_i p_i \ln w_i + \sum_i p_i \ln a_i
\]

subject to \(\Sigma a_i = 1\). The \(a_i\) only appear in the second term, and constrained optimization yields \(a_i = p_i\) for each \(i\). Hence, lotteries are chosen on the basis of maximizing the following (choosing a convenient re-normalization):

\[
\sum_i p_i \ln w_i + \lambda \sum_i p_i \ln p_i = \sum_i p_i \ln w_i - \lambda (-\sum_i p_i \ln p_i)
\]

The term \((-\sum_i p_i \ln p_i)\) is the entropy discount for event uncertainty\footnote{Arrow, 1971.}. So unlike Latané, and consistent with Samuelson’s critique, we do not make the geometric mean do all the

\footnote{See Latané (1959, p. 147) and Samuelson (1971, p. 2494).}
work. We use the geometric mean of wealth to measure reward, but we subtract the log of the geometric mean of probabilities to measure risk, that is, we use entropy as our risk measure.

Entropy has been useful in information theory (Shannon 1948), inequality measures (Theil, 1972), and psychology (Norwich, 1993), but has yet to gain any traction—parallel with Latané’s geometric mean criteria—in the economics of uncertainty and finance. By trading off entropy against the log of the geometric mean, we develop a simple one-parameter index for lotteries. The entropy discount for a lottery’s uncertainty,

$$-\lambda \left( \sum_{\sigma \in S} -p(\sigma) \ln(p(\sigma)) \right),$$

where $\lambda$ is a positive parameter, implies a probability-partitioning effect in lotteries. Indeed, we view the partition of the state space as a central feature of the consumer’s perceptual apparatus regarding uncertain outcomes. A lottery initially presented to a consumer is perceived as different from that same initial lottery but where lottery branches have been perturbed with respect to probabilities: whether branches with the same outcome have been merged across probabilities, or whether new lottery branches are generated by dividing up a given probability for a given outcome into several new branches—such perturbations are new lotteries as far as the consumer is concerned. In particular, any division of $p(\sigma)$ into smaller probabilities with the same outcome (e.g. $(p_1(\sigma),...,p_N(\sigma))$ into $(\Sigma_{i=1}^{N_1} p_1(\sigma)),...,\Sigma_{i=1}^{N_N} p_N(\sigma))$ such that $\Sigma_{i=1}^{N} p_i(\sigma) = p_j(\sigma)$) increases entropy and lowers the ranking of the lottery. Lotteries thus “simplified” or “expanded” across probabilities (for given outcomes) are different lotteries given consumers geometropic tastes (and the partitioning effect due to entropy), and thus ranking formulas implicitly assuming such simplifications, such as the first order stochastic dominance criteria, will not provide a reliable guide to individuals’ actual lottery rankings. In particular, a lottery with (wealth, probability) pair $(w(\sigma);p(\sigma))$ will be preferred to a lottery in which $(w(\sigma);p(\sigma))$ is replaced with the pair $(w(\sigma) + \epsilon;p(\sigma)/2), (w(\sigma) + \epsilon;p(\sigma)/2)$ for some $\epsilon > 0$ even though such a choice violates first order stochastic dominance.

It follows that the independence axiom, which drives many of the results from expected utility theory, does not hold as it depends on the simplification property holding. In particular, merging lottery branches with the same outcome by adding their probabilities, or dividing an outcome into several new lottery branches by dividing up its probability, changes lottery ordering in our model but not in the expected utility model. So our model explicitly departs from the expected utility framework. We show the index is empirically relevant in explaining real world lottery choices. We also extend the analysis to create an index for portfolios.

Section 2 formally introduces our index. It also extends the analysis to apply to endogenously constructed portfolios. In section 3, we show that many of the desirable properties of a lottery index hold for our index, including some of the basic insurance

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5For one partial exception, see Yang and Qi (2005)
6See Birnbaum(2005) for many examples of choices that violate the first order stochastic dominance (FOSD) criteria. We are grateful to Mark Machina for pointing this out to us. Note that the probability partitioning property of entropy implies that not only is FOSD an inappropriate metric for consumer’s ranking of risky choices, but all moment indices of risk where probabilities might be so partitioned.
7For prior departures see, for example, Machina (1989), Starmer (2000), Kahneman and Tversky (1979), and Tversky and Kahneman, 1992.
results from expected utility theory. Market equilibrium is discussed in Section 4. Section 5 provides the estimation framework for the trade-off between geometric means and entropy. Section 6 concludes.

# 2 Geomentropic Preferences

## 2.1 Geomentropic Preferences over Lotteries

A lottery is a list $L = (S, S, p, w)$ where $S$ is a finite state space, $S$ is a partition of $S$, $p : S \rightarrow [0, 1]$ assigns probabilities to partition sets, and $w : S \rightarrow R_{++}$ assigns payoffs to partition sets. The geometric mean of the lottery $L$ is $G(L) = \prod_{\sigma \in S} w(\sigma)^{p(\sigma)}$ and its entropy is $N(L) = -\sum_{\sigma \in S} p(\sigma) \ln p(\sigma)$.

Let $\succeq$ be a preference ordering over lotteries. If for two lotteries $L$ and $L'$, both $L \succeq L'$ and $L' \succeq L$ then we write $L \sim L'$. If for two lotteries $L$ and $L'$, $L \succeq L'$ but it is not the case that $L' \succeq L$ then we write $L \succ L'$.

Our lottery index can be derived from preference axioms in straightforward fashion. The first two axioms, A1 and A2, are standard:

A1. (Completeness and Transitivity) For all lotteries $L$ and $L'$, either $L \succeq L'$ or $L' \succeq L$. Furthermore, for all lotteries $L$, $L'$, and $L''$, if $L \succeq L'$ and $L' \succeq L''$ then $L \succeq L''$.

By a slight abuse of notation, let $(S, S, 1, w)$ denote the the zero-entropy lottery that assigns $w$ to every state. Specifically, $S = S$ is meant to suggest that the partition contains only one set: the entire state space, $S$. This lottery is analogous to a risk-free lottery in the mean-variance framework.

A2. (Continuity) For any lottery $L$, the sets $\{(S, S, 1, w) \succeq L\}$ and $\{L \succeq (S, S, 1, w)\}$ are closed.

A consequence of A1 and A2 is the existence of a zero-entropy equivalent, that is, $G^*(L) \in R_{++}$ such that $(S, S, 1, G^*(L)) \sim L$. A3 makes explicit the trade-off between geometric mean and entropy.

A3(a). (Geometric Mean Monotonicity) If $N(L) = N(L')$ and $G(L) > G(L')$ then $L \succ L'$.

A3(b). (Entropy Monotonicity) If $G(L) = G(L')$ and $N(L) < N(L')$ then $L \succ L'$.

Under A1-A3, lotteries can be ranked by their zero-entropy equivalents, as shown in the following proposition:

Proposition 1: If preferences satisfy A1-A3 then $L \succ L'$ iff $G^*(L) > G^*(L')$.

Proof: First suppose $L \succ L'$. Let $K$ and $K'$ be the respective zero-entropy lotteries that are equally preferred to $L$ and $L'$. (These exist by A2.) By A1 $K \succ K'$. Since $N(K) = N(K') = 0$, A3 implies $G(K) > G(K')$ and hence $G^*(L) > G^*(L')$.

Shannon (1948) is the most significant early advocate for this entropy measure.
Now suppose $G^*(L) > G^*(L')$. Recall that $G^*(L)$ and $G^*(L')$ are the geometric means of the risk-free lotteries $K$ and $K'$ that are equally preferred to $L$ and $L'$, respectively. Since $K$ and $K'$ are risk-free, and $G^*(L) > G^*(L')$, by A3, $L > L'$. QED

Although more flexibility is possible, a constant rate of substitution between risk and reward is a reasonable benchmark, and can be thought of as a first order approximation to actual preferences. It will also be seen to be general enough for many purposes.

A4. (Constant Marginal Rate of Substitution) There exists a constant $\lambda > 0$ such that, for any two lotteries $L$ and $L'$, if $L \sim L'$ then $\ln G(L) - \ln G(L') = \lambda [N(L) - N(L')]$.

Proposition 2: If preferences satisfy A1-A4 then there exists $\lambda > 0$ such that $L > L'$ iff $\ln G(L) - \lambda N(L) > \ln G(L') - \lambda N(L')$.

Proof: For any two lotteries $L$ and $L'$, $G^*(L)$ and $G^*(L')$ are well defined by A2. By A4, there exists $\lambda > 0$ such that $G^*(L) = \ln G(L) - \lambda N(L)$ and $G^*(L') = \ln G(L') - \lambda N(L')$. Proposition 1 ensures $G^*(L) > G^*(L')$ iff $L > L'$. So, by A1, $\ln G(L) - \lambda N(L) > \ln G(L') - \lambda N(L')$ iff $L > L'$. QED

We will refer to the index constructed in Proposition 2 as the \textit{geomentropic} index and the preferences described by A1-A4 as geomentropic preferences. In contrast to expected utility preferences, geomentropic preferences do not imply an index that is linear in the probabilities of the various outcomes. This linearity is a consequence of A2 along with Neumann’s and Morgenstern’s independence axiom: $L \succeq L'$ iff $\alpha L + (1 - \alpha) L'' \succeq \alpha L' + (1 - \alpha) L''$ for all $\alpha \in [0, 1]$ and all lotteries $L''$. The independence axiom has long had its critics\footnote{See, for example, Allais (1953), Machina (1982), and Machina (1989).} and we do not impose it here.

\subsection*{2.2 Geomentropic Preferences over Compound Lotteries}

Let $\{S_k, S_k, p_k, w_k\}_{k=1}^K$, be a list of lotteries. The associated compound lottery with probability weights $\alpha = \{\alpha_k\}_{k=1}^K$ is of the form $\{\cup_k S_k, \cup_k S_k, \alpha p, w\}$, where $\alpha p$ assigns $\alpha_k p(\sigma)$ to all $\sigma \in S_k$ and $w$ assigns $w_k(\sigma)$ to all $\sigma \in S_k$. This compound lottery can be interpreted as first randomly choosing a component lottery and then randomly generating the outcome of the chosen lottery. The geometric mean of the compound lottery $L$ is $G(L) = \prod_k \prod_{\sigma \in S_k} w_k(\sigma)^{\alpha_k p_k(\sigma)}$ and its entropy is defined as $N(L) = -\sum_k \sum_{\sigma \in S_k} \alpha_k p_k(\sigma) \ln \alpha_k p_k(\sigma)$.

Thus the geomentropic index distributes lottery weights. However, it does not merge branches with the same outcomes by adding probabilities, nor does it divide one branch outcome into two or more by dividing the probabilities. Because it does not exhibit \textit{scalability} over outcomes, it does not satisfy the reduction of compound lottery assumption implicit in the independence axiom. For example, an expected utility maximizer is indifferent between the compounding of two identical lotteries with probability weights $\alpha$ and $(1 - \alpha)$ and the same lottery with certainty. By contrast, the geomentropic agent prefers the simple lottery $(S, S, w, p)$ to the compounding of two identical simple lotteries due to the higher entropy of the compound lottery (the probability-partitioning effect). For the non-expected utility maximizer, the two lotteries are not equivalent, a point made years ago by Mark Machina (1987). Geomentropic preferences assume a trade-off between the
log of the lottery’s geometric mean and its entropy, instead of the reduction of compound lottery assumption inherent in the independence axiom of expected utility.

In summary, under geomentropic preferences, history matters in the sense that lottery presentation matters, with individuals perceiving subsequent algebraic manipulations involving outcomes as different lotteries. Scalability over outcomes implies non-unique geomentropic preferences. Harrison et al (2012) find that when choosing among many alternatives, the choices of experimental subjects do not support the reduction of compound lotteries assumption.

2.3 Geomentropic Preferences over Portfolios

A portfolio is a non-negative unit vector $v = (v_1, ..., v_K)$ where $v_k$ is interpreted as the fraction of the portfolio value $W$ invested in the $k^{th}$ asset. The $k^{th}$ asset is a lottery $(S_k, S_k, p_k, w_k)$. Given $\sigma = (\sigma_1, ..., \sigma_K)$, where $\sigma_k \in S_k$ for all $k$, let $\gamma(\sigma)$ be the joint probability that $s_k \in \sigma_k$ for all $k \in \{1, ..., K\}$. The geometric mean of the one-period payoff from a portfolio is

$$G_p(v) = \prod_{\sigma \in \times_k S_k} \left[ \sum_k \left( \frac{v_k W}{\pi_k} \right) w_k(\sigma_k) \right]^{\gamma(\sigma)}$$

where $W$ is the wealth invested and $\pi_k$ is the price of a unit of the $k^{th}$ asset. Portfolio risk is derived from the risks of the individual assets, as measured by the entropy of their marginal distributions, and their respective importance in the portfolio. Formally, define the marginal probabilities $p_k(\sigma_k)$ for the marginal distribution of asset $k$ by

$$p_k(\sigma_k) = \Sigma_{\ell \neq k} \Sigma_{\sigma_\ell \in S_\ell} \gamma(\sigma)$$

Define the entropy of the $k^{th}$ asset as

$$N_k = - \sum_{\sigma_k \in S_k} p_k(\sigma_k) \ln p_k(\sigma_k).$$

Then the entropy of the portfolio is defined as

$$N_p(v) = \sum_k v_k N_k$$

Specifically, portfolio risk consists of the arithmetic average of individual asset uncertainty ($N_k$), weighted by their presence in the portfolio. Unlike the standard portfolio model based on mean-variance analysis, unsystematic risk does not go to zero as the number of assets increases. Rather, the portfolio risk converges to the mean asset risk.

It is common in the finance literature to assume the existence of a risk-free asset. The geomentropic analog is an asset that exhibits only the trivial partition, that is, the partition $S$ that contains only the entire space $S$. Such an asset has entropy of zero and exhibits one outcome in all states of the world. Denote the price of a unit of the risk-free asset (cash) as $\pi_C$. It is straightforward to show that as the share of the riskless asset increases, holding the proportions of the other assets constant, portfolio entropy falls. If only the risk-free asset is held in the portfolio, then portfolio entropy is zero.
2.4 Dynamic Geomentropic Preferences

Suppose the investor behaves as if the stochastic process governing the states of the world is ergodic, that is, there exists \( \gamma : \times_k S_k \rightarrow [0, 1] \), such that \( \sum_{\sigma \in \times_k S_k} \gamma(\sigma) = 1, \gamma(\sigma) > 0 \) for all \( \sigma \), and \( \lim_{t \to \infty} p_{t\sigma} = \gamma(\sigma) \) for all \( \alpha \) and for all \( \sigma \), where \( p_{t\sigma} \) are the transition probabilities from state \( \alpha \) to \( \sigma \) after \( t \) periods. We assume the dynamic geomentropic investor acts as if investing to maximize relative to this ergodic distribution, and thus write the investor’s objective function as:

\[
\ln G_p(v) - \lambda N_p(v) = \sum_{\sigma \in \times_k S_k} T \gamma(\sigma) \ln \left[ \sum_k \left( \frac{w_k W}{w_k} \right) w_k(\sigma) \right] - \lambda N_p(v)
\]

for the \( T \)-period horizon reinvestment problem when all available assets are risky. We assume that the appropriate entropy-metric for the portfolio, is the entropy of this ergodic distribution. Under these assumptions, dynamic geomentropic preferences when there is a risk-free asset can be written as:

\[
\ln G_p(v) - \lambda N_p(v) = \ln \left\{ \prod_{\sigma \in \times_k S_k} \left[ \sum_k \left( \frac{w_k W}{w_k} \right) w_k(\sigma) \right] + \left( 1 - \sum_m v_m \right) \frac{W}{W^C} \right\}^{T \gamma(\sigma)} - \lambda \left[ \sum_k w_k N_k \right]
\]

A lottery is a portfolio with one risky asset.

3 Implications for Individual Behavior

3.1 Geomentropic Risk Aversion

An expected utility maximizing agent who strictly prefers to receive the expected value of a lottery rather than the lottery itself is said to be risk averse. Analogously, we will define geomentropic risk aversion as a strict preference for receiving the geometric mean of a lottery rather than receiving the lottery itself. Given A3b, geomentropic agents exhibit geomentropic risk aversion. This follows directly from the entropy associated with a non-degenerate lottery.

3.2 Geomentropic Complexity Aversion

We will say that an agent is complexity averse if \( L' \succ L \) whenever \( w(\sigma) = w'(\sigma) \) for all \( \sigma \in S \) and the partition \( S' \) is coarser than the partition \( S \). Then the definitions immediately imply the following theorem:

Theorem: Geomentropic agents are complexity averse.

Note, by contrast, that expected utility maximizers are not inclined to complexity aversion. This is because of the scalability properties of expected utility. Thus compound lotteries are equally preferred to their expected utility equivalents, a property not shared by geomentropic preferences over lotteries.

Financially, investors with geomentropic preferences would rather hold a treasury bill than stock with an option that yields the same return.

\[\text{Shiryayev (1984, pp. 116-118, the Ergodic Theorem and page 545, Theorem 3) presents conditions for existence of an ergodic distribution.}\]
3.3 Geomentropic Uncertainty Aversion

Geomentropic preferences exhibit a property that is analogous to the notion of uncertainty aversion introduced by Gilboa and Schmeidler (1989). Given two equally preferred lotteries, geomentropic individuals prefer a convex combination of the two lotteries over the chance of getting either one. The result relies on the distinction between taking a convex combination in which the weighting variable, call it \( \nu \), is a choice variable in the sense that its value is realized a priori and hence treated as non-random, and a true lottery in which one of the lotteries is received at random. In this latter case say the chance of getting the first lottery is \( \gamma \), and as a random variable it induces some entropy. Uncertainty aversion means that any portfolio of the two lotteries, whatever the value of \( \nu \), is preferred to a lottery with those two lotteries as prizes, whatever the value of \( \gamma \).

Proposition: For all \( \nu \in (0, 1) \), for all \( \gamma \in (0, 1) \), and for all pairs of lotteries satisfying \( (X; p) \sim (Y; q) \), geomentropic preferences satisfy:

\[

\nu(X; p) + (1 - \nu)(Y; q) > ((X; p), (Y; q); \gamma, (1 - \gamma)) = (X, Y; \gamma p, (1 - \gamma)q).

\]

Proof: This is equivalent to \( \nu \Sigma_i p_i \ln X_i + \nu \Sigma_i q_i \ln Y_i + (1 - \nu) \Sigma_i q_i \ln q_i > \Sigma_i p_i \ln X_i + \Sigma_i q_i \ln q_i \ln(p_i \gamma) + \Sigma_i q_i (1 - \gamma) \ln Y_i + \lambda \Sigma_i q_i (1 - \gamma) \ln(q_i (1 - \gamma)) \).

Subtracting terms on the right hand side with similar terms on the left hand side, we can rewrite the last inequality as \( (\nu - \gamma) \Sigma_i p_i \ln X_i + \lambda \Sigma_i q_i \ln Y_i - \lambda \Sigma_i q_i \ln q_i > \lambda \gamma \ln \gamma + (1 - \gamma) \ln(1 - \gamma) \).

Since the two lotteries are equally preferred, the terms in the square brackets sum to zero, and the last inequality becomes \( 0 > \lambda (\gamma \ln \gamma + (1 - \gamma) \ln(1 - \gamma)) \), which is always true because \( \gamma \) is bounded between 0 and 1. QED

3.4 Geomentropic Prudence

Gollier (2001, p 236) defines prudence, a concept due to Kimball (1992), by designating an agent as prudent if adding an uninsurable zero mean risk to his future wealth raises his optimal level of saving. In other words, an increase in risk for a given arithmetic mean causes the prudent agent to increase holdings of the riskless asset. The geomentropic analogue is for an increase in the entropy of the risky asset for a given geometric mean to cause an agent to increase holdings of the risk-free asset.

Consider a two-asset world where asset 1 is a risky asset with price \( \pi_1 \) and asset 2 is a riskless asset with price \( \pi_C \). The payoffs associated with the two assets are \( w_1(\sigma) \) and \( w_C \), respectively. The partition of the state space associated with the risk-free asset is the trivial partition containing only the state space and thus its entropy is \( N(L_2) = -\ln(1) = 0 \). If \( v \) is the share of the risky asset in the portfolio then the geomentropic index of the portfolio \((v, 1 - v)\) is

\[

\ln G_p(v) - \lambda N_p(v) = \sum_{\sigma \in \bigtimes S_h} T \gamma(\sigma) \ln[v \frac{W}{\pi_1} w(\sigma) + (1 - v) \frac{W}{\pi_C} w^C] - \lambda v N_1.

\]

For any horizon length, \( T \), the geomentropic index is concave in \( v \). If real asset prices are such that there is an interior solution—that is, if the optimal \( v \) is in the interior of \((0, 1)\)—then the first order condition for the optimal choice of \( v \) is

\[

8
\]
\[ \sum_{\sigma \in \times_k S_k} T^\gamma(\sigma) \left\{ \frac{W}{\pi_1} w_1(\sigma) - \frac{W}{\pi_C} w_C(\sigma) \right\} \bigg/ \left\{ v \frac{W}{\pi_1} w_1(\sigma) + (1 - v) \frac{W}{\pi_C} w_C \right\} = \lambda N_1 \]

Inspection of this first order condition for fixed asset prices and fixed distribution of returns reveals that a decrease in risk tolerance—that is, an increase in \( \lambda \)—must imply a decrease in \( v \frac{W}{\pi_1} w_1(\sigma) + (1 - v) \frac{W}{\pi_C} w_C \) and (assuming the mean return of the risky asset exceeds that of the riskless asset) a decrease in \( v \), the fraction of the portfolio invested in the risky asset, and thus an increase in risk-free investment.

### 3.5 Fanning and the Allais Paradox with Geomentropic Preferences

To explain some empirical anomalies (including the Allais Paradox) for the expected utility model, Machina (1992) employs a three outcome lottery model with \( p = (p_1, p_2, p_3) \) and corresponding fixed wealth outcomes \( w = (w_1, w_2, w_3) \), where \( w_3 > w_2 > w_1 \). In the probability triangle with \( p_3 \) on the vertical axis, and \( p_1 \) on the horizontal axis (with \( p_2 \)'s value implicitly determined by the choice of the other two probabilities within the triangle), under expected utility sets of indifferent lotteries lie on parallel straight lines with slopes equal to

Expected Utility indifference curves slope = \(-\frac{u'(w_1) - u'(w_2)}{u'(w_3) - u'(w_2)}\)

Given the linearity in probability assumptions under expected utility preferences (the result of the independence assumption in expected utility, see for example, Gollier (2001, section 1.3)), lotteries lying further to the northwest receive higher probability weights on the better outcome (\( p_3 \) is larger and \( p_1 \) is smaller, holding \( p_2 \) constant), and hence, are preferred to all lotteries on indifference curves to the southeast.

Geomentropic indifference curves exhibit more complex patterns within the Machina probability triangle than expected utility. Geomentropic indifference curves additionally depend on the log-odds ratios as follows:

Geomentropic indifference curves slope = \(-\frac{\ln(w_1) - \ln(w_2) + \ln(p_1/p_2)}{\ln(w_3) - \ln(w_2) + \ln(p_3/p_2)}\)

Geomentropic indifference curves tend to “fan out” (that is, their slopes increase as \( p_3 \) increases, holding \( p_2 \) constant) when lottery prizes are skewed to the right (that is, when \( \ln(w_3) - \ln(w_2) > \ln(w_2) - \ln(w_1) \)). Unlike the expected utility indifference curves, geomentropic indifference curves may be negative for low values of \( p_2 \) or high values of \( \lambda \) as they “fan” out.

In particular, this section shows that for values of \( \lambda \) that are consistent with empirical observation (roughly \( 3 > \lambda > .4 \)), geomentropic preferences are consistent with Allais paradox choices. This is in contrast to expected utility theory, which is inconsistent with such choices, regardless of the form of the utility function.

The Allais paradox considers choices between two paired lotteries (Machina, 1989), which can be appropriately regarded as choices between lotteries changing the wealth level of participant. (Recall that Latané’s maximum chance principle is essentially a wealth investment principle.) Given initial wealth \( W > 0 \), consider four lotteries:

\( L_1: \)
1.00 chance of $1,000,000 + W

$L_2$:
.10 chance of $5,000,000 + W
.89 chance of $1,000,000 + W
.01 chance of $0 + W

$L_3$:
.10 chance of $5,000,000 + W
.90 chance of $0 + W

$L_4$:
.11 chance of $1,000,000 + W
.89 chance of $0 + W

Most people claim to prefer $L_1$ over $L_2$ while preferring $L_3$ over $L_4$. However, whatever the utility function used, this is an inconsistent set of choices under the expected utility model because of the reduction of compound lotteries assumption (see Machina, 1989, the diagram on the middle of the right hand column of p. 1629 and corresponding discussion).

By contrast, for feasible values of wealth $W$, and $\lambda$, geomentropic preferences are consistent with experimentally observed preferences of $L_1$ over $L_2$, and at the same time, preferences for $L_3$ over $L_4$.

$L_1$ is preferred to $L_2$ if

$$\ln($1m/W + 1) + \lambda(1\ln(1)) > .1\ln($5m/W + 1) + .89\ln($1m/W + 1) + .01\ln($0 + W) + \lambda(.1\ln(.1) + .89\ln(.89) + .01\ln(.01))$$

or, upon factoring out $\ln(W)$ from both sides, if

$$\ln($1m/W + 1) > .1\ln($5m/W + 1) + .89\ln($1m/W + 1) + \lambda(.1\ln(.1) + .89\ln(.89) + .01\ln(.01))$$

For geomentropic preferences, preference of $L_1$ over $L_2$ indicate the certain relative wealth term $\ln($1m/w + 1) is preferred to the uncertain income $.1\ln($5m/W + 1) + .89\ln($1m/W + 1), discounted by the entropy associated with the uncertainty $\lambda(.1\ln(.1) + .89\ln(.89) + .01\ln(.01))$. Simulations indicate that for all values of wealth $W$, $L_1$ is preferred to $L_2$ whenever $\lambda$ is .34 or greater. This critical value of $\lambda$ falls monotonically as $W$ increases. For $W = $1m, the critical value is $\lambda = .28$; for $W = $5m, the critical value is $\lambda = .13$; and for $W = $10m, $\lambda = .08$. When $\lambda$ is above these critical values $L_1$ is always preferred to $L_2$.

At the same time, $L_3$ is preferred to $L_4$ if

$$\.1\ln($5m/W + 1) + \lambda(.1\ln(.1) + .9\ln(.9)) > .11\ln($1/W + 1) + \lambda(.11\ln(.11) + .89\ln(.89))$$
Figure 1: Wealth-\(\lambda\) values Consistent With Allais Choices

The upper area in figure 1 indicates that for every wealth level, when \(\lambda > .34\), geomentropic preferences are consistent with typical Allais preferences \((L_1 > L_2\) and \(L_3 > L_4)\). Low levels of wealth are consistent with Allais choices, even for very low values of \(\lambda\) as indicated. The highest value of the lower bound for \(\lambda\) of .34 (achieved at a wealth level of about $80,000) monotonically declines as wealth increases. This establishes the existence of an open set of parameters for which Allais-style preferences are consistent with geomentropic preferences.

3.6 Insurance Decisions with Geomentropic Preferences

The results regarding fair pricing and insurance demand that result from expected utility maximization also result from geomentropic preferences.

Specifically, let \(I^*\) denote the amount of insurance an agent purchases at unit price \(s\). The agent faces a loss of \(I\) with probability \(p\); with probability \((1 - p)\) there is no loss. The agent chooses \(I^*\) to maximize

\[
p \ln (W - sI^* - I + I^*) + (1 - p) \ln (W - sI^*) + p \ln p + (1 - p) \ln (1 - p)
\]

The first order condition is

\[
p(1 - s)/(s(1 - p)) = (W - sI^* - I + I^*)/(W - sI^*)
\]

With actuarilly fair insurance, so \(s = p\), the first order condition implies \(I^* = I\); thus the agent purchases full coverage.
When \( s > p \), so insurance prices are not actuarially fair, the left side of the first order condition above is less than one so the right side must be less than one at the optimum. Inspection reveals this to be true only if \( I^* < I \), so the agent purchases only partial insurance is purchased. This differs somewhat from standard expected utility, which can generate a demand for full insurance coverage, even with unfair prices, if the consumer is sufficiently risk adverse.

4 Dynamic Geomentropic Preferences and Asset Prices

4.1 Equilibrium and Re-Investment Certainty Equivalence

Assume that all investors have the same marginal rate of substitution, \( \lambda \). As explained in Section 2.4, and consistent with Latanés wealth reinvestment principle, we assume geomentropic investors behave as if maximizing over their time horizon, \( T \), with respect to the ergodic distribution, \( \gamma \).

Investors’ re-investment horizon has not been settled in the literature, though given Modigliani’s (1966) life cycle saving/investing model, values for an individual’s investment lifetime would plausibly be in the range of 20 to 40 years. The empirical sections below assume a 30-year horizon, but the linearity of key identifying restrictions implies that the estimates will be quite robust for the 20-40 year range. For long investment horizons, the value of a portfolio will grow close to its long-run geometric mean growth path with high probability. Hence, the compound growth of the risky portfolio over \( T \) periods less the \( \lambda \)-weighted entropy of the risky portfolio should roughly equal the compound growth of the risk-free portfolio. In particular, the index of risky and risk-free rates will satisfy the following equality because of re-investment certainty equivalence:

\[
\left\{ T \sum_{\sigma \in \times_k S_k} \gamma(\sigma) \ln \left[ \sum_k v_k \left( \frac{W}{\pi_k} w_k(\sigma) \right) \right] \right\} - T \ln \left[ \prod_{\sigma \in \times_k S_k} \left( \sum_k v_k \left( \frac{W}{\pi_k} w_k(\sigma) \right) \right) \right] = \lambda N_p
\]

where \( w_C \) is the risk-free return. The empirical sections below report estimates of \( \lambda \) using the certainty equivalencies of a risky and risk-free portfolio. These estimates lie between 1 and 2, values that account for the equity premium puzzle.

4.2 Financial Market Equilibrium with Homogeneous Agents

A financial market is a triple, \((x, W, \phi)\), where \( x = (x_1, \ldots, x_K) \) is the exogenous supply of each asset, \( W = (W^1, \ldots, W^I) \) is the exogenous initial wealth of each investor, and \( \phi(G, N) \) is the investors’ geomentropic preference index. A financial equilibrium is a pair \((\pi, v)\) such that \( v \) maximizes the geomentropic index and the market clearing conditions hold for each asset: \( v \in \arg\max \{ \phi(G_p(v), N_p(v)) \} \), \( \sum_{i=1}^I \frac{v_i W^i}{\pi_C} = x_k \) for all risky assets \( k \), and \( \sum_{i=1}^I (1 - \sum_k v_k) W^i / \pi_C = x_C \) for the risk-free asset. The first order conditions associated with the maximization of

\[
\ln G_p(v) - \lambda N_p(v) = \ln \left\{ \left\{ \prod_{\sigma \in \times_k S_k} \left[ \sum_k (v_k \frac{W^i}{\pi_k} w_k(\sigma)) + (1 - \sum_m v_m) \frac{W^i}{\pi_C} w^C \right] \right\} \right\} - \lambda \left[ \sum_k v_k N_k \right]
\]

set the partial derivatives of the objective function equal to zero: namely,
\[ T \sum_{\sigma \in S_k} \gamma(\sigma) \left[ \sum_{j=1}^J \left( \frac{W^i w_j(\sigma)}{\pi_j} - \frac{W^i w_C}{\pi_C} \right) \right] / \left( \sum_k (v_k \frac{W^i w_k(\sigma)}{\pi_k} + (1 - \sum k v_k) \frac{W^i w_C}{\pi_C}) \right) - \lambda N_j = 0 \]

for all \( j \in \{1, \ldots, K\} \). An aggregate first order condition can be derived by multiplying the \( j^{th} \) first order condition by \( v_j \), summing over \( j \), and canceling the common wealth terms:

\[ T \sum_{\sigma \in S_k} \gamma(\sigma) \sum_j v_j \left[ \frac{w_j(\sigma)}{\pi_j} - \frac{w_C}{\pi_C} \right] / \left( \sum_k (v_k \frac{w_k(\sigma)}{\pi_k} + (1 - \sum m v_m) \frac{w_C}{\pi_C}) \right) = \lambda N_p(v) \]

For the portfolio as a whole, the net portfolio returns (given on the right side of the previous equation) are the average risky returns above the risk free return, normalized by the weighted returns of the whole portfolio. Investors set portfolio returns equal to \( \lambda \) times portfolio entropy, \( \lambda \sum k v_k N_k = \lambda N_p(v) \). As shown in the appendix, geometric preferences are concave in shares (that is, concave in \( v \)).

Substituting the market clearing conditions into the first order conditions, and simplifying, yields

\[ T \sum_{\sigma \in S_k} \gamma(\sigma) \sum_j v_j \left[ \frac{W^i w_j(\sigma)}{\pi_j} - \frac{W^i w_C}{\pi_C} \right] / \left( \sum_k (w_k(\sigma)x_k + w_Cx_C) \right) = \lambda N_p(v) \]

where \( W = \sum_{i=1}^I W^i \), the sum of wealth invested in the market by all the \( I \) investors.

The representative investor holds less of every risky asset in equilibrium given an exogenous increase in \( \lambda \)—that is, the greater the investors’ taste for certainty, the more risk-free asset they hold in their equilibrium portfolio (the lower the share of risky assets, \( \sum k v_k \)). This is the content of the following theorem:

Theorem: The share of each risky asset is decreasing in \( \lambda \).

Proof: Substitute the market clearing condition into the first order conditions:

\[ T \sum_{\sigma \in S_k} \gamma(\sigma) \left[ \sum_j v_j \left( \frac{W^i w_j(\sigma)}{\pi_j} - \frac{W^i w_C}{\pi_C} \right) \right] / \left( \sum_k (w_k(\sigma)x_k + w_Cx_C) \right) = \lambda N_j \]

Designating the price of the risk-free asset as numeraire, as \( \lambda \) increases on the right side, the price of the \( j^{th} \) asset, \( \pi_j \), must fall to maintain equilibrium. From the market clearing condition for that asset, \( \sum_{i=1}^I \frac{v_j W^i}{\pi_j} = x_j \), a decrease in the price of the risky asset must be accompanied by a proportional decrease in the share of the asset. QED

For an individual investor, any portfolio satisfying the portfolio maximization condition

\[ T \sum_{\sigma \in S_k} \gamma(\sigma) \sum_j v_j \left[ \frac{w_j(\sigma)}{\pi_j} - \frac{w_C}{\pi_C} \right] / \left( \sum_k (v_k \frac{w_k(\sigma)}{\pi_k} + (1 - \sum m v_m) \frac{w_C}{\pi_C}) \right) = \lambda N_p \]

is equivalent, and different investors may choose different portfolios as long as the compounded, normalized excess returns to the risky assets in the portfolio equaled \( \lambda \) times that portfolio’s entropy. Portfolios with higher entropy must also have higher geometric returns.
Figure 2 illustrates the relationship between portfolio entropy (derived from the choices of shares of each asset on the simplex lying above the horizontal plane) given as the dotted simplex, and the necessary portfolio net returns for each level of portfolio entropy as given on the vertical axis. In Figure 2, the implied value of $\lambda$ is unity; other values for $\lambda$ scale the portfolio entropy up (as $\lambda$ increases) or down (as $\lambda$ decreases). The portfolio net returns at the origin equal the net portfolio returns with only the risk-free asset in the portfolio. As the first order conditions indicate, the price or returns for any given asset does not increase linearly as entropy increases (even if asset shares are held constant).

5 Estimating the Marginal Rate of Substitution

Recall the “first order” condition for the aggregate portfolio, with risky assets resulting in wealth levels $w_k(\sigma)$ per share for the $k$th asset, when the $\sigma$ state of the world is realized, and wealth levels of $w^C$ per share for the risk-free asset in all states of the world:

$$\sum_j v_j \sum_{\sigma \in S_k} T\gamma(\sigma) \left[ \frac{w_j(\sigma)}{\pi_j} - \frac{w^C}{\pi^C} \right] / \left[ \sum_k (v_k w_k(\sigma)) + (1 - \sum_m v_m) \frac{w^C}{\pi^C} \right] = \lambda N_p$$

This is the equation we estimate by regressing the normalized excess returns (essentially, $\sum_k v_k (E(r_k) - r^C)$, where $E$ is the expectation operator), normalized by the average portfolio returns across all states of the world ($\sum_k (v_k w_k(\sigma)) + (1 - \sum_k v_k) \frac{w^C}{\pi^C}$), on portfolio entropy in order to estimate the $\lambda$ that optimally trades off asset uncertainty against asset returns.

In somewhat more standard, financial notation, $w_k(\sigma) = 1 + r_k(\sigma)$, where $r_k(\sigma)$ is the percent return per share in the specified time period. This more usual, financial market notation corresponds better to reported data, so that the aggregate first order condition can be written as:
\[
\sum_j v_j \sum_{\sigma \in \times_k S_k} T \gamma(\sigma) \left[ \left\{ \frac{1 + r_j(\sigma)}{\pi_j} - \frac{1 + r_C}{\pi_C} \right\} - \frac{(1 - \sum_m v_m)(1 + r_C)}{\pi_C} \right] = \lambda N_p
\]

Letting \( R_k^* \) be the relative returns to the \( k \)th risky asset net of the risk-free rate, the above first order equation for the \( k \)th asset can be written as

\[
R_k^* = \lambda N_k,
\]

and for the portfolio, the first order condition can be written as:

\[
R_p^* = \lambda N_p
\]

To approximate this first order condition for empirical purposes, we make three assumptions. Like other researchers, we assume that the past matters in the sense of indicating what the future entropy and geometric mean returns are likely to be. Related to the first assumption, we also assume that average market returns of risky assets, less the risk free rate (suitably normalized), approximates the left side term above. Finally, we assume that the denominator in the left side sum, which is the average portfolio return, can be approximated by the average returns over the whole of the market period,

\[
\sum_k \sum_{\sigma \in \times_k S_k} v_k \gamma(\sigma)(1 + r_k(\sigma)) + (1 - \sum_m v_m)(1 + r_C)).
\]

This assumes that the error in measurement of the left side variable (so this is not a case of attenuation bias) can be treated as approximately zero:

\[
\sum_k \sum_{\sigma \in \times_k S_k} v_k T \gamma(\sigma)(1 + r_k(\sigma)) + (1 - \sum_m v_m)(1 + r_C)).
\]

With these simplifications, we regress the daily (table 2) and monthly (table 3) average market returns less the risk free returns on the entropy of the market returns, that is, we regress \( r_{mkt} - r_C \), divided by the average portfolio returns over the respective period (and assuming \( v_r = v_C = .5 \)), compounded for 30 years, upon the entropy for the market returns (that is, for our all estimates, that half the portfolio is in risky assets when calculating the portfolio from the risky asset returns). As Table 1 indicates, the mean of this normalized return is slightly greater than one (1.04). Note also that the correlation between entropy and the standard deviation of market returns is very high, suggesting either measure of uncertainty may fit the data as well as the other one.

The entropy coefficient estimates \( \lambda \), as indicated by the first order condition above. We calculate entropy from the history of average recent returns, using different window lengths, and then again, using different numbers of support points in the entropy distribution. None of those alternative specifications differed much from the results reported here. Correcting for first order autocorrelation also had virtually no effect on the estimates presented in Tables 2 through 5.

Because we regress \( r_{mkt} - r_C \) divided by a constant (average portfolio returns over the whole period) on entropy, we can simulate market ‘mean/variance’ regressions by regressing the same dependent variable on estimated standard deviation from market returns (this is akin to fitting the efficient frontier relationship as specified say, in Elton and Gruber, 1995, p.88, after suitable substitutions) for comparison. The coefficients on our normalized dependent variable may be different from a standard mean/variance model.
without the normalization, but the fit (regression F-statistics, probability significance for coefficients, and R-square) statistics will be unaffected. This allows us to compare the explanatory power of entropy against the more traditional measure of uncertainty, standard deviation of portfolio returns. Again, because of the relatively high correlation between these uncertainty measures, it would not be surprising that they fit the data similarly in terms of regression R-square.

5.1 Tables 2 and 3

The data came from the Fama/French factors available from the Wharton Business School. Data used in the first two tables was for the January 1, 1927 to July 31, 2013. The results in Table 2 are divided into two ‘columns’. The ‘Entropy $R^2$ F-Stat’ on the left hand side are the results for entropy estimates (regressing $R^*_m$ on the market entropy, $N_m$, of results, calculated returns in the indicated window), the results under the headings ‘SD $R^2$ F-Stat’ on the right hand side are for standard deviations (regressing $R^*_p$ on the standard deviations calculated from the respective market window). The data represent the returns from the value-weighted returns of all CRSP firms (Center for Research in Security Prices data base) incorporated in the US and listed on the NYSE, NASDAQ, or AMEX. It only reports information on ordinary common shares which either haven’t been or don’t need to be further defined (no firms incorporated outside the US, no trust components, no closed end funds).

The estimates in Tables 2 through 5 are OLS regressions without an intercept, as implied by the geomentropic model’s first order condition for portfolios (given in section 4). The estimated value of $\lambda$ using a 4-week market window (entropy and the standard deviation of market returns constructed from the last 20 days of trades) over the whole sample period from 1927 to 2013 is about 1.7, statistically significant at better than the one percent level (significance level is .0002). The overall fit using entropy as the dependent variable is not much different statistically from the fit using the market standard deviation as the measure of risk/uncertainty: the R-square is the same on the left and right hand sides (.0006), and the F-statistic for model fit is virtually the same (13.75 for the entropy model, and 14.07 for the standard deviation model).

Many things are apparent from the 4-week window models in the upper panel: very little of the variation in the normalized net returns is explained by either measure of uncertainty (one tenth of one percent at most), and entropy generally does as good a job at explaining the returns as does the standard deviation (SD). Indeed, in the post-war models (last three lines), the entropy R-square and F-statistics are better than the standard deviations, though the difference is not statistically important.

In the estimates, we examine the overall period with and without information from World War II (WWII), and also examine just changes in the post war period to see whether risk tolerance has changed over time. The slight increase in $\lambda$ suggests less tolerance for risk after the war, but when placed in the context of a shift in a spline function, the upward shift in $\lambda$ towards less tolerance for uncertainty is statistically insignificant at the 10 percent level (it is significant at the 15 percent level). Perhaps the slight postwar increase in $\lambda$ is associated with lingering concerns resulting from the stock market crash of the 1930s.
Similarly, the division of the post war period into pre- and post-ERISA (Employee Retirement Income Act) periods was to test whether the expansion of stock market participation after the enactment of ERISA in 1974, significantly changed the observed $\lambda$: either because of a change in perceived market transparency—which may cause $\lambda$ to fall, or a change in the marginal investor’s tolerance for uncertainty—which may cause $\lambda$ to fall or rise. ERISA encouraged corporate pension fund investments in stocks by requiring that pensions be appropriately funded, and prohibited plans from holding more than 10 percent of the corporation’s own stock. Institutional stock ownership also increased as individuals shifted from direct investment in equities to holding mutual fund shares and as public pension funds and nonprofit endowments abandoned their traditional policies against investments in equities. For example, public funds in California and 15 other states did not invest in any stocks until 1968. After ERISA, there was a dramatic increase in stock ownership by public entities (Bhide, 1993). Using either the 4-week window estimates in the upper panel (where $\lambda$ falls from about 2 to 1.87) or the 6-week estimates in the lower panel (where $\lambda$ falls from 2 to 1.85), the estimated $\lambda$ falls. However, the fall is not statistically significant when tested as a shift in a spline function occurring with the post-ERISA enactment.

Results for the entropy coefficient indicate reasonable values for $\lambda$, from about 1.6 to 2 when estimating first order conditions using daily returns. These results were virtually unchanged when the model was estimated by first order autocorrelation regressions. (The estimated value of $\lambda$ is obviously sensitive to our assumption that half the portfolio is in the risk-free asset: if the share of the risk-free asset fell to zero, the estimated value of $\lambda$ tends to one.)

Table 3 results for monthly returns again indicate $\lambda$ values generally slightly less than two, for monthly returns, between 1.5 and 1.8. (Monthly entropy and standard deviation measures were generated by averaging the daily measures over the last 20 trading days before the end of the respective month.) The monthly estimates of $\lambda$, as with the daily estimates, are always statistically significant. While estimates of the effect of standard deviations on normalized net returns in Table 2 were generally statistically significant, the standard deviation variates are only statistically significant at the five percent level for the monthly returns in Table 3 in the lower panel for the overall period (the statistical significance is .0470). With respect to the monthly returns, entropy fits the returns much better than standard deviations do in every comparison.

Overall, the results suggest reasonable approximations for portfolio $\lambda$s given the first order conditions derived from our theoretical model, and, entropy measures of uncertainty fit the data as well as variance (standard deviation) measures of uncertainty.

### 5.2 Table 4: Entropy or Standard Deviation?

Table 1 indicates a high degree of correlation between the standard deviation and entropy, as measures of uncertainty in the portfolio outcomes. This seems to be generally confirmed by the similar regression fits (given as F-statistics for the model fit, or model R-squared) when normalized difference between the risky and risk-free rates (the “normalized equity premium”) are regressed alternatively on either measure.

In Table 4, the normalized equity premium (suitably compounded) is regressed on both measures simultaneously, in an effort to find which one offers the better empirical fit. The
upper panel exhibits variations in the normalized daily equity premium (corresponding to Table 2) as a function of entropy and the standard deviation in the rates, while the lower panel exhibits variations in the normalized monthly equity premium (corresponding to the results in Table 3). The results presented here are for estimates of the uncertainty based on 6 week market window (i.e., the last 30 trading days) and 6 probability buckets, but similar results were obtained for other market windows, and alternative support point specifications.

Entropy generally fits better than the standard deviation when both are in the regression in two ways: entropy always exhibits the expected positive sign, and the estimated \( \lambda \) values in the upper panel have a magnitude similar to the magnitude found in Tables 2 and 3 (as also below for alternative data sets): roughly between 1 and 2. In the presence of the entropy measure for uncertainty, the standard deviation always has a counterintuitive negative sign for all post-war specifications.

Second, the entropy measure appears to provide a better empirical fit in that it is generally more statistically significant than the standard deviation measure as indicated by the probability significance levels. Especially for variation in the monthly returns, entropy fits the data better than the standard deviation measure of uncertainty.

### 5.3 Table 5: an Alternative US Data Set

Like the data for Tables 2 and 3, these data also come from CRSP firms. However, while these are also value weighted returns, it includes distributions. These data were pulled only from firms on the NYSE, and doesn’t exclude non-trust stocks or closed end funds as does the data discussed in the last subsection. Data used for these last two tables were for the January 1, 1966 to December 31, 2012 period. The results in Table 4 are again divided into two ‘columns’. The ‘Entropy \( R^2 \) F-Stat’ on the left hand side are the results for entropy estimates (regressing \( R^*_p \) on the entropy of results, calculated returns in the indicated window), the results under the headings ‘SD \( R^2 \) F-Stat’ on the right hand side are for standard deviations (regressing \( R^*_p \) on the standard deviations calculated from the respective market window).

During this period, with returns for this sample, the overall fit is statistically significant (with probability significance of 1.9 percent), and not much different from the fit using the market standard deviation as the measure of risk/uncertainty. Again, whether using the 4-week window to calculate the uncertainty variable on the right hand side, or employing daily or monthly returns, the fit of the entropy and standard deviation tells pretty much the same story: not much of the variation in returns is explained by either variable, but entropy does just about as good a job as standard deviation in explaining the market returns.

Interestingly, though all the \( \lambda \) values associated with entropy are reasonable, and are consistent with the entropy values that explain Allais paradox rankings, namely from 1.4 to 2.4, the \( \lambda \) associated with the daily returns is only about half the value as the \( \lambda \) associated with monthly returns. It is not clear why there is this difference. Since we have daily risk and riskfree returns, we continue to use the daily data to construct entropy and standard deviations for all the models in Table 5 (we average all daily values over the previous 20 trading days, or 4 week window, and match them to the end of the month.
average values that correspond to the net normalized returns). These results were virtually unchanged when the model was estimated by first order autocorrelation regressions.

5.4 Table 6: International Estimates Using Monthly Returns Data

We do not have any daily returns for the data in Table 6, so all the entropy and standard deviations here are constructed from the last twelve months of market returns using only the monthly return data. This means that there is likely more noise in our uncertainty measures in Table 6 than there is in Tables 2-5. Nonetheless, the estimated $\lambda$ are surprisingly similar to those estimated with the prior data sets.

This data comes Marmi (2013), available for 16 developed economies. Again, the distinguishing feature of these data is that only monthly returns are available. The starting time for each data set varies- the data for “developed” markets starts in 1988, and “emerging” markets (South Korea, India, Brazil, and China) have later starting dates (1992, 1993, 1995, and 1998, respectively), as reflected in the sample sizes. Marmi (2013) used the 1000 largest companies in the market for at least some of the estimates. Excluding Japan (whose estimated $\lambda$ value is very imprecise, even among all the imprecision in Table 5), estimates for $\lambda$ range generally between 1.4 and 2.0, except for China, Germany, and South Korea, which were a little bit lower (and hence, seem to discount information uncertainty a little bit more than other countries in this period).

5.5 Table 7: Robustness Check Employing US Real Gross Domestic Product

The empirical results in Tables 2 through 6 indicate that consumers tend to weigh entropy (uncertainty about payoffs, as a measure of risk) one to two times more heavily than the log of geometric returns when evaluating investment opportunities (or more generally, lotteries). Another way to test whether these estimated ‘first order condition’ results are sensible, is to see how individuals weight these factors (the geometric means vs. the entropy of a portfolio) in the aggregate with respect to growth in the US economy.

Growth, rather than levels, is the appropriate outcome given the Latané (1959) logic: in a many-period reinvestment process where gains are reinvested proportionately, the portfolio with the greatest geometric mean results in the highest terminal wealth. Here, we assume that real aggregate wealth is roughly proportional to real gross domestic product (RGDP). In each period, individuals add to RGDP by following geomentropic preferences. If this is true, then we should find growth to be negatively related to financial market entropy, and the (log of) geometric mean levels of wealth to have a positive relationship with RGDP. Moreover, the results above suggests a smaller effect for the log of the geometric mean (in terms of its coefficient) than entropy (in terms of its coefficient in an aggregate growth equation). In the prior results, for example, the entropy effect was about twice as important as the log of geomentropic means effect, given the estimated $\lambda$ values in Tables 2 through 6.

We don’t measure wealth, but assume that aggregate real wealth is proportional to RGDP. This factor of proportionality will only rescale the right hand side coefficients in the regressions where growth is in terms of changes in the levels, i.e., where
the dependent variable is $\Delta RGDP_t = RGDP_t - RGDP_{t-1}$. But regression fit (R-squared), and the relative size of the coefficients, will be unaffected by this scaling. This factor of proportionality will factor out and affect only the intercept value where growth is measured as proportional changes in levels, i.e., where the dependent variable is $\%\Delta RGDP_t = \log(RGDP_t) - \log(RGDP_{t-1})$, again, leaving the overall regression fit and relative size of the coefficients unchanged. To allow for the possibility that that factor of proportionality (between income and wealth) gradually shifts over time, we also include a trend variable in all our specifications.

To proxy investor wealth levels (their geometric means, as well as arithmetic means) as the appropriate right hand side predictors, the geometric mean resulting from investments in risky portfolios is calculated as for wealth levels, as $\zeta \prod_i (PDI \ast (1 + rmt_i))^{p_i}$, where $\zeta$ is again an unknown fraction whose value does not affect our robustness tests. Here, PDI=average quarterly real disposable income per capita, $rmt_i=$median value of risky market returns in the $i$th bucket assuming a uniform distribution of returns, $p_i=$ probability of risky market returns falling in the $i$th bucket. (Hence, mean wealth, reported in the upper two specifications of Table 7, is $\sum_i p_i (PDI \ast (1 + rmt_i))$).

Following our specification for geomentropic preferences given in Tables 2 through 6, namely $\log G_p(v) - \lambda N_p(v)$, the relative results given in the second and third panels of Table 6 represent our geomentropic proxy results. In the far right hand columns of those bottom two panels, we present the F-statistic off the null hypothesis that $2 * \beta \log($geometric mean$) = -\beta \text{entropy}$, as implied by the results in Tables 2 through 6. Since the dependent variable in these equations represent aggregate growth variables, the log of geometric mean is given on the basis of average daily returns. (Obviously, compounding the daily returns into quarterly values (by multiplying the log of the geometric mean by 60), would reduce the coefficient of the geometric mean proportionately, and increase the importance of entropy relative to the long of the geometric means.) The tests on the restriction results in F-values (.47, .05, 1.12, and .36 respectively) indicate that the null hypothesis cannot be rejected at the usual levels of probability significance (respectively, as indicated, .50, .82, .29 and .55), consistent with the results in Tables 2 through 6.

Two other results are notable in Table 7. The measure of investment risk, whether the standard deviation in the wealth returns or the entropy of the investment portfolio, always exhibit significant negative coefficients as expected. This suggests that Samuelson’s critique of Latané is appropro: investors behave as if risk matters to them. The second result is that entropy and geometric means explain as much of the quarterly growth in output as does the standard deviation of wealth returns and the associated arithmetic means. The is, the R-square and regression F-statistics are slightly better for the $\%\Delta RGDP$ specifications using geomentropic preferences than the usual standard deviation and arithmetic mean specifications, but slightly worse using the $\Delta RGDP$ specifications, whether considering the 6 support-points results in the middle panel of Table 7, or the 10-support-point results in the lower panel.

5.6 Table 8: “Certainty Equivalencies” Over A Lifespan

Data on the annualized risky rates, risk free rates, as well as real disposal gross domestic product per capita are available from 1929. We assume, following the literature, that the ex post trends in the last 30 years of returns approximate the ex ante returns, and use 30-year lags to calculate the geometric mean of risky returns, the probability distribution
of risky returns (we use a 6-bucket decomposition in Tables 7 and 8) and entropy. (As a practical matter, we also employ the (log of) geometric mean for the riskfree rates over the prior 30 years. The results using the arithematic mean instead over the period yielded virtually the same results.) The certainty equivalency of the risky, and risk-free, portfolios condition allows for a robustness check on the data, as it implies a particular value for \( \lambda \) (here, \( p(\sigma) \) are derived from the observed aggregate returns of risky assets):

\[
\lambda = \frac{\ln(\prod_{\sigma \in S}(1 + r(\sigma))^{p(\sigma)^{30}}) - \ln((1 + r_C)^{30})}{N_p}
\]

Or,

\[
\lambda = 30\ln(\prod_{\sigma \in S}(1 + r(\sigma))^{p(\sigma)}) - \ln((1 + r_C))/N_p
\]

For example, by 1959, the previous 30 years of data indicated that the annual log of the geometric mean of the risky asset was .06308, and the annual log of the geometric mean of the riskfree asset was .012533. Hence, the implied lambda consistent with the certainty equivalence condition is thirty times this difference (30(.06308 − .012533)) divided by the entropy value for this period, 1.69337, which equals .89550, as indicated in the middle column of estimates.

The data for remaining years suggests \( \lambda \) consistent with the empirical findings in Tables 2 through 5, values that also consistent with Allais preferences. Overall, \( \lambda \) values are around 1.5, generally bounded between 1 and 2.

To compare our reconciliation of the risky and risk-free rates with the expected utility model, we also present implied relative risk aversion coefficients (RRA) using the same data (prior 30 years means for the risky and risk-free rates, whose differences represent the equity premium for risk) using the simple model of Gollier (2001, section 5.2). The “equity premium” for 1959, the difference in risky and risk-free rates that compensates the investor for holding the risky assets, for example, is .109 − .011 = .098. Employing Gollier’s formula, we find the RRA factor is 100, an implausibly large value. Indeed for all years, these estimates of the RRA imply that an investor is so risk averse that she is willing to give up 29 percent of her wealth to avoid a gamble in which she receives a 30 percent wealth gain half the time, and a 30 percent wealth reduction half the time.

The four right hand columns suggest that as entropy rises, so does the equity premium. Indeed, a simple regression indicates as entropy goes up, so does the equity premium (a standard deviation increase in entropy results in almost half a standard deviation increase in the equity premium), with an R-square of .18.

### 6 Concluding Remarks

Extending Latané’s (1959) maximum chance, or wealth reinvestment principle, we examine a very simple one-parameter “maximum choice” model, which we call geomomentropic preferences. This non-expected utility model trades off the log of the geometric mean of wealth against entropy, weighted by the parameter \( \lambda \). Geomentropic preferences explain many of the commonly observed characteristics of consumer behavior under uncertainty, including the Allais paradox.

When we extend this model to portfolio choice, and allow for reinvestment of the assets, we find geomomentropic preferences fit the data as well as, and sometimes better than,
standard mean-variance models. Robustness checks suggest that a geometropoic characterization of the financial markets tracks aggregate growth as predicted by analyses using daily and monthly returns. Moreover, certainty equivalence under wealth reinvestment, given geometropoic preferences, explains the equity premium puzzle of the the expected utility model, which implies unfeasibly high relative risk aversion.

Our model explains lottery and portfolio choices likely to be observed in the market place. However, like other non-expected utility choice models, it may not describe the behavior of non-market outcomes or unfeasible lotteries, say of the type embedded in perpetual-wealth enhancing lotteries (Machina, 1989).
References
Heidegger, Martin (1927) Being and Time (German: Sein und Zeit, 1927, Berlin, Max Niemeyer verlag)


NBER webpage on business cycles. [www.nber.org/cycles.html]


Tallon, Jean-Marc and Jean-Christophe Vergnaud (2008) “Beliefs and Dynamic consistency”, HAL Archives, //hal.archives-ouvertes.fr/hal-00306458/document


### Table 1: Means, Ranges, and Correlation Matrix, 1927-2013

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<tr>
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<th>Means</th>
<th>Ranges</th>
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<th>entropy30</th>
<th>mktsd20</th>
<th>mktsd30</th>
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<td>.75</td>
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<td>std dev–6 week</td>
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<td>(.0016, .0539)</td>
<td>.75</td>
<td>.79</td>
<td>.96</td>
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Note: N=22,858. All correlations were statistically significant at better than the .0001 level.
### Table 2: Daily Normalized Market Returns less Risk Free Rates, Regressed on Entropy or Standard Deviations (of Market Rates), 1927-2013, (Probability Significance)

#### 4-week market window

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<th>Year</th>
<th>Entropy</th>
<th>$R^2$</th>
<th>F-Stat</th>
<th>SD</th>
<th>$R^2$</th>
<th>F-Stat</th>
<th>N</th>
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<td>(.0004)</td>
<td>(.0013)</td>
<td>(.0006)</td>
<td>(.0009)</td>
<td>(.013)</td>
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<tr>
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<td>.0005</td>
<td>11.49</td>
<td>89.640</td>
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<td>12.35</td>
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<td>(.0007)</td>
<td>(.0004)</td>
<td>(.646)</td>
<td>(.0056)</td>
<td>(.0006)</td>
<td>(.0056)</td>
<td>(.013)</td>
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<tr>
<td>Post-WWII</td>
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<td>.0009</td>
<td>16.20</td>
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<td>13.85</td>
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<td>(.0012)</td>
<td>(.646)</td>
<td>(.0056)</td>
<td>(.0056)</td>
<td>(.0008)</td>
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<td>(.0006)</td>
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<td>(.0056)</td>
<td>(.013)</td>
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<td>7.09</td>
<td>9693</td>
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#### 6-week market window

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<th>SD</th>
<th>$R^2$</th>
<th>F-Stat</th>
<th>N</th>
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<td>Overall (1927-2013)</td>
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<td>(.0005)</td>
<td>(.0001)</td>
<td>(.0006)</td>
<td>(.0005)</td>
<td>(.013)</td>
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<td>21747</td>
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<td>(.0005)</td>
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<td>(.0005)</td>
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<td>(.0005)</td>
<td>(.013)</td>
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<td>Post-WWII</td>
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<td>107.275</td>
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<td>(.0002)</td>
<td>(.0008)</td>
<td>(.0002)</td>
<td>(.013)</td>
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<td>9.48</td>
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<td>(.0007)</td>
<td>(.0103)</td>
<td>(.013)</td>
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Note: The intercept is restricted=0. The entropy and standard deviation (SD) were constructed using lags of the indicated market window length. Using a 2 week window also resulted in virtually the same pattern of results, as did constructing the entropy on 4- or 10-support point probability distributions instead of the 6-point support used here. Correcting for first order autocorrelation yielded virtually the same estimates. Using French, Fama/French factors.
Table 3: Monthly Normalized Market Returns less Risk Free Rates, Regressed on Entropy or Standard Deviations (of Market Rates), 1927-2013, (Probability Significance)

### 4-week market window

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<tr>
<th>Year</th>
<th>Entropy</th>
<th>$R^2$</th>
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<th>SD</th>
<th>$R^2$</th>
<th>F-Stat</th>
<th>N</th>
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<tr>
<td>Overall (1927-2013)</td>
<td>1.5216</td>
<td>.0078</td>
<td>8.10</td>
<td>36.340</td>
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<tr>
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<td>(.0045)</td>
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<td>Overall less WWIII</td>
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<td>(.0096)</td>
<td>(.2708)</td>
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<td>Post-WWII</td>
<td>1.5791</td>
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<td>34.399</td>
<td>.0016</td>
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<td>814</td>
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<td>(.0015)</td>
<td>(.2470)</td>
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<td>Post-WWII to ERISA</td>
<td>1.5206</td>
<td>.0103</td>
<td>3.66</td>
<td>56.744</td>
<td>.0031</td>
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<td>(.0566)</td>
<td>(.2999)</td>
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<tr>
<td>ERISA to 2013</td>
<td>1.6092</td>
<td>.0137</td>
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<td>.0012</td>
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<tr>
<td></td>
<td>(.0019)</td>
<td>(.4613)</td>
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### 6-week market window

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<th>SD</th>
<th>$R^2$</th>
<th>F-Stat</th>
<th>N</th>
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<td>Overall (1927-2013)</td>
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<td>(.0470)</td>
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<td>(.0049)</td>
<td>(.0702)</td>
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<tr>
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<td>.0152</td>
<td>12.56</td>
<td>53.327</td>
<td>.0040</td>
<td>3.27</td>
<td>814</td>
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<td></td>
<td>(.0004)</td>
<td>(.0710)</td>
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<tr>
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<td>(.2711)</td>
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Note: The intercept is restricted=0. The entropy and standard deviation (SD) were constructed using lags of the indicated market window length. Using a 2 week window also resulted in virtually the same pattern of results, as did constructing the entropy on 4- or 10-support point probability distributions instead of the 6-point support used here. Correcting for first order autocorrelation yielded virtually the same estimates. Using French, Fama/French factors.
Table 4: Normalized Market Returns less Risk Free Rates, Regressed on Entropy and Standard Deviations (of Market Rates), 1927-2013, (Probability Significance)

6-week market window, Daily Rates

<table>
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<tr>
<th>Year</th>
<th>Entropy</th>
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<th>$R^2$</th>
<th>N</th>
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6-week market window, Monthly rates

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<td>(&lt;.0001)</td>
<td>(.0001)</td>
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Note: The intercept is restricted=0. The entropy and standard deviation (SD) were constructed using lags of the indicated market window length. Using a 2 week window also resulted in virtually the same pattern of results, as did constructing the entropy on 4- or 10-support point probability distributions instead of the 6-point support used here. Correcting for first order autocorrelation yielded also didn’t change the results. Using French, Fama/French factors.
Table 5: Daily and Monthly Market Returns less Risk Free Rates, Regressed on Entropy or Standard Deviations (of Market Rates), 1966-2012, (Probability Significance)

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<th>SD</th>
<th>$R^2$</th>
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<th>N</th>
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Note: The intercept is restricted=0. The entropy and standard deviation (SD) were constructed using lags of the indicated market window length. Using a 2 week window also resulted in virtually the same pattern of results, as did constructing the entropy on 4- or 10-support point probability distributions instead of the 6-point support used here. Correcting for first order autocorrelation yielded virtually the same estimates.

<table>
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<th>Country</th>
<th>Entropy</th>
<th>$R^2$</th>
<th>F-Stat</th>
<th>SD</th>
<th>$R^2$</th>
<th>F-Stat</th>
<th>N</th>
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</table>

Note: The intercept is restricted=0. The entropy and standard deviation (SD) were constructed using lags of the monthly values over the last year. Because only 12 observations were used to construct the entropy values, we report the results using a 4-support point probability distribution. Results using a 6-point support instead yielded the same results except for Japan (whose entropy coefficient became negative). Correcting for first order autocorrelation yielded virtually the same estimates.
### Table 7: Changes in US Real Domestic Product, Quarterly Data, Regressed on Geometric mean wealth and Entropy, or Arithmetic Returns and Standard Deviations, 1947-2013 (Probability Significance) [F-stat for Regression]

**Arithmetic Means and Variances as Financial Market Proxies**

<table>
<thead>
<tr>
<th>Specification</th>
<th>Intercept</th>
<th>Arithmetic mean</th>
<th>Stand. Dev.</th>
<th>$R^2$</th>
<th>F-Restriction</th>
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</thead>
<tbody>
<tr>
<td>$\Delta RGDP_t$</td>
<td>12.15</td>
<td>.0031</td>
<td>-.5896</td>
<td>.1758</td>
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<tr>
<td>$%\Delta RGDP_t$</td>
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<td>3E-7</td>
<td>-5E-5</td>
<td>.0659</td>
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<td>[&lt;.0001] [.7867] [.0017] [6.21]</td>
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</table>

**Geomomentropic Components as Financial Market Proxies: 6 Support Points**

<table>
<thead>
<tr>
<th>Specification</th>
<th>Intercept</th>
<th>Ln(Geom. mean)</th>
<th>Entropy</th>
<th>$R^2$</th>
<th>F-Restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta RGDP_t$</td>
<td>-698.89</td>
<td>106.52</td>
<td>-122.81</td>
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<td>.47</td>
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<td>$%\Delta RGDP_t$</td>
<td>-.0796</td>
<td>.0138</td>
<td>-.0233</td>
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**Geomomentropic Components as Financial Market Proxies: 10 Support Points**

<table>
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<th>Ln(Geom. mean)</th>
<th>Entropy</th>
<th>$R^2$</th>
<th>F-Restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta RGDP_t$</td>
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<td>-92.09</td>
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<td>[.1093] [.0744] [.0016] [11.52] [.2915]</td>
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<tr>
<td>$%\Delta RGDP_t$</td>
<td>-.0897</td>
<td>.0157</td>
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<td>[.1951] [.0987] [&lt;.0001] [9.92] [.5506]</td>
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</table>

Notes: N=268 for all models. A ‘trend’ coefficient based on quarter was also included in each specification but is not reported here (excluding it did not change any of the substantive results reported). The entropy and geometric mean ‘wealth’ variables were constructed using 60 trading day-lags from the first day of the relevant quarter, using the indicated number of support points (either 6 or 10, for the estimated probabilities for the entropy and geometric mean measures). The geometric mean was calculated for wealth levels, as $\prod_i (PDI \times (1 + rmt))^p_i$, where PDI=average quarterly real disposable income per capita, rmt=median value of risky market returns in the relative bucket assuming a uniform distribution of returns, $p_i$=probability of risky market returns falling in the $ith$ bucket. Calculations for the arithmetic mean and standard deviation were based on all 60 lagged trading days, and were also calculated using consumer’s real disposable income per capita, so they measure wealth levels using the same variables as the geomomentropic variables.
Table 8: Reinvested Wealth: Arbitrage values for $\lambda$ with a 30-year Reinvestment Horizon ($\lambda^1$) or Discounted Infinite Horizon ($\lambda^2$), with Implied Relative Risk Aversion Values (RRA)

<table>
<thead>
<tr>
<th>year</th>
<th>ln(mktgeo)</th>
<th>ln(rfgeo)</th>
<th>entropy</th>
<th>$\lambda^1$</th>
<th>$\lambda^2$</th>
<th>mktmean</th>
<th>rfmean</th>
<th>RRA</th>
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<td>1.68301</td>
<td>1.03982</td>
<td>3.03154</td>
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<td>0.012213</td>
<td>1.62457</td>
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<td>1.61668</td>
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<td>0.015481</td>
<td>1.56934</td>
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<td>3.75370</td>
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<td>3.19418</td>
<td>0.14433</td>
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<td>1.43879</td>
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<td>2.27991</td>
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<td>1.36616</td>
<td>0.98412</td>
<td>0.12304</td>
<td>0.046273</td>
<td>279.719</td>
</tr>
</tbody>
</table>

Notes: All calculations from annual data, employing the previous 30 years of values of the respective variables in their calculations (S&P 500 annual returns for as a proxy for risky returns-mkt; 3-month treasury t-bill rate annualized for the proxy for risk-free returns–rf, and real gross domestic product per capita in the calculation of the relative risk aversion parameter (using the simplified model of Gollier (2001, section 5.2). Calculations for $\lambda^2$ used an infinite horizon with discounting by the risk-free rate (mean risk-free returns over the relevant 30 year period). Calculations for odd years yielded similar values.
A Geomentropic Preferences with Additive, Zero-Mean Errors

A.1 Geomentropic Prudence

We show that an increase in the uncertainty of returns to both the risky asset and the same asset with a zero-mean, symmetrically distributed, uninsurable risk, increases savings (via an increase in the share of the risk-free, cash asset). This extends geomentropic preferences to the usual analysis of risk changes employed using expected utility models.

Denote uninsurable risk as $\epsilon$. We then examine what is required in terms of shift in savings share to maintain its geomentropic value relative to the case with no stochastic variation in the returns to future wealth: the returns for the risky asset goes from $(1+r_i)/\pi$ to $(1+r_i)/\pi + \epsilon$, and the returns to the safe asset goes from $1+r^C$ to $1+r^C + \epsilon$ where $\epsilon$ is the uninsurable perturbation in the returns as noted. (The proof also works for the case where $\epsilon$ only affects the risky asset.) To ensure that wealth is non-negative, we assume that $\epsilon$ is small relative to the average portfolio returns, $v(1+r^i)/\pi + (1-v)(1+r^C)$, where $v$ is the share of the risky asset in the portfolio, $\pi$ is the price of the risky asset, and the price of the risk-free asset has been set to 1. Though the consumer takes $\pi$ as given here, the low-down theorem above is a prudence result arising from asset equilibrium.

Our approach here is to examine the change in the share of the risky asset, $\Delta v$, that equates the geomentropic values with and without the uninsurable risk (we assume that the consumer is not always at the same corner solution):

$$E_{\epsilon} \sum_{\sigma \in S} \{ p(\sigma) \ln[(v + \Delta v)W((1+r(\sigma))/\pi + \epsilon) + (1-(v+\Delta v))W(1+r^C+\epsilon)] \} - \lambda N_p = \sum_{i} \{ p_i \ln[(vW(1+r(\sigma))/\pi + (1-v)W(1+r^C))] \} - \lambda N_p$$

Canceling common terms and moving both expressions to the left hand side of the equation yields the equivalent condition that

$$E_{\epsilon} \sum_{\sigma \in S} \{ p(\sigma) \ln[((v+\Delta v)W(1+r(\sigma))/\pi + \epsilon) + (1-(v+\Delta v))W(1+r^C+\epsilon))] - \ln[(vW(1+r(\sigma))/\pi + (1-v)W(1+r^C))] \} = 0$$

Noting that the left-hand, square-bracketed term can be rewritten as

$$\{ \Delta v((1+r(\sigma))/\pi - (1+r^C)) + \epsilon \} + \{ (vW(1+r(\sigma))/\pi + (1-v)W(1+r^C) \}$$

Factoring out the common $vW(1+r(\sigma))/\pi + (1-v)W(1+r^C)$ terms and canceling them, and then factoring out and canceling the $W$ terms, the expression for geomentropic prudence becomes

$$E_{\epsilon} \sum_{\sigma \in S} \{ p(\sigma) \ln[1 + [\Delta v((1+r(\sigma))/\pi - (1+r^C)) + \epsilon]/[(v(1+r(\sigma))/\pi + (1-v)(1+r^C))] \} = 0$$

To take expectation of the left-hand term, we split up the symmetrically distributed, zero-mean $\epsilon$ random variable into intervals:

$$\int_{-\infty}^{0} \sum_{\sigma \in S} \{ p(\sigma) \ln[1 + [\Delta v((1+r(\sigma))/\pi - (1+r^C)) + \epsilon]/[(v(1+r(\sigma))/\pi + (1-v)(1+r^C))] \} \phi(\epsilon) d\epsilon$$
\[
\int_0^\infty \sum_{\sigma \in S} \{ p(\sigma) \ln \left\{ 1 + \left[ \Delta v((1 + r(\sigma))/\pi - (1 - r^C)) \right] + \epsilon \right\} / \left\{ [(v(1 + r(\sigma))/\pi + (1 - v)(1 + r^C))] \right\} \phi(\epsilon) d\epsilon
\]

Since \( \epsilon \) is symmetric (hence, \( \phi(\epsilon) = \phi(-\epsilon) \)), we can replace these integrals with

\[
\int_0^\infty \sum_{\sigma \in S} \{ p(\sigma) \ln \left\{ 1 + \left[ \Delta v((1 + r(\sigma))/\pi - (1 - r^C)) \right] - \epsilon \right\} / \left\{ [(v(1 + r(\sigma))/\pi + (1 - v)(1 + r^C))] \right\} \phi(\epsilon) d\epsilon +
\]

\[
\int_0^\infty \sum_{\sigma \in S} \{ p(\sigma) \ln \left\{ 1 + \left[ \Delta v((1 + r(\sigma))/\pi - (1 - r^C)) \right] + \epsilon \right\} / \left\{ [(v(1 + r(\sigma))/\pi + (1 - v)(1 + r^C))] \right\} \phi(\epsilon) d\epsilon
\]

Combining these integrals, and letting \( D = [(v(1 + r(s))/\pi + (1 - v)(1 + r^C))] \), the defining geomentropic prudence equation becomes

\[
\int_0^\infty \sum_{\sigma \in S} \{ p(\sigma) \ln \left\{ 1 + \left[ \Delta v((1 + r(\sigma))/\pi - (1 - r^C)) \right] / D + [(\Delta v^2(r(\sigma)/\pi - r^C)^2)] / D^2 - [\epsilon^2]/D^2 \} \phi(\epsilon) d\epsilon = 0
\]

By our requirement that \( \epsilon \) be small relative to the returns (so that there be no negative wealth values), the \([\epsilon^2]/D^2\) term on the far right hand side will be small relative to the other terms, so that our equality will only be satisfied if the positive term middle term, \([\Delta v^2(r(\sigma)/\pi - r^C)^2]/D^2\), is offset by the left hand term \([2\Delta v((1 + r(\sigma))/\pi - (1 - r^C))]/D\), which given that on average the risky returns will be greater than the returns to cash, will only be the case if \( \Delta v < 0 \), that is, if the share of the risky asset falls and the share of savings increases. QED

From the form of the proof, it is obvious that it readily extends to several risky assets in our portfolio framework.

### A.2 "Weak" Temperance

Eeckhoudt and Schlesinger (2006) define temperance (we call this “arithmetic-mean temperance”) to be \(((W - \epsilon_1), (W + \epsilon_2); .5, .5) \succ ((W, W - \epsilon_1 + \epsilon_2); .5, .5)\). They indicate that “...temperance shows a type of preference for disaggregation of two independent zero-mean random variables.” Geomentropic preferences are consistent with arithmetic-mean temperance, as we show below.

An analogous, alternative definition for “geometric-mean temperance” would be \(((W^{1-\epsilon_1}), (W^{1+\epsilon_2}); .5, .5) \succ ((W, W^{1-\epsilon_1+\epsilon_2}); .5, .5)\). But this last preference ordering doesn’t hold for equivalent geometric means, like it will for additive, zero-mean errors. Rather, in every case where the geometric means are the same, the geomentropic evaluation for the two alternative lotteries is equivalent as entropy for the comparison lotteries are also equal. Hence, \(((W^{1-\epsilon_1}), (W^{1+\epsilon_2}); .5, .5) \sim ((W, W^{1-\epsilon_1+\epsilon_2}); .5, .5)\), and there can be no geometric-mean temperance with geomentropic preferences.

“Arithmetic-mean” temperance, on the other hand, leads people facing unavoidable risk to reduce exposure to another independent risk when preferences are geometric. We assume \( \epsilon_1, \epsilon_2 \) are independent random, symmetric variates with zero mean. Entropy terms cancel, so that arithmetic-mean temperance holds under our index if \( E_{\epsilon_1}E_{\epsilon_2}(\ln(W + \epsilon_1) + \ln(W + \epsilon_2)) > E_{\epsilon_1}E_{\epsilon_2}(\ln(W) + \ln(W - \epsilon_1 + \epsilon_2)) \) or if
$E_1 E_2 (\ln((W + \epsilon_1)(W + \epsilon_2))) > E_1 E_2 (\ln((W)(W + \epsilon_1 + \epsilon_2))).$ Noting that $(W + \epsilon_1)(W + \epsilon_2) = (W)(W + \epsilon_1 + \epsilon_2) + \epsilon_1 \epsilon_2$, and canceling similar terms from both lottery indices, we get temperance if

$$E_1 E_2 \ln(1 + (\epsilon_1 \epsilon_2)/\gamma) > 0 \text{ where } \gamma = W^2 + \epsilon_1 W + \epsilon_2 W.$$

Since our variates are symmetric, $\phi(\epsilon) = \phi(-\epsilon)$ for all $\epsilon_1, \epsilon_2$. Hence, $E_1 E_2 \ln(1 + (\epsilon_1 \epsilon_2)/\gamma) =$

$$\int_0^\infty \int_0^\infty \ln(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W)) \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2 +$$

$$\int_{-\infty}^0 \int_0^\infty \ln(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W)) \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2 +$$

$$\int_0^\infty \int_{-\infty}^0 \ln(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W)) \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2 +$$

$$\int_0^\infty \int_0^\infty \ln(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W)) \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2 =$$

$$\int_0^\infty \int_0^\infty \ln((1 + (\epsilon_1 \epsilon_2)/(W^2 - \epsilon_1 W - \epsilon_2 W))(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W - \epsilon_2 W)) + (1 - \epsilon_1 \epsilon_2)/(W^2 - \epsilon_1 W + \epsilon_2 W))(1 + (\epsilon_1 \epsilon_2)/(W^2 + \epsilon_1 W + \epsilon_2 W))) \phi(\epsilon_1) d\epsilon_1 \phi(\epsilon_2) d\epsilon_2$$

There will be arithematic-mean temperance with geomentropic preferences if this last integral is positive. Algebraic manipulations reduces the “$\ln(\cdot)$” in the integral to $\ln[(1 + (\epsilon_4^4 + 2W^2\epsilon_2^2(3W^2 - \epsilon_1^2 - \epsilon_2^2))/\text{denominator}] > 0$, where denominator $(W^2 - W(\epsilon_1 + \epsilon_2))(W^2 - W(\epsilon_2 - \epsilon_1))(W^2 + W(\epsilon_2 - \epsilon_1))(W^2 + W(\epsilon_1 + \epsilon_2))$ as both denominator and numerator terms in the ratio term are positive as long as wealth is large relative to the stochastic gains and losses. QED.

### A.3 Losses Hurt More than Equivalent Gains Help

To show the general nature of the result, we place the random shift in a portfolio context. Specifically, suppose that our portfolio has one risky asset (with share $v$ and asset price $\pi$) and one risk-free asset (with share $1 - v$ and nominal returns per share of $1 + r^C$). Consider the change in geomentropic outcomes when the wealth levels achieved from the risky asset go from $(vW)/\pi(1 + r(\sigma))$ if state $s$ is realized to $(vW)/\pi(1 + r(\sigma) + \epsilon)$ if state $\sigma$ is realized, where $\epsilon$ is a mean zero random variable (this could be the case for one or more states of the world). Then under geomentropic preferences, greater certainty in returns is preferred to uncertainty in returns for a portfolio (the results are obviously more general than this example of a 1 risky asset, 1 riskless asset portfolio, if

$$\sum_{\sigma \in S} p(\sigma) \ln[((vW)/\pi)(1 + r(\sigma)) + (1 - v)W(1 + r^C)] + \lambda v \sum_{s \in S} p(\sigma) \ln p(\sigma) >$$

$$E_1 \sum_{\sigma \in S} p(\sigma) \ln[((vW)/\pi)(1 + r(\sigma) + \epsilon) + (1 - v)W(1 + r^C)] + \lambda v \sum_{s \in S} p(\sigma) \ln p(\sigma)$$

Factoring out common $W$ terms, and common entropy terms from both sides, yields the equivalent condition
\[ \sum_{\sigma \in S} p(\sigma) \ln((v/\pi)(1 + r(\sigma)) + (1 - v)(1 + r^C)) > E_r \sum_{\sigma \in S} p(\sigma) \ln((v/\pi)(1 + r(\sigma) + \epsilon) + (1 - v)(1 + r^C)) \]

As
\[ E_r \sum_{\sigma \in S} p(\sigma) \ln(((v/\pi)(1 + r(\sigma) + \epsilon) + (1 - v)(1 + r^C)) = \sum_{\sigma \in S} p(\sigma) \ln((v/\pi)(1 + r(\sigma)) + (1 - v)(1 + r^C)) + \sum_{\sigma \in S} p(\sigma) E_r \ln(1 + ((v/\pi)\epsilon)/((v/\pi)(1 + r(\sigma)) + (1 - v)(1 + r^C))) , \]

we cancel the “certainty” terms (roughly, those not involving \( \epsilon \)) from both sides of the above inequality, and greater certainty is returns is preferred to uncertainty in returns if
\[ 0 > \sum_{\sigma \in S} p(\sigma) E_r \ln(1 + ((v/\pi)\epsilon)/((v/\pi)(1 + r(\sigma)) + 1 - v(1 + r^C)) \]

since by Jensen’s inequality for concave functions,
\[ E \ln((1 + (v/\pi)\epsilon)/((v/\pi)(1 + r(\sigma)) + (1 - v)(1 + r^C))) < \ln(1 + (v/\pi)E\epsilon)/\{v(1 + r(\sigma)) + (1 - v)(1 + r^C)) \} = \ln(1) = 0 \text{ as } E(\epsilon) = 0. \text{ QED.} \]

That is, more randomness in the returns to the risky asset reduces its geometric mean \((G_r)\) falls, lowering the geomentropic value of the portfolio.

### A.4 Geomentropic Preferences are Concave in Shares

**Proof of Concavity of Geomentropic Preferences**

To show that geomentropic preferences are concave in the shares, rewrite \( \ln G_p(v) - \lambda N_p(v) \) as
\[ \ln[\prod_{\sigma \in \times k} \left[ \frac{W_C}{\pi K} w^C + \sum_k v_k (\frac{W_k}{\pi K} w_k(\sigma) - \frac{W_C}{\pi K} w^C) \right]^{T(\gamma(\sigma))} - \lambda \sum_k v_k N_k \]

Let \( \frac{W_C}{\pi K} w^C = \phi \) and \( \frac{W_k}{\pi K} w_k(\sigma) - \frac{W_C}{\pi K} w^C = \theta_k \), so that geomentropic preferences can be written as
\[ \ln[\prod_{\sigma \in \times k} [\phi + \sum_k v_k \theta_k]^{T(\gamma(\sigma))} - \lambda \sum_k v_k N_k ] \]

Factor out \( \phi \) from both terms in the product, and take logarithms to get
\[ \sum_{\sigma \in \times k} T(\gamma(\sigma)) \ln(\phi[1 + \sum_k v_k \theta_k']) - \lambda \sum_k v_k N_k ] \]

where \( \theta_k' = \theta_k/\phi \). This is concave in shares if for \( v_k \neq v_k' \), the following holds for all \( 0 < t < 1 \)
\[ \sum_{\sigma \in \times k} T(\gamma(\sigma)) \ln(\phi[1 + \sum_k (tv_k + (1-t)v_k') \theta_k']) - \lambda \sum_k (tv_k + (1-t)v_k') N_k ] > t\{ \sum_{\sigma \in \times k} T(\gamma(\sigma)) \ln(\phi[1 + \sum_k v_k \theta_k']) - \lambda \sum_k v_k N_k ] \}

Canceling the common \( \phi \) and entropy terms from both sides of the inequality, the inequality is seen to hold if
\[ \sum_{\sigma \in \times k} T(\gamma(\sigma)) \ln((1 + \sum_k (tv_k + (1-t)v_k') \theta_k')) > t\{ \sum_{\sigma \in \times k} T(\gamma(\sigma)) \ln((1 + \sum_k v_k \theta_k')) \} + (1-t)\{ \sum_{\sigma \in \times k} T(\gamma(\sigma)) \ln((1 + \sum_k v_k \theta_k')) \} \]
Note that this inequality holds for any state of the world (that is, holding \( \sigma \) fixed) as the arithmetic mean exceeds the geometric mean:

\[
\ln([1 + \sum_k (tv_k + (1 - t)v_k')\theta_k']) > t \ln([1 + \sum_k v_k\theta_k']) + (1 - t) \ln([1 + \sum_k v_k'\theta_k'])
\]

since

\[
(t[1 + \sum_k v_k\theta_k'] + (1 - t)[1 + v_k'\theta_k']) > [1 + \sum_k v_k\theta_k'][1 + \sum_k v_k'\theta_k']^{(1-t)}
\]

Since the inequality holds for each state of the world, it holds for all states of the world when summed over both sides of the inequality. (QED)