Matching with Continuous Bidirectional Investment

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Abstract

We develop a one-to-one matching game where men and women (interns and managers, etc.) exert costly efforts to produce benefits for their partners. We prove the existence and Pareto optimality of interior stable allocations, and we characterize the relationship between players' costs, efforts, benefits, and payoffs in such allocations. We find, for instance, that men and women with lower marginal costs of effort choose to provide their partners with higher benefits by exerting more effort; in return, they receive higher benefits from their partners and attain higher payoffs.

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1 Introduction

Partners usually exert costly efforts to produce benefits for each other. For instance, when a man and a woman date or marry, the man may benefit the woman by cooking her favorite meals and helping her strategize future business meetings, while the woman may benefit the man by planning their vacations. Likewise, when a firm's manager and an intern match, the manager may benefit the intern by teaching her about the firm and how it does business, while the intern may benefit the manager (and the firm) by working on certain projects. Similarly, when a senior executive, like a Chief Research Officer (CRO), and a firm match, the CRO may benefit the firm by bringing in and managing talented scientists, while the firm may benefit the CRO via a salary and unique research opportunities (e.g., proprietary data and occasions to conduct experiment by changing business practices).[1]

These benefits (and the associated efforts) are not exogenous. Rather, the benefit a person chooses to provide to their partner depends directly on (i) their own cost of effort and (ii) the benefit their partner provides to them. It also depends indirectly on (iii) their own and their partner's outside opportunities, which themselves depend on the benefits and partners chosen by the other men and women. (For instance, if a man's partner is desired by other men, then he needs to provide a higher benefit in order to keep her; yet, he would never choose to provide a benefit so high as to make him worse off than he could be with some other attainable woman.) Our goal is to characterize how these forces shape the benefits that people produce and receive. In particular, we ask two questions. First, how does the benefit a person produces compare to the benefit produced by their partner? Second, how does a person's cost of effort influence the benefits they produce and receive?

To answer these questions, we first study a general one-to-one matching game, called the General Game, and we prove several results concerning the existence and Pareto optimality of stable allocations. Subsequently, we develop a special version of this game, called the Effort Game, in which heterogeneous men and women (inters and managers, etc.) pair with each other and exert costly efforts to benefit their partners, and we answer our main questions.

In the General Game, a finite number of men and women pair with each other. When a man and a woman match, they select an agreement that specifies their individual and joint actions from an (un-)countable set of feasible agreements.[2] The man's and the woman's

[1]Other examples where partners exert costly effort for each other's benefit include: mentors and mentees, masters and apprentices (including graduate students and professors), deans and colleges, lawyers/consultants and clients (or other buyers and sellers of customized services/goods), and so on.

[2]The notion that partners select agreements is natural. Every partnership is governed by a collection of behavioral rules (e.g., "we have a date night once a week), which are implicitly and explicitly selected by the partners over the course of their relationship. Since these rules determine the partners' individual and joint

payoffs are determined by their identities and the agreement they select. Our solution concept is a stable matching and vector of agreements, which we call a stable allocation. In a stable allocation, (i) each player earns at least the value of his or her outside option and (ii) no two players can do strictly better by pairing and selecting a new agreement, i.e., no two players "block" the allocation.

We first show that a stable allocation exists when the payoffs are continuous in the agreement and the set of feasible agreements is compact (Proposition 1). (The proof illuminates a connection between our game and the class of "Deferred Acceptance" algorithms.) We also show that a strongly Pareto optimal stable allocation exists (Proposition 2). Subsequently, we give intuitive sufficient conditions for the existence of and the Pareto optimality of stable allocations with interior agreements (Proposition 3 and 4).

We next develop the Effort Game. In the Effort Game, when a man and a woman match, they come to an agreement about the amount of effort each exerts. Their efforts produce benefits for each other and are chosen from a compact interval. The man's payoff is the benefit produced by the woman's effort less the cost of his own effort. Analogously, the woman's payoff is the benefit produced by the man's effort less the cost of her own effort. For simplicity, all players have the same benefit production function, which is increasing in effort.[3] Each player is also endowed with a type (e.g., ability) that affects his or her cost of effort.

To address our questions, we focus on stable allocations with interior agreements. Under the natural assumptions that players dislike exerting extremely high effort and dislike exerting effort when their partners don't exert any effort, these interior stable allocations exist, are Pareto optimal, and are the only stable allocations where players are matched (Corollary 1).

As to our first question on how partners' benefits compare, we find that players match based on the benefits they produce in *any* stable allocation with interior agreement.[4] Specifically, a man who produces the l-th highest benefit among men matches to a woman who produces the l-th highest benefit among women (Proposition 5); the analogous result holds for women. The intuition is that players "compete" with each other for higher benefit partners. To illustrate, suppose two men m and m' both produce the same benefit, but m' has

actions, the partners effectively select an agreement. Our simplification is that these rules are selected at the start of a relationship as opposed to being selected over the course of the relationship.

[3]Our results, save Proposition 9, continue to hold when men have one benefit function and women have a different benefit function. Thus, our core results extend to settings where one side pays the other for a service, e.g., consultants/lawyers and clients. Additionally, Proposition 5 continues to hold when the benefit a player produces depends on his or her identity and effort. See Section 4 for details.

[4]There are usually a multiplicity of interior stable allocations. Unless mentioned otherwise, our findings pertain to every interior stable allocation.

partner w' who produces a strictly higher benefit than the partner of m. Then m finds it best to increase his effort by an arbitrarily small amount in order to produce a slightly larger benefit and "win" w' away from m'. Man m is able to win w' in this fashion because she desires the largest benefit possible. He's willing to win w' because he gains a strictly higher benefit in exchange for a smaller increase in his cost. Thus, m and w' block. It follows that a necessary condition of stability is that men who produce the same benefit are matched to women who produce the same benefit.

Since the benefit production function is common across players, it follows that players match based on the effort they exert (Corollary 2). Interestingly, Rammstedt and Schupp [18] and Watson et al. [26] find empirical support for this prediction: they both find that conscientious people more frequently date and marry other conscientious people, while lazy people more frequently date and marry other lazy people. Our result provides a novel rationalization for this observation.

To answer our second question on how players' benefits and types compare, we assume that the *marginal cost* of effort is decreasing in type/ability. We find that this is sufficient to ensure that higher ability players produce larger benefits. Specifically, we show that if man m' has a strictly higher type than another man m, then m' produces a benefit at least as large as the benefit produced by m (Proposition 6); the analogous result holds for women. The intuition is that higher ability players can "outcompete" lower ability players because their lower marginal costs allow them to profitably offer slightly higher benefits. Thus, the competition for partners drives them to provide higher benefits. It follows from Proposition 6 that higher ability (i) players always exert more effort and (ii) receive higher benefits from their partners. For instance, (i) m' always exerts at least as much effort as m (Corollary 3) and (ii) the partner of m' always produces a benefit at least as high as the partner of m (Corollary 4).

Under the additional assumption that all players share a common cost of zero effort, we find that higher ability players have higher payoffs (Proposition 7). This additional assumption ensures that higher ability players have strictly lower costs. Thus, they profitably "imitate" and outcompete lower ability players whenever lower ability players do strictly better. Hence, a necessary condition of stability is that higher ability players earn at least as much as lower ability players.

Our findings suggest that higher ability players match with other higher ability players. While this need not happen in every interior stable allocation (we discuss why in Section 5), we establish that there is at least one interior stable allocation where it happens (Proposition 8). Interestingly, Belot and Francesconi [3] and Hitsch et al. [11] find that more educated people date each other. Since more educated people often have a lower cost of effort (e.g., Regan et al. [19]), their findings lend support to this prediction.

We close by examining how decreases in players' opportunity costs of effort, perhaps due to reductions in work/family responsibilities and more sleep, affect their decisions and outcomes. We proceed by assuming that men and women have a symmetric endowment of types, and we select the unique "symmetric" stable allocation. (This allocation is focal because it treats equals equally and maximizes social welfare (Lemma 11).) We find that decreases in players' marginal costs of effort, i.e., increases in their types, increase their efforts, the benefits they produce and receive, and their payoffs in this allocation (Proposition 9). This result follows from the symmetry of the type endowment, the submodularity of the cost function, and the fact that all players have a common cost of zero effort.

Related Literature

Our work makes economic and technical contributions to three literatures. The first literature examines matching with agreements – e.g., Demange and Gale [8], Hatfield and Milgrom [10], and Roth [20]. We contribute to it via the General Game by weakening many of the traditional assumptions and by giving a new and general existence proof. The second literature examines pre-match investment – e.g., Burdett and Coles [4], Cole et al. [6], and Peters [15]. We contribute to it via the Effort Game by allowing investment/effort to be co-determined with the matching, by examining how effort cost affects the benefits players produce and receive, and by developing comparative statics. The third literature examines when players match assortatively in their endowed types – e.g., Becker [2] and Legros and Newman [14]. We contribute to it via the Effort Game by providing simple conditions on payoffs that guarantee assortative matching in types and by focusing on the relationship between benefits and types, instead of the relationship between matched players' types.

There are three seminal games in the matching literature: the "Marriage Game" of Gale and Shapley [9]; the "Assignment Game" as articulated by Demange and Gale [8]; and the "Generalized Marriage Game" as articulated by Hatfield and Milgrom [10].[5] In the Marriage Game, men and women match with each other and receive payoffs based only on the identities of their partners. The Assignment Game generalizes the Marriage Game by allowing men and women to agree to a (real-valued) monetary transfer when they match. Thus, players' payoffs depend on their partners' identities and their transfers; these payoffs are either quasi-linear or weakly monotone in money. Both the Marriage Game and the Assignment Game are special cases of the General Game; see Example 3 (in Section 2). The Generalized Marriage Game extends the Marriage Game by allowing for many-to-one matching – e.g., a firm hiring multiple workers – and by allowing a worker and a firm to select an agreement from a *finite* set of possible agreements when they match. Agreements are more than simple

[5]See Roth and Sotomayor [21] and Sönmez and Ünver [23] for the histories of these games.

transfers; in addition to salaries, they may specify vacation time, job responsibilities (e.g., teaching loads), and a myriad of factors. Hence, players' payoffs depend on the identities of their matches and their agreement. When firms can only hire one worker, the Generalized Marriage Game is a special case of the General Game.

One might think that the Effort Game is an Assignment Game. This isn't the case: an Assignment Game only allows each matched couple to agree to a single point in $\mathbb{R}$, whereas the Effort Game allows each matched couple to agree to a point in $\mathbb{R}^2$. Thus, the Effort Game is suitable for examining the efforts that both players exert, while an Assignment Game is not. Also, unlike an Assignment Game, the Effort Game makes no assumptions about the monotonicity of payoffs. Furthermore, the Effort Game is not a Generalized Marriage Game since its set of possible agreements is a product of intervals; this uncountability initially motivated our study of the General Game.

Our existence results, Propositions 1 and 2, are related to the existence results of (i) Hatfield and Milgrom [10] and Roth [20], (ii) Crawford and Knoer [7] and Quinzii [17], and (iii) Alkan and Gale [1] and Kaneko [13]. Hatfield and Milgrom [10] and Roth [20] both prove existence for the Generalized Marriage Game by giving "Deferred Acceptance" algorithms that halt at stable allocations in finite time. Crawford and Knoer [7] prove existence for a quasi-linear Assignment Game by employing a contradiction argument that illuminates a connection between the class of Deferred Acceptance algorithms and the existence of stable allocations in their game. Quinzii [17] generalizes Crawford and Knoer's result by proving existence for a weakly monotone Assignment Game via Scarf's Balancedness Theorem.

Kaneko [13] considers a more abstract, one-to-one matching game between men and women. In his game, each man m and each woman w have a set of achievable payoffs V_{mw} from which they pick some point when they're matched. He shows that if this set satisfies certain conditions (e.g., if $\boldsymbol{v}' \in V_{mw}$, then $\boldsymbol{v} \leq \boldsymbol{v}'$ implies $\boldsymbol{v} \in V_{mw}$), then Scarf's Balancedness Theorem implies a stable allocation exists. Alkan and Gale [1] build on Kaneko's environment by assuming that each man m and each woman w's set of achievable payoffs is given by their Pareto frontier. When this frontier is described by a bounded, strictly decreasing, and continuous function that intersects both the horizontal and vertical axes, they describe an algorithm that computes a stable allocation in finite time.

We differ from these studies in our assumptions and in our method of proof. Consequently, our result applies to natural games like Example 2 (in Section 2) where payoffs are non-monotone, Kaneko's technical properties don't hold, and the Pareto frontier doesn't intersect either axis. In addition, our method of proof is novel and it illuminates a connection between the class of Deferred Acceptance algorithms and the existence of stable allocations in the General Game.

Our finding on how players' benefits (and efforts) compare, i.e., Proposition 5 and Corollaries 1 and 2 are related to the works of Burdett and Coles [4], Cole et al. [6], and Peters [15] who examine games with pre-match investments. Burdett and Coles draw on search theory to build a two-stage game, with an infinite number of heterogeneous players, where players first invest in themselves (e.g., go to school and the gym) to increase their own attractiveness and then enter a matching market with frictions. A player's payoff is the attractiveness of his or her partner. They find that, when players are impatient, they invest just enough to attract the best matches they can and that these investments may be inefficient. Cole et al. [6] draw on matching theory to build a two-stage game where players first invest in themselves and then enter a quasi-linear Assignment Game, where each pair's joint surplus is a function of types and investments. In equilibrium, they show that players who invest more match with each other, a result similar to Corollary 2, and they show that equilibrium investments can be inefficient. Peters [15] considers a game analogous to Cole et al., save that second stage resembles a Marriage Game, where players' payoffs are linear in the investments of their partners. In large economies, he shows that players usually over-invest.

Our work is complementary to these papers. Like them, we study the effects of investment. However, in the Effort Game, investment/effort is match-specific since it's determined at the time players match, whereas pre-match investments are the same for every possible match. This distinction is both intuitive and economically meaningful since it leads to Pareto optimal investment per Corollary 1. We also explore how players' costs of effort are related to (i) the benefits they produce and receive and (ii) their payoffs; none of the papers discussed above examine these relationships.

There are also games that study post-match investment. For instance, Kaya and Vereshchagina consider a three-stage "roommates" game with transferable utility where: (i) players pair off, (ii) each pair writes a contract that divides the profit from their joint venture, and (iii) each player decides how much costly and unobserved effort to contribute to the venture. They show that efforts/investments are inefficient, due to the moral hazard problem, and they develop conditions under which players match assortatively in their endowed types/productivities. Our work is complementary because we allow players to verifiably choose and compete on the basis of effort/benefit and, unlike Kaya and Vereshchagina, we examine the relationship between benefit and type.

Speaking of assortative matching in types, our finding in this area, Proposition 8, is also related to the work of Becker [2] and Legros and Newman [14].[6] In his seminal paper, Becker

[6]There is a rich literature on assortative matching in types in transferable utility games – e.g., Chiappori et al. [5] and Pycia [16]. Additionally, there is a large matching-with-frictions literature on assortative matching in type – see Smith [22] for an overview.

uses a quasi-linear Assignment Game to show that players match assortatively when their joint surplus function is supermodular in their types. Legros and Newman consider a more general, non-transferable utility matching game that's akin to the game of Alkan and Gale [1]. They develop general conditions on the Pareto frontiers of each man-woman pair that guarantees higher types match with each other. Our work is complementary since (i) we develop natural conditions on the primitive payoff functions under which higher types match with each other and (ii) we focus on the relationship between the benefit and type, instead of the relationship between matched players' endowed types.

2 Description of the General Game

This section describes the General Game, defines a stable allocation, and discusses several examples.

Environment

There are two finite sets of players, men $\mathcal{M} = \{1, \dots, M\}$ and women $\mathcal{W} = \{M + 1, \dots, N\}$, with $N > M > 0$. Let $\mathcal{N} = \mathcal{M} \cup \mathcal{W}$. We write m for the m-th man, w for the w-th woman, and i for the i-th player (regardless of gender).

Each player may either be single or may match with a member of the opposite sex. We adopt the convention that a single player is matched to himself or herself. A **matching** is a function that specifies each player's match, i.e., is a $\phi : \mathcal{N} \to \mathcal{N}$ such that: (i) for each man m, $\phi(m) \in \mathcal{W} \cup \{m\}$; (ii) for each woman w, $\phi(w) \in \mathcal{M} \cup \{w\}$; and (iii) for each man m and each woman w, $\phi(m) = w \iff \phi(w) = m$. We say player i is **partnered** if $\phi(i) \neq i$. We write Φ for the finite set of all matchings.

When a man and a woman match, they select an **agreement** $\boldsymbol{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$, where $k \geq 1$. Their agreement specifies their individual and joint **actions** $x_1, \dots, x_k$. For instance, x_1 and x_2 may give the number of hours m and w spend at work each week respectively, x_3 may give the number of days m and w spend camping every year, and so on. Also, each single player has an agreement $\boldsymbol{x} \in \mathbb{R}^k$ with himself or herself.

Given a $\phi \in \Phi$, we write $\boldsymbol{x}^i$ for the agreement player i has with either (i) his or her partner or (ii) himself or herself. We write $\bar{\boldsymbol{x}} = (\boldsymbol{x}^1, \dots, \boldsymbol{x}^M, \boldsymbol{x}^{M+1}, \dots, \boldsymbol{x}^N)$ for the vector of players' agreements. Since $\boldsymbol{x}^i = \boldsymbol{x}^{\phi(i)}$ for each player i,[7] we have $\bar{\boldsymbol{x}} \in A(\phi) = \{(\tilde{\boldsymbol{x}}^1, \dots, \tilde{\boldsymbol{x}}^N) \in \mathbb{R}^{kN} \,|\, \tilde{\boldsymbol{x}}^i = \tilde{\boldsymbol{x}}^{\phi(i)}$ for all $i \in \mathcal{N}\}$, where $A(\phi)$ as the set of possible agreement vectors for the matching ϕ. For each $\phi \in \Phi$, we have that $A(\phi)$ is a nonempty vector subspace of $\mathbb{R}^{kN}$. An

[7] If player i is partnered, then i and his or her match $\phi(i)$ have *an* agreement $\boldsymbol{x}$, so $\boldsymbol{x}^i = \boldsymbol{x}$ and $\boldsymbol{x}^{\phi(i)} = \boldsymbol{x}$. If player i is single, then he or she as an agreement $\boldsymbol{x}$ and $\phi(i) = i$, so $\boldsymbol{x}^{\phi(i)} = \boldsymbol{x}^i = \boldsymbol{x}$.

allocation is a $(\phi, \bar{\boldsymbol{x}})$ such that $\phi \in \Phi$ and $\bar{\boldsymbol{x}} \in A(\phi)$, i.e., is a matching and a vector of agreements. We write $\mathcal{A} = \{(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN} | \bar{\boldsymbol{x}} \in A(\phi)\}$ for the set of allocations.

A player's payoff depends (only) on the identity of his or her match and their agreement. Formally, each man m has a payoff function $u_m : \{\mathcal{W} \cup \{m\}\} \times \mathbb{R}^k \to \mathbb{R}$ over his possible matches and agreements. Likewise, each woman w has a payoff function $u_w : \{\mathcal{M} \cup \{w\}\} \times \mathbb{R}^k \to \mathbb{R}$. We normalize the value of being single to zero, i.e., for every $\boldsymbol{x} \in \mathbb{R}^k$, we have $u_i(i, \boldsymbol{x}) = 0$ for each player i. Let $(\phi, \bar{\boldsymbol{x}}) = (\phi, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^i, \ldots, \boldsymbol{x}^N) \in \Phi \times \mathbb{R}^{kN}$, in a slight abuse of notation we write $u_i(\phi, \bar{\boldsymbol{x}})$ for the payoff of player i in $(\phi, \bar{\boldsymbol{x}})$, i.e., $u_i(\phi, \bar{\boldsymbol{x}}) \equiv u_i(\phi(i), \boldsymbol{x}^i)$.

Stable Allocations

The next four definitions develop the idea of a stable allocation. Let $X \subset \mathbb{R}^k$ be the (nonempty) set of "feasible" agreements.[8]

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *feasible* if agreements are in X, i.e., $\bar{\boldsymbol{x}} \in X^N$.

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *individually rational* if every player gets at least the value of being single, i.e., $u_i(\phi, \bar{\boldsymbol{x}}) \geq 0$ for each player i.

Definition. A man m and a woman w *block* a $(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN}$ if they can obtain strictly higher payoffs together than they obtain in $(\phi, \bar{\boldsymbol{x}})$, i.e., if there exists an $\boldsymbol{x} \in X$ such that

$$u_m(w, \boldsymbol{x}) > u_m(\phi, \bar{\boldsymbol{x}}) \text{ and } u_w(m, \boldsymbol{x}) > u_w(\phi, \bar{\boldsymbol{x}}).$$

Definition. An allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is *stable* if (i) it is feasible, (ii) individually rational, and (iii) no man and woman block it.

Stable allocations are our solution concept. When an allocation is stable: (i) no player can do strictly better by choosing to be single (per individual rationality) and (ii) no two players can do strictly better by matching with each other and choosing a new agreement instead of following $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ (per no blocking). As in Gale and Shapley [9], we might imagine that a stable allocation is the outcome of a bargaining process where players try to maximize their own payoffs. When an allocation is stable, no player can do strictly better by (i) opting out or (ii) by trying to strike a new bargain with some other player j, as j would reject this bargain since it doesn't make him or her strictly better off. (After bargaining concludes, we imagine that players match with their agreed upon partners and take their agreed upon actions, and then receive their payoffs.[9])

[8] We take X to be the same for all men and women for notational simplicity.

[9] Players may execute their agreements because (i) there is community enforcement and players punish each other for defections (e.g., Kandori [12]), (ii) because the agreements constitute informal, yet enforceable contracts (e.g., a verbal contract or an "implied-in-fact" contract), or (iii) they have a preference for doing so (e.g., a sense of responsibility).

Remark. Since there are usually many stable allocations, we'll focus on results about the set of stable allocations or specific selections thereof.

Remark. The set of stable allocations and the core coincide since the payoffs of a matched man and woman only depend on their identities and their agreement.

Examples

Since the General Game is quite abstract, it's helpful to give a few examples.

Example 1. A Simple Effort Game.

Suppose there are four players, two men and two women, i.e., $\mathcal{M} = \{1, 2\}$ and $\mathcal{W} = \{3, 4\}$. Let each player i have a type $\theta_i \in R$, e.g., innate ability. We set $\theta_1 = \theta_4 = 2$ and $\theta_2 = \theta_3 = 1$. Let $X = [0, 2]^2$ be the set of feasible agreements. When a man m and a woman w are matched (to each other), their payoffs to agreement $(x_1, x_2) \in \mathbb{R}^2$ are

$$u_m(w, x_1, x_2) = x_2 - \frac{(x_1)^2}{\theta_m} - {}^1\!/_8 \text{ and } u_w(m, x_1, x_2) = x_1 - \frac{(x_2)^2}{\theta_w} - {}^1\!/_8.$$

Recall that single players get zero. For $(x_1, x_2) \in X$, we think of x_1 and x_2 as the man's and the woman's efforts respectively. Thus, both players exert costly effort to make each other happy: m exerts effort x_1 to produce a benefit of x_1 for w and incurs a cost of $(x_1)^2/\theta_m - \frac{1}{8}$ for doing so, while w exerts effort x_2 to produce a benefit of x_2 for m and incurs a cost of $(x_2)^2/\theta_w - \frac{1}{8}$ for doing so. (The assumption that each player's effort is in $[0, 2]$ reflects the idea that people can only work so hard as they face time and energy limitations.)

One stable allocation of this game is $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \boldsymbol{x}^{2^\star}, \boldsymbol{x}^{3^\star}, \boldsymbol{x}^{4^\star})$, where $\phi^\star(1) = 4$ and $\boldsymbol{x}^{1^\star} = \boldsymbol{x}^{4^\star} = (1, 1)$, and $\phi^\star(2) = 3$ and $\boldsymbol{x}^{2^\star} = \boldsymbol{x}^{3^\star} = ({}^1\!/_2, {}^1\!/_2)$. That is, man 1 matches with woman 4 and they both exert effort 1, while man 2 matches with woman 3 and they both exert effort ${}^1\!/_2$.

Let's verify $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable. To do this, we need to show that it's feasible, individually rational, and not blocked. Feasibility is automatic since $(1, 1)$ and $({}^1\!/_2, {}^1\!/_2)$ are in X. Individual rationality requires a bit more work. We have $u_1(\phi^\star, \bar{\boldsymbol{x}}^\star) = 1 - {}^1\!/_2(1)^2 - {}^1\!/_8 = {}^3\!/_8$. By analogous calculations $u_2(\phi^\star, \bar{\boldsymbol{x}}^\star) = {}^1\!/_8$, $u_3(\phi^\star, \bar{\boldsymbol{x}}^\star) = {}^1\!/_8$, and $u_4(\phi^\star, \bar{\boldsymbol{x}}^\star) = {}^3\!/_8$. It follows that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is individually rational, i.e., no player can do better by choosing to be single.

We need to make sure that there are no blocking pairs. The definition of blocking makes no assumptions about which players block an allocation: players who are matched may block, as may players who are not matched. Thus, we need to check four pairs.

Consider man 1 and woman 3. They block if there is a $(x_1, x_2) \in X$ such that $u_1(3, x_1, x_2) >$

$^3/_8$ and $u_3(1, x_1, x_2) > {^1/_8}$, i.e., if

$$x_2 - \frac{1}{2}(x_1)^2 > \frac{1}{2} \text{ and } x_1 - (x_2)^2 > \frac{1}{4}.$$

It is readily verified that this system has no real solutions. Consequently, man 1 and woman 3 can't block, i.e., man 1 and woman 3 can't do better by matching and choosing a new agreement instead of following $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. An analogous argument shows that man 2 and woman 4 can't block.

Consider man 1 and woman 4. They block if there is a $(x_1, x_2) \in X$ such that $u_1(4, x_1, x_2) > {^3/_8}$ and $u_4(1, x_1, x_2) > {^1/_8}$, i.e., if

$$x_2 - (x_1)^2 > \frac{1}{4} \text{ and } x_1 - \frac{1}{2}(x_2)^2 > \frac{1}{2}.$$

Since this system has no real solutions, man 1 and woman 4 can't block. Symmetry gives that man 2 and woman 3 can't block. It follows that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable. △

Example 2. Hiking.

Let $\mathcal{M} = \{1, 2\}$, let $\mathcal{W} = \{3, 4\}$, and let $X = [0, 1]$. When a man m and a woman w are matched, their payoffs to agreement $x \in \mathbb{R}$ are

$$u_m(w, x) = \begin{cases} x & \text{if } x \leq \frac{1}{2} \\ 1 - x & \text{if } x > \frac{1}{2} \end{cases} \text{ and } u_w(m, x) = \begin{cases} 2x & \text{if } x \leq \frac{1}{3} \\ 1 - x & \text{if } x > \frac{1}{3}, \end{cases}$$

Recall that single players get zero. For $x \in X$, we think of x as the percentage of their Sundays m and w spend hiking – w only wants to go for a third of the day, while m wants to go for half the day.

Let V_{mw} be the set of payoffs that man m and woman w can achieve when matched, i.e., $V_{mw} = \{(v_m, v_w) \in \mathbb{R}^2 |$ there is a $x \in X$ s.t. $u_m(w, x) = v_m$ and $u_w(m, x) = v_w\}$. This set is the blue triangle plotted in Figure 2.1; it's bounded and has empty interior. The Pareto frontier is the northeastern edge of V_{mw}; it's highlighted in gray in the figure. While the frontier is strictly decreasing, it never intersects the horizontal or vertical axes. Thus, V_{mw} does not meet the assumptions of Alkan and Gale [1] or Kaneko [13].

There is a continuum of stable allocations. One set is described by $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, where $\phi^\star(1) = 3$, $\phi^\star(2) = 4$, and $\bar{\boldsymbol{x}}^\star = (x, x, x, x)$ with $x \in [\frac{1}{3}, \frac{1}{2}]$. That is, man 1 matches with woman 3, man 2 matches with woman 4, and all players hike for a common proportion of their Sunday x, with $x \in [\frac{1}{3}, \frac{1}{2}]$.

Such an allocation is trivially feasible and individually rational. In such an allocation,

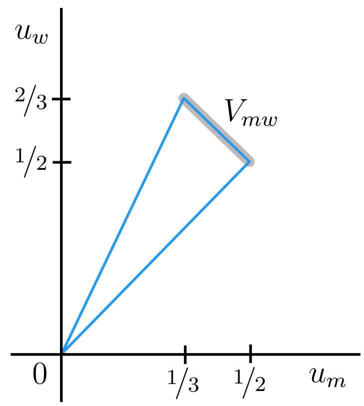

Figure 2.1: Achievable Payoffs in Example 2 for Man m and Woman w

(a) man 1 and woman 3 or (b) man 1 and woman 4 can't block as an increase in the duration makes woman 3 or woman 4 worse off and a decrease in the duration makes man 1 worse off. Likewise, (c) man 2 and woman 3 or (d) man 2 and woman 4 can't block. △

The General Game nests both the Marriage and Assignment Games. Since payoffs in the Marriage Game only depend on the identities of players' matches, we can embed it in the General Game by taking X to be a singleton. The Assignment Game requires a bit more work; the next example illustrates with a quasi-linear version of this game.

Example 3. A Quasi-Linear Assignment Game.

In a quasi-linear Assignment Game, men and women receive some surplus from matching, which they split via a monetary transfer. For each man m and each woman w, let $f_m(w) \geq 0$ give m's surplus from matching with w and let $f_w(m) \geq 0$ give w's surplus from matching with m. When m and w match, they agree to a monetary transfer $x \in \mathbb{R}$; m earns his surplus less the transfer $x \in \mathbb{R}$ and w earns her surplus plus x, i.e.,

$$u_m(w, x) = f_m(w) - x \text{ and } u_w(m, x) = f_w(m) + x.$$

Recall that single players get zero.

Let $X = [-\bar{\alpha}, \bar{\alpha}]$, where $\bar{\alpha} = \max_{m,w}\{f_m(w) + f_w(m)\}$. To digress, usually an Assignment Game allows players to agree to any transfer in $\mathbb{R}$. Yet, restricting transfers to $[-\bar{\alpha}, \bar{\alpha}]$ is without loss. If man m and woman w have a transfer $x \notin [-\bar{\alpha}, \bar{\alpha}]$, then either m or w has a negative payoff. Thus, in any individually rational allocation (and, by inclusion, any stable allocation), all transfers are in $[-\bar{\alpha}, \bar{\alpha}]$.[10]

[10] In general, Demange and Gale [8] allow players to agree to any transfer in $\mathbb{R}$. However, they require, that for each man m and each woman w, there is an finite α'_{mw} such that if m and w's transfer is greater than α'_{mw} or less than $-\alpha'_{mw}$, then either m or w has a negative payoff. Thus, every transfer is in $[-\bar{\alpha}', \bar{\alpha}']$

It is readily verified that a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ satisfies Demange and Gale's [8] definition of stability. △

Remark. While the General Game subsumes both the Marriage and Assignment Games, it doesn't preserve many of their properties. For instance, there is usually no "man-preferred" stable allocation in the General Game because players may be indifferent to some matches and agreements. We discuss this in the Supplement.

3 Results for the General Game

In this section, we state and prove our results for the General Game. We defer the proofs to the end of each subsection in order to discuss the results.

Existence of Stable Allocations

In this subsection, we prove that stable allocations exist under the following assumption.

Assumption 1. Compactness and Continuity.
The set of feasible agreements X is compact and, for each man m and each woman w, the payoffs $u_m(w', \boldsymbol{x})$ and $u_w(m', \boldsymbol{x})$ are continuous in $\boldsymbol{x}$ for all women w' and for all men m' respectively.

Proposition 1. Existence of a Stable Allocation.
Let Assumption 1 hold, then a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exists.

Proposition 1 gives, for instance, that there is a stable allocation in Examples 1 and 2, in the (Generalized) Marriage Game, and in the Assignment Game since inspection shows that Assumption 1 holds in each. We'll prove Proposition 1 by contradiction. Specifically, we'll show that if there is no stable allocation when X is compact, then there is no stable allocation when agreements are restricted to a finite subset of X. This contradicts a well-known result that there is always a stable allocation when the set of feasible agreements is finite.[11] The key insight of our proof is that under the contradiction hypothesis and Assumption 1, we can use the Heine-Borel Theorem to ensure the existence of this finite subset of X. We need four preliminary results before we make this argument precise.

Lemma 1. Finite Existence.
Let the set of feasible agreements X be finite, then a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exists.

in any individually rational allocation, where $\bar{\alpha}' = \max_{m,w}\{\alpha'_{mw}\}$. So restricting x to be in the feasible set $X = [-\bar{\alpha}', \bar{\alpha}']$ is without loss.

[11] The intuition here is similar to that of Crawford and Knoer [7]; however, our game, assumptions, and formal approach are quite different from theirs.

Results like Lemma 1 are standard in the one-to-one and many-to-one matching literatures when payoffs are such that players can strictly order matches and agreements; for instance, Hatfield and Milgrom's Theorem 3 [10] and Roth's Theorem 1 [20]. Both papers prove existence by giving variants of Gale and Shapley's [9] Deferred Acceptance algorithm that find a stable allocation in finite time. The problem is that players in our game may be indifferent. Fortunately, these algorithms can be modified to allow for indifferences via the inclusion of a "tie-breaking" rule – see Roth and Sotomayor's Theorem 2.8 [21] for an example of this kind of modification. (In the Supplement, we give a proof of Lemma 1 that's based on tie-breaking.)

Lemma 2. Continuity of $u_i(\phi, \bar{\boldsymbol{x}})$.
Let Assumption 1 hold and let $\phi \in \Phi$, then $u_i(\phi, \bar{\boldsymbol{x}})$ is continuous in $\bar{\boldsymbol{x}}$ for each player i.

Proof. This is almost obvious. Let $\{\bar{\boldsymbol{x}}(j)\}_{j=1}^{\infty} \to \bar{\boldsymbol{x}}$, with $\bar{\boldsymbol{x}}(j)$ and $\bar{\boldsymbol{x}}$ in $\mathbb{R}^{kN}$. Write $\bar{\boldsymbol{x}}(j) = (\boldsymbol{x}^1(j), \ldots, \boldsymbol{x}^i(j), \ldots, \boldsymbol{x}^N(j))$ and $\bar{\boldsymbol{x}} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^i, \ldots, \boldsymbol{x}^N)$. Then, we have $\{\boldsymbol{x}^i(j)\} \to \boldsymbol{x}^i$ for each $i \in \mathcal{N}$. Hence, for each $i \in \mathcal{N}$, we have

$$\lim_{j\to\infty} u_i(\phi, \bar{\boldsymbol{x}}(j)) = \lim_{j\to\infty} u_i(\phi(i), \boldsymbol{x}^i(j)) = u_i(\phi(i), \boldsymbol{x}^i) = u_i(\phi, \bar{\boldsymbol{x}}),$$

where the first and third equalities are definitional, and the second equality is due to Assumption 1. □

To prove Proposition 1, we need to establish that (i) the set of feasible and individually rational allocations is representable as a collection of compact sets and (ii) that the set of allocations a man and woman can block is open. To these ends, we introduce the following notation. Let $\phi \in \Phi$ and let

$$F(\phi) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} \,|\, \bar{\boldsymbol{x}} \in X^N \cap A(\phi) \text{ and } u_i(\phi, \bar{\boldsymbol{x}}) \geq 0 \text{ for all } i \in \mathcal{N}\}$$

be the (possibly empty) set of agreement vectors in $\mathbb{R}^{kN}$ such that the pair $(\phi, \bar{\boldsymbol{x}})$ is a feasible and individually rational allocation for each $\bar{\boldsymbol{x}} \in F(\phi)$. Let $\Phi_F = \{\phi \in \Phi | F(\phi) \neq \emptyset\}$ be the set of matchings such that, for each $\phi \in \Phi_F$, there is an $\bar{\boldsymbol{x}}$ so that $(\phi, \bar{\boldsymbol{x}})$ is feasible and individually rational. Notice that an allocation $(\phi, \bar{\boldsymbol{x}})$ is feasible and individually rational if an only if $\phi \in \Phi_F$ and $\bar{\boldsymbol{x}} \in F(\phi)$. Since players may always be matched to themselves (i.e., be single), we have that $\Phi_F \neq \emptyset$, i.e., there is a feasible and individually rational allocation.

Lemma 3. Compactness of $F(\phi)$.
Let Assumption 1 hold and let $\phi \in \Phi$, then $F(\phi)$ is compact.

Proof. We take $F(\phi)$ to be nonempty, since the empty set is trivially compact. We have that $F(\phi)$ is bounded as X^N is compact. We also have that $F(\phi)$ is closed because (i) X^N and $A(\phi)$ are closed and (ii) the function $u_i(\phi, \bar{\boldsymbol{x}})$ is continuous in $\bar{\boldsymbol{x}}$ for each $i \in \mathcal{N}$ per Lemma 2. □

When a man m and a woman w block a $(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN}$ with agreement $\boldsymbol{x} \in X$, we say $(m, w, \boldsymbol{x})$ **blocks** $(\phi, \bar{\boldsymbol{x}})$. Let $C = \mathcal{M} \times \mathcal{W} \times X$. For a $\phi \in \Phi$ and $c = (m, w, \boldsymbol{x}) \in C$, let

$$D_\phi(c) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} \,|\, u_m(w, \boldsymbol{x}) > u_m(\phi, \bar{\boldsymbol{x}}) \text{ and } u_w(m, \boldsymbol{x}) > u_w(\phi, \bar{\boldsymbol{x}})\}$$

be the set of vectors in $\mathbb{R}^{kN}$ such that c blocks $(\phi, \bar{\boldsymbol{x}})$ for all $\bar{\boldsymbol{x}} \in D_\phi(c)$.

Lemma 4. Openness of $D_\phi(c)$.
Let Assumption 1 hold, let $\phi \in \Phi$, and let $c \in C$, then $D_\phi(c)$ is open.

Proof. We take $D_\phi(c)$ to be nonempty, since the empty set is trivially open. Let $(m, w, \boldsymbol{x}) = c$. Since $u_m(\phi, \bar{\boldsymbol{x}})$ and $u_w(\phi, \bar{\boldsymbol{x}})$ are continuous in $\bar{\boldsymbol{x}}$ per Lemma 2, we have that $\{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_m(\phi, \bar{\boldsymbol{x}}) < u_m(w, \boldsymbol{x})\}$ and $\{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_w(\phi, \bar{\boldsymbol{x}}) < u_w(m, \boldsymbol{x})\}$ are open sets. Since $D_\phi(c) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_m(\phi, \bar{\boldsymbol{x}}) < u_m(w, \boldsymbol{x})\} \cap \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_w(\phi, \bar{\boldsymbol{x}}) < u_w(m, \boldsymbol{x})\}$, it follows that $D_\phi(c)$ is open. □

Proof of Proposition 1.[12] Suppose that there is no stable allocation. Since $\Phi_F \neq \emptyset$, every feasible and individually rational allocation is blocked by some man and woman. Let $\phi \in \Phi_F$. Then, for every $\bar{\boldsymbol{x}} \in F(\phi)$, we have that $(\phi, \bar{\boldsymbol{x}})$ is blocked by a $c \in C$, which implies that

$$F(\phi) \subset \cup_{c \in C} D_\phi(c).$$

Since $D_\phi(c)$ is open by Lemma 4, we have $\{D_\phi(c)\}_{c \in C}$ is an open cover of $F(\phi)$. Since $F(\phi)$ is compact by Lemma 3, the Heine-Borel Theorem gives the existence of a finite sub-cover $\{D_\phi(c_{\phi j})\}_{j=1}^{l_\phi}$. Thus, for every $\bar{\boldsymbol{x}} \in F(\phi)$, we have $(\phi, \bar{\boldsymbol{x}})$ is blocked by some element of $\{c_{\phi j}\}_{j=1}^{l_\phi}$. Repeating this argument for all matchings in Φ_F gives a set

$$E = \cup_{\phi \in \Phi_F} \{c_{\phi j}\}_{j=1}^{l_\phi}$$

such that every feasible and individually rational allocation is blocked by an element of E. Since Φ_F is finite as Φ is finite, E is finite. Let

$$E_X = \{\boldsymbol{x} \in X \,|\, (m, w, \boldsymbol{x}) \in E \text{ for some } (m, w) \in \mathcal{M} \times \mathcal{W}\}$$

be the set of agreements associated with E; we have that E_X is finite.

[12]This proof benefited greatly from discussions with Asaf Plan.

To establish the contradiction, suppose the set of feasible agreements is E_X. We establish that every feasible and individually rational allocation $(\phi, \bar{\boldsymbol{x}})$ is blocked, i.e., there is a $(m, w, \boldsymbol{x}) \in \mathcal{M} \times \mathcal{W} \times E_X$ for whom $u_m(w, \boldsymbol{x}) > u_m(\phi, \bar{\boldsymbol{x}})$ and $u_w(w, \boldsymbol{x}) > u_w(\phi, \bar{\boldsymbol{x}})$. Given this, there is no stable allocation. Yet, since E_X is finite, Lemma 1 gives that there is a stable allocation, a contradiction.

Let $(\phi, \bar{\boldsymbol{x}})$ be a feasible and individually rational allocation, i.e., let $(\phi, \bar{\boldsymbol{x}}) \in \mathcal{A}$ such that $\bar{\boldsymbol{x}} \in (E_x)^N \cap A(\phi)$ and $u_i(\phi, \bar{\boldsymbol{x}}) \geq 0$ for each player i. Since $(\phi, \bar{\boldsymbol{x}})$ remains feasible and individually rational when the set of feasible agreements is X (as $\bar{\boldsymbol{x}} \in (E_X)^N \subset X^N$ and the payoff functions don't change), we have $\phi \in \Phi_F$ and $\bar{\boldsymbol{x}} \in F(\phi)$. Thus, there is a $(m', w', \boldsymbol{x}') \in E$ such that $u_{m'}(w', \boldsymbol{x}') > u_{m'}(\phi, \bar{\boldsymbol{x}})$ and $u_{w'}(w', \boldsymbol{x}') > u_{w'}(\phi, \bar{\boldsymbol{x}})$. Since $\boldsymbol{x}' \in E_X$, we have that man m' and woman w' block $(\phi, \bar{\boldsymbol{x}})$ when the set of feasible agreements is E_X, i.e., $(m', w', \boldsymbol{x}') \in \mathcal{M} \times \mathcal{W} \times E_X$ with $u_{m'}(w', \boldsymbol{x}') > u_{m'}(\phi, \bar{\boldsymbol{x}})$ and $u_{w'}(w', \boldsymbol{x}') > u_{w'}(\phi, \bar{\boldsymbol{x}})$. □

Remark. It's possible to weaken a few of our assumptions and still obtain existence. In particular, we may (i) allow different men and women to have different sets of feasible agreements, (ii) allow for heterogeneous values to being single (or remove the option to be single), and (iii) replace continuity with upper-semicontinuity (or even weaker continuity assumptions). We may also generalize the agreement space to be a closed subset of a compact metric space. We cannot, however, easily dispense with the compactness of X. We discuss these issues in the Supplement.

Pareto Optimality

In this subsection, we prove the existence of stable allocations that are Pareto optimal. Such allocations are relevant for social welfare and have suggested as an alternative solution concept in two-sided matching games (e.g., Sotomayor [24]).

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *Pareto optimal* if there is no other feasible allocation $(\phi', \bar{\boldsymbol{x}}')$ such that (i) all players do weakly better in $(\phi', \bar{\boldsymbol{x}}')$ than $(\phi, \bar{\boldsymbol{x}})$, i.e., $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi, \bar{\boldsymbol{x}})$ for each player i, and (ii) at least one player does strictly better in $(\phi', \bar{\boldsymbol{x}}')$ than $(\phi, \bar{\boldsymbol{x}})$, i.e., $u_i(\phi', \bar{\boldsymbol{x}}') > u_i(\phi, \bar{\boldsymbol{x}})$ for some player i. If a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is Pareto optimal, we say it is a Pareto stable allocation.

We focus on "strong" Pareto optimality instead of "weak" Pareto optimality since (i) every stable allocation is weakly Pareto optimal and (ii) strong Pareto optimality is more intuitive. (Notice that Pareto optimal allocations won't generally maximize social welfare because payoffs aren't quasi-linear.)

Proposition 2. Existence of a Pareto Stable Allocation.
Let Assumption 1 hold, then a Pareto stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exists.

Proposition 2 gives, for instance, that there is a Pareto stable allocation in Examples 1 and 2, in the Marriage Game, and in the Assignment Game. We'll prove Proposition 2 by showing that the set of stable allocations is "compact" given Assumption 1. Thus, there is an allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ that maximizes total welfare on this set. We'll argue that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is Pareto optimal via contradiction. Indeed, if it weren't, then there would be another feasible allocation $(\phi', \bar{\boldsymbol{x}}')$ where all players do weakly better and one player does strictly better. But, then $(\phi', \bar{\boldsymbol{x}}')$ is stable. Hence, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ does not maximize total welfare on the set of stable allocations, a contradiction. We need three preliminary results before we make this argument precise.

Let $\mathcal{S}$ denote the set of stable allocations. We need to partition $\mathcal{S}$ into a collection of compact subsets. To these ends, for each $\phi \in \Phi$, let $\mathcal{S}(\phi) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | (\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}\}$ be the (possibly empty) set of agreement vectors that yield a stable allocation when paired with ϕ. Let $\Phi_{\mathcal{S}} = \{\phi \in \Phi | \, \mathcal{S}(\phi) \neq \emptyset\}$ be set of matchings which are part of a stable allocation. An allocation $(\phi, \bar{\boldsymbol{x}})$ is stable if and only if $\phi \in \Phi_{\mathcal{S}}$ and $\bar{\boldsymbol{x}} \in \mathcal{S}(\phi)$. We write $B_r(\bar{\boldsymbol{x}})$ for the open ball of radius r in $\mathbb{R}^{kN}$ around $\bar{\boldsymbol{x}}$, i.e., $B_r(\bar{\boldsymbol{x}}) = \{\bar{\boldsymbol{x}}' \in \mathbb{R}^{kN} \, | \, ||\bar{\boldsymbol{x}} - \bar{\boldsymbol{x}}'|| < r\}$, where $|| \cdot ||$ denotes the Euclidian norm.

Lemma 5. Compactness of $\mathcal{S}(\phi)$.
Let Assumption 1 hold, then $\mathcal{S}(\phi)$ is compact for each $\phi \in \Phi$.

Proof. We take $\mathcal{S}(\phi)$ to be nonempty, as every empty set is trivially compact. Since $\mathcal{S}(\phi) \subset X^N$, it is trivially bounded. Hence, we need only show that $\mathcal{S}(\phi)$ is closed. We prove this by establishing that the complement of $\mathcal{S}(\phi)$ is open.

Let $V(\phi) = \mathbb{R}^{kN} \backslash \mathcal{S}(\phi)$. For each $\bar{\boldsymbol{x}} \in V(\phi)$, we show that there is a $B_r(\bar{\boldsymbol{x}}) \subset V(\phi)$, for some $r > 0$. There are two cases, $\bar{\boldsymbol{x}} \in F(\phi)$ or $\bar{\boldsymbol{x}} \notin F(\phi)$. If $\bar{\boldsymbol{x}} \in F(\phi)$, then $(\phi, \bar{\boldsymbol{x}})$ is feasible and individually rational, but is not stable. Thus, $(\phi, \bar{\boldsymbol{x}})$ is blocked by some man and woman, i.e., $\bar{\boldsymbol{x}} \in D_\phi(m, w, \boldsymbol{x})$ for some $(m, w, \boldsymbol{x}) \in C$. Since $D_\phi(m, w, \boldsymbol{x})$ is open by Lemma 4, there is a $B_r(\bar{\boldsymbol{x}}) \subset D_\phi(m, w, \boldsymbol{x})$. Since $\bar{\boldsymbol{x}}' \in D_\phi(m, w, \boldsymbol{x})$ implies $(\phi, \bar{\boldsymbol{x}}')$ is not stable, we have that $B_r(\bar{\boldsymbol{x}}) \subset V(\phi)$. If $\bar{\boldsymbol{x}} \notin F(\phi)$, then there is a $B_r(\bar{\boldsymbol{x}}) \subset \mathbb{R}^{kN} \backslash F(\phi)$ for some $r > 0$ since $F(\phi)$ is compact by Lemma 3. Since $\mathcal{S}(\phi) \subset F(\phi)$, we have that $\mathbb{R}^{kN} \backslash F(\phi) \subset \mathbb{R}^{kN} \backslash \mathcal{S}(\phi) = V(\phi)$, implying $B_r(\bar{\boldsymbol{x}}) \subset V(\phi)$. □

The next lemma establishes that there is a stable allocation that maximizes total welfare on the set of stable allocations. Given a $(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN}$, let

$$T(\phi, \bar{\boldsymbol{x}}) = \Sigma_{i \in \mathcal{N}} u_i(\phi, \bar{\boldsymbol{x}})$$

be the total payoff to all players in $(\phi, \bar{\boldsymbol{x}})$.

Lemma 6. Total Welfare and Stable Allocations.
Let Assumption 1 hold, then there is a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *such that* $T(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq T(\phi', \bar{\boldsymbol{x}}')$ *for every* $(\phi', \bar{\boldsymbol{x}}') \in \mathcal{S}$.

Proof. By Proposition 1, we have that $\mathcal{S}$ is nonempty. Thus, $\Phi_{\mathcal{S}}$ is nonempty (and finite), as is $\mathcal{S}(\phi)$ for each $\phi \in \Phi_{\mathcal{S}}$. For each $\phi \in \Phi_{\mathcal{S}}$, we have the existence of an $\bar{\boldsymbol{x}}_\phi \in \mathcal{S}(\phi)$ that solves $\max_{\bar{\boldsymbol{x}} \in \mathcal{S}(\phi)} T(\phi, \bar{\boldsymbol{x}})$. Simply, the Extreme Value Theorem applies as $T(\phi, \bar{\boldsymbol{x}})$ is continuous in $\bar{\boldsymbol{x}}$ by Lemma 2, $\mathcal{S}(\phi)$ is nonempty, and $\mathcal{S}(\phi)$ is compact by Lemma 5. Let $I = \cup_{\phi \in \Phi_{\mathcal{S}}} \{(\phi, \bar{\boldsymbol{x}}_\phi)\}$ and let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ solve $\max_{(\phi, \bar{\boldsymbol{x}}) \in I} T(\phi, \bar{\boldsymbol{x}})$. A solution exists since I is finite.

We now establish that $T(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq T(\phi, \bar{\boldsymbol{x}})$ for all $(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}$. Suppose not, then there is a $(\phi', \bar{\boldsymbol{x}}') \in \mathcal{S}$ with $T(\phi', \bar{\boldsymbol{x}}') > T(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, $\mathcal{S}(\phi') \neq \emptyset$ and $(\phi, \bar{\boldsymbol{x}}_{\phi'}) \in I$. Since, $T(\phi', \bar{\boldsymbol{x}}_{\phi'}) \leq T(\phi^\star, \bar{\boldsymbol{x}}^\star)$, we have $T(\phi', \bar{\boldsymbol{x}}_{\phi'}) < T(\phi', \bar{\boldsymbol{x}}')$, a contradiction of the maximality of $\bar{\boldsymbol{x}}_{\phi'}$ on $\mathcal{S}(\phi')$. $\square$

Remark. It's clear from the proof that there is also a stable allocation that maximizes the total payoff of any group of players (e.g., the total payoff of women).

Let $(\phi, \bar{\boldsymbol{x}})$ be a feasible allocation and let

$$P(\phi, \bar{\boldsymbol{x}}) = \{(\phi', \bar{\boldsymbol{x}}') \in \mathcal{A} | \, \bar{\boldsymbol{x}}' \in X^N \text{ and } u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi, \bar{\boldsymbol{x}}) \text{ for all } i \in \mathcal{N}\}$$

be the set of feasible allocations that are at least as good as $(\phi, \bar{\boldsymbol{x}})$. This set is non-empty as $(\phi, \bar{\boldsymbol{x}}) \in P(\phi, \bar{\boldsymbol{x}})$.

Lemma 7. Two Useful Results.
*L*7.1 : *Let* $(\phi', \bar{\boldsymbol{x}}')$ *be a stable allocation, then every element of* $P(\phi', \bar{\boldsymbol{x}}')$ *is also a stable allocation.*
*L*7.2 : *Let* $(\phi', \bar{\boldsymbol{x}}')$ *be a stable allocation and let* $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *be a solution to*

$$\max_{(\phi, \bar{\boldsymbol{x}}) \in P(\phi', \bar{\boldsymbol{x}}')} T(\phi, \bar{\boldsymbol{x}}).$$

Then $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *is a Pareto stable allocation.*

Proof. This is almost obvious. We first establish $L7.1$. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi', \bar{\boldsymbol{x}}')$. By construction of $P(\cdot)$, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is feasible and individually rational, so we only need to show that it can't be blocked. We do this by contradiction. Suppose a man m and a woman w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ with agreement $\boldsymbol{x} \in X$, then we have $u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_m(\phi', \bar{\boldsymbol{x}}')$ and $u_w(m, \boldsymbol{x}) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_w(\phi', \bar{\boldsymbol{x}}')$ where the weak inequality follows from the definition of $P(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, m and w also block $(\phi', \bar{\boldsymbol{x}}')$, a contradiction.

Now we establish $L7.2$. Since $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi', \bar{\boldsymbol{x}}')$, we have $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable by $L7.1$. We only need to show that there is no other feasible allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ such that (i) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$ and (ii) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for some $i \in \mathcal{N}$. We do this by contradiction. Suppose there is such a $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$, then $T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > T(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ does not solve the maximization problem, a contradiction. $\square$

Proof of Proposition 2. By Lemma 6, there is a $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in \mathcal{S}$ with $T(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq T(\phi, \bar{\boldsymbol{x}})$ for all $(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}$. Since $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and since $P(\phi^\star, \bar{\boldsymbol{x}}^\star) \subset \mathcal{S}$ per Lemma 7, we have that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ solves

$$\max_{(\phi, \bar{\boldsymbol{x}}) \in P(\phi^\star, \bar{\boldsymbol{x}}^\star)} T(\phi, \bar{\boldsymbol{x}}).$$

Thus, Lemma 7 gives that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is a Pareto stable allocation. $\square$

Remark. It's possible to replace our assumption of continuity with upper-semicontinuity and still obtain the existence of a Pareto optimal stable allocation.

Interior Stable Allocations

In this subsection, we give conditions that ensure the existence of a stable allocation with interior agreements. We'll use these conditions to establish that the Effort Game has a solution.

For a set $S \subset \mathbb{R}^k$, we write ∂S for the boundary of S and $\text{int}(S)$ for the interior of S.

Definition. An allocation $(\phi, \bar{\boldsymbol{x}}) = (\phi, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^N)$ is *interior* if some player is partnered, i.e., $\phi(i) \neq i$ for some $i \in \mathcal{N}$, and all partnered players have agreements on the interior of X, i.e., $\phi(i) \neq i$ implies $\boldsymbol{x}^i \in \text{int}(X)$ for all $i \in \mathcal{N}$. If a (Pareto) stable allocation is interior, we say it is an interior (Pareto) stable allocation.

We establish the existence of interior stable allocations under the following assumption.

Assumption 2. Sufficient Conditions for an Interior Stable Allocation.
The set of feasible agreements X has nonempty interior and:
(i) There is a man m, a woman w, and an agreement $\boldsymbol{x} \in \text{int}(X)$ such that $u_m(w, \boldsymbol{x}) \geq 0$ and $u_w(m, \boldsymbol{x}) \geq 0$.
(ii) For each man m, each woman w, and each $\boldsymbol{x} \in \partial X$, we have at least one of the following:
(a) Either $u_m(w, \boldsymbol{x}) < 0$ or $u_w(m, \boldsymbol{x}) < 0$; or
(b) There is a $\boldsymbol{x}' \in \text{int}(X)$ such that $u_m(w, \boldsymbol{x}') > u_m(w, \boldsymbol{x})$ and $u_w(m, \boldsymbol{x}') > u_w(m, \boldsymbol{x})$.

Part (i) ensures that being partnered is not worse than being single for at least one man and one woman; we refer to this as **agreeability**. Part (ii) gives that each boundary agreement is either (a) very undesirable to the man or the woman and/or (b) payoff dominated by some other feasible agreement. If (a) holds for all men, all women, and all $\boldsymbol{x} \in \partial X$, then we say **intolerability** holds. If (b) holds for all men, all women, and all $\boldsymbol{x} \in \partial X$, then we say that **blockability** holds.

While intolerability and blockability are strong assumptions, they are often reasonable. In fact, we'll make heavy use of intolerability in the next section because it embeds the intuitive idea that a player is very unhappy when (i) he or she exerts extreme effort β or (ii) he or she exerts positive effort and his or her match does not; see the continuation of Example 1 below.

Proposition 3. Existence of an Interior Stable Allocation.
Let Assumptions 1 and 2 hold. Then, (i) an interior Pareto stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *exists and (ii) any stable allocation where some player is partnered is an interior stable allocation.*

Proposition 3 gives, for instance, that there is an interior Pareto stable allocation in Examples 1 and 2 since Assumptions 1 and 2 hold. It's easily seen that Assumption 1 hold in both examples, so we only need to verify that Assumption 2 holds. We do this next by showing that agreeability and intolerability/blockability hold.

Example 1 (Continued). Intolerability.
We want to verify that agreeability and intolerability hold. We have that agreeability holds since $u_1(4, \boldsymbol{x}) = {}^3\!/\!{}_8 > 0$ and $u_4(1, \boldsymbol{x}) = {}^3\!/\!{}_8 > 0$ when $\boldsymbol{x} = (1, 1)$.

Showing intolerability takes a bit more work: for each man m and each woman w, we need to walk the boundary of $[0, 2]^2$ and show that either m or w has a negative payoff. Focus on m and recall that $u_m(w, x_1, x_2) = x_2 - \frac{1}{\theta_m}(x_1)^2 - {}^1\!/\!{}_8$. For all $y \in [0, 2]$, we have that (i) $u_m(w, y, 0) = -{}^1\!/\!{}_8 - \frac{1}{\theta_m}y^2 < 0$, i.e., m gets a negative payout when he exerts effort and w does not, and (ii) $u_m(w, 2, y) = y - {}^1\!/\!{}_8 - \frac{1}{\theta_m}4 < 0$ since $\theta_w \in \{1, 2\}$, i.e., m gets a negative payout if he exerts extreme effort. By symmetry, we have (iii) $u_w(m, 0, y) < 0$ for and (iv) $u_w(m, y, 2) < 0$ for all $y \in [0, 2]$. Thus, for every $(x_1, x_2) \in \partial[0, 2]^2$, we've either $u_m(w, x_1, x_2) < 0$ or $u_w(m, x_1, x_2) < 0$. $\triangle$

Example 2 (Continued). Blockability.
We want to verify that this example satisfies agreeability and blockability. We have that agreeability holds since $u_m(w, x) = 0$ and $u_w(m, x) = 0$ for each man m and each woman w when $x = 0$. We also have that blockability holds as m and w's payoffs are strictly increasing at $x = 0$ and strictly decreasing at $x = 1$. $\triangle$

We'll prove Proposition 3 in two steps. First, we'll use a contradiction argument to establish that any stable allocation with a partnered player is an interior stable allocation. The key insight is that if a matched man and woman have a boundary agreement then Assumption 2 ensures either (i) the agreement isn't individually rational or (ii) they can block. In either case, we obtain the requisite contradiction. In the second step, we'll leverage this result and a construction argument to show the existence of an interior Pareto stable allocation.

Proof of Proposition 3. We begin by proving the that if $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ is a stable allocation where some player is partnered, then it's interior. We do this by showing that all partnered players have agreements on the interior of X, i.e., $\phi^\star(i) \neq i$ implies $\boldsymbol{x}^{i^\star} \in \text{int}(X)$ for all $i \in \mathcal{N}$.

Without loss, consider a man m with $\phi^\star(m) \neq m$; the argument is analogous for a woman. Let $w = \phi^\star(m)$. We argue by contradiction. If $\boldsymbol{x}^{m^\star} \in \partial X$, then, since Assumption 2 holds, we have: (a) $u_m(w, \boldsymbol{x}^{m^\star}) < 0$ or $u_w(m, \boldsymbol{x}^{m^\star}) < 0$ or (b) there is a $\boldsymbol{x}' \in X$ such that $u_m(w, \boldsymbol{x}') > u_m(w, \boldsymbol{x}^{m^\star})$ and $u_w(m, \boldsymbol{x}') > u_w(m, \boldsymbol{x}^{m^\star})$. If (a), we have that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is not individually rational, a contradiction. If (b), m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ with agreement $\boldsymbol{x}'$, another contradiction. It follows that $\boldsymbol{x}^{m^\star} \in \text{int}(X)$.

Now, we prove the existence of a Pareto interior stable allocation. By Proposition 2, there is a Pareto stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$. There are two cases (i) some man and woman are matched by $\phi^\star$ or (ii) no man and woman are matched by $\phi^\star$. If case (i), then $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior allocation by the above paragraph. Consequently, we proceed under (ii).

By Assumption 2, there is a man m, a woman w, and an agreement $\tilde{\boldsymbol{x}} \in \text{int}(X)$ such that m and w both get at least zero at agreement $\tilde{\boldsymbol{x}}$. We construct a candidate interior Pareto stable allocation $(\phi', \bar{\boldsymbol{x}}')$ from $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ by taking m and w, matching them, and assigning them agreement $\tilde{\boldsymbol{x}}$, while leaving everyone else's match and agreement alone. Formally, we construct $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \ldots, \boldsymbol{x}^{N'})$ from $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ as follows. First, set $\phi'(m) = w$, set $\phi'(w) = m$, set $\boldsymbol{x}^{m'} = \tilde{\boldsymbol{x}}$, and set $\boldsymbol{x}^{w'} = \tilde{\boldsymbol{x}}$. Second, set $\phi'(i) = \phi^\star(i)$ and set $\boldsymbol{x}^{i'} = \boldsymbol{x}^{i^\star}$ for all $i \in \mathcal{N} \backslash \{m, w\}$.

Clearly, $(\phi', \bar{\boldsymbol{x}}')$ is an interior allocation. Since $u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) = 0$ for all $i \in \mathcal{N}$, we've weakly increased everyone's payoff, i.e., $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$. Hence, Lemma 7 gives that $(\phi', \bar{\boldsymbol{x}}')$ is also a stable allocation. It remains to prove that $(\phi', \bar{\boldsymbol{x}}')$ is Pareto optimal. To these ends, suppose it were not. Then there is another feasible allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ such that (i) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) \geq u_i(\phi', \bar{\boldsymbol{x}}')$ for all $i \in \mathcal{N}$ and (ii) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > u_i(\phi', \bar{\boldsymbol{x}}')$ for some $i \in \mathcal{N}$. Since $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$, (i) and (ii) imply that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is not Pareto optimal, a contradiction. □

Interior Stable Allocations and Pareto Optimality

In this subsection, we give conditions that guarantee *every* interior stable allocation is Pareto optimal. We give these conditions to tie the last two subsections together and to obviate the welfare analysis of the Effort Game.

We write $B_r(\boldsymbol{x})$ for the open ball of radius r in $\mathbb{R}^k$ around $\boldsymbol{x}$, i.e., $B_r(\boldsymbol{x}) = \{\boldsymbol{x}' \in \mathbb{R}^k \mid ||\boldsymbol{x} - \boldsymbol{x}'|| < r\}$.

Definition. For a man m and a woman w, we say that $u_m(w, \boldsymbol{x})$ (or $u_w(m, \boldsymbol{x})$) is *locally* *nonsatiated in* $\boldsymbol{x}$ if, for each agreement $\boldsymbol{x} \in \mathbb{R}^k$, there is always a nearby agreement that m (w) strictly prefers, i.e., there is a $\boldsymbol{x}' \in B_r(\boldsymbol{x})$ such that $u_m(w, \boldsymbol{x}) < u_m(w, \boldsymbol{x}')$ for each $r > 0$ (a $\boldsymbol{x}'' \in B_r(\boldsymbol{x})$ such that $u_w(m, \boldsymbol{x}) < u_w(m, \boldsymbol{x}'')$ for each $r > 0$).

The next assumption gives our sufficient conditions.

Assumption 3. Sufficient Conditions for Interior Stable Allocations to be Pareto Optimal. The set of feasible agreements X has nonempty interior, intolerability holds, and, for each man m and each woman w, the payoffs $u_m(w', \boldsymbol{x})$ and $u_w(m', \boldsymbol{x})$ are continuous and locally nonsatiated in $\boldsymbol{x}$ for all women w' and for all men m' respectively.

Proposition 4. Interior Stable Allocations are Pareto Optimal.
Let Assumption 3 hold, then every interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *is Pareto optimal.*

Proposition 4 gives, for instance, that the stable allocation we found in Example 1 is Pareto optimal. Simply, for each man m and each woman w, we have that $u_m(w, x_1, x_2)$ and $u_w(m, x_1, x_2)$ are strictly increasing in either x_1 or x_2 and so are locally nonsatiated. We'll prove Proposition 4 by using a "bribery" argument. The idea is that local nonsatiation allows a man (woman) to bribe an indifferent woman (man) to form a blocking pair. Thus, if a stable allocation is not Pareto optimal, the player who does strictly better in the improving allocation bribes his or her match in the improving allocation into blocking the initial stable allocation, a contradiction.

Proof of Proposition 4. We argue by contradiction. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and suppose that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is not Pareto optimal. Then there is another feasible allocation $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \ldots, \boldsymbol{x}^{N'})$ with (i) $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$ and (ii) $u_i(\phi', \bar{\boldsymbol{x}}') > u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for some $i \in \mathcal{N}$. Suppose that man m is the player who does strictly better; this is without loss as an analogous argument applies when a woman does strictly better. We'll show that m and some woman block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. But first, we need two preliminary facts.

Fact one: m is matched to some woman under ϕ'. If not, then $u_m(\phi', \bar{\boldsymbol{x}}') = 0$. Since $u_m(\phi', \bar{\boldsymbol{x}}') > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$, we have $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$, a contradiction of the stability of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. In light of this, let $w = \phi'(m)$.

Fact two: $\boldsymbol{x}^{m'} \in \text{int}(X)$. If not, then $\boldsymbol{x}^{m'} \in \partial X$, so either $u_m(\phi', \bar{\boldsymbol{x}}') = u_m(w, \boldsymbol{x}^{m'}) < 0$ or $u_w(\phi', \bar{\boldsymbol{x}}') = u_w(m, \boldsymbol{x}^{m'}) < 0$ by Assumption 3. Since $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$, either $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$ or $u_w(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$, a contradiction as $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable.

We now establish that m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., we establish that there is an $\boldsymbol{x} \in X$

such that

$$u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \tag{3.1}$$

$$u_w(m, \boldsymbol{x}) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star). \tag{3.2}$$

Since (i) payoffs are continuous in $\boldsymbol{x}$ by Assumption 3 and (ii) $u_m(w, \boldsymbol{x}^{m'}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$, there is an $r > 0$ such that equation (3.1) holds for all $\boldsymbol{x} \in B_r(\boldsymbol{x}^{m'})$. Since $\boldsymbol{x}^{m'} \in \text{int}(X)$, we take r to be sufficiently small such that $B_r(\boldsymbol{x}^{m'}) \subset X$. Recall that $u_w(m, \boldsymbol{x}^{m'}) = u_w(\phi', \bar{\boldsymbol{x}}') \geq u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since $u_w(\cdot)$ is locally nonsatiated, there is an $\boldsymbol{x} \in B_r(\boldsymbol{x}^{m'})$ with $u_w(m, \boldsymbol{x}) > u_w(m, \boldsymbol{x}^{m'})$. Thus, equations (3.1) and (3.2) hold at $\boldsymbol{x}$, i.e., m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. □

Remark. We cannot meaningfully swap blockability for intolerability in Assumption 3. Simply, blockability and local nonsatiation are mutually exclusive conditions when X is compact and we usually take X to be compact.[13]

4 Description of the Effort Game

In this section, we describe the Effort Game and show that it has a solution.

Environment

Let $\Theta \subset \mathbb{R}_+$ be a finite set of types ordered in the usual way. We endow each player i with a type θ_i (e.g., innate ability) from Θ, and suppose that types are commonly known. Let $b : \mathbb{R} \to \mathbb{R}$ be a continuous and strictly increasing function, with $b(0) \geq 0$. We call b the "benefit" function because if a player exerts effort y, then he or she provides benefit $b(y)$ to his or her match. Let $c : \Theta \times \mathbb{R} \to \mathbb{R}$ be continuous in it's second argument, with $c(\theta, 0) \geq 0$ for all $\theta \in \Theta$. We call c the "cost" function because if a type θ player exerts effort y, he or she incurs cost $c(\theta, y)$. We suppose that, due to time and energy limitations, players can only exert efforts between 0 and β, where $0 < \beta < \infty$. Thus, the set of feasible agreements/efforts is $X = [0, \beta]^2$.

When a man m and a woman w are matched (to each other), their payoffs from agreement $(x_1, x_2) \in \mathbb{R}^2$ are

$$u_m(w, x_1, x_2) = b(x_2) - c(\theta_m, x_1) \text{ and } u_w(m, x_1, x_2) = b(x_1) - c(\theta_w, x_2).$$

[13] To see this, suppose local nonsatiation holds, blockability holds, payoffs are continuous in $\boldsymbol{x}$, and X is compact. Let m be a man, and let w be a woman. The Extreme Value Theorem gives that $\alpha = \arg\max_{\boldsymbol{x} \in X} u_m(w, \boldsymbol{x})$ is nonempty. Since blockability gives that every point in ∂X is payoff dominated by a point in $\text{int}(X)$, we have $\alpha \subset \text{int}(X)$. But then, for each $\boldsymbol{x} \in \alpha$, there is a $B_r(\boldsymbol{x}) \subset X$ for some $r > 0$, so local nonsatiation gives that there is a $\boldsymbol{x}' \in B_r(\boldsymbol{x})$ with $u_m(w, \boldsymbol{x}') > u_m(w, \boldsymbol{x})$. Thus, $\boldsymbol{x} \notin \alpha$, a contradiction.

Recall that single players get zero. As in Example 1, for $(x_1, x_2) \in X$, we think of x_1 and x_2 as the man's and the woman's efforts respectively. Thus, m and w exert effort to produce a benefit for each other.

It will usually be the case that a player's effort diminishes his or her payoff, i.e., $c(\theta, y)$ is positive and increasing in y. However, sometimes a player's effort provides him or her with a "personal" benefit, e.g., in cooking a delicious dinner for his wife, a husband also cooks himself a delicious dinner. We capture this personal benefit by allowing $c(\theta, y)$ to be non-monotone and so be negative for certain efforts.

Remark. While all players prefer matches who produce higher benefits, they *do not* have a common preference over matches and agreements.[14]

The benefit function is homogenous for simplicity. Our results, save for Proposition 9, continue to hold when the benefit a player produces depends on his or her sex/side. This observation is important because it allows us to extend our core results to environments where one side, say clients, pays the other side, say lawyers, for a service, e.g., a vigorous defense. We'd model this environment by letting the client's benefit and cost functions be linear in effort/money, e.g., $b_c(y) = y$ and $c_c(\theta, y) = y$, while letting the lawyers' benefit and cost functions be non-linear, e.g., $b_l(y) = y^2$ and $c_l(\theta, y) = y/\theta$. (Of course, with these particular payoffs, our results are only interesting for the lawyers.) Additionally, Proposition 5 continues to hold when (i) the benefit a player produces depends on his or her identity/type or (ii) when payoffs aren't additively separable, but are continuous, strictly increasing in benefit, and strictly decreasing in cost.

Interior Stable Allocations

We focus on interior stable allocations since we maintain the following natural assumptions throughout the remainder of the paper.

Assumption 4. Agreeability and Intolerability.
The benefit and cost functions, as well as players' endowed types are such that agreeability and intolerability hold.

This assumption embeds the intuitive idea that a player is very unhappy when (i) he or she exerts extreme effort β or (ii) he or she exerts positive effort and his or her match does not. The next corollary motivates our focus on interior stable allocations.

[14] To illustrate, consider two men m and m' who have a choice between (a) woman w and agreement (x_1, x_2) or (b) woman w' and agreement (x_1, x_2'). If it's the case that $x_2' > x_2$, then both men prefer (b) to (a) because (i) they incur the same cost in both (a) and (b) and (ii) woman w' generates a strictly larger benefit than woman w. However, if we replace the agreement in option (b) with agreement (x_1', x_2'), then the men may have different preferences over (a) and (b) even though $x_2' > x_2$. For instance, when $(x_1, x_2) = (1/4, 9/10)$, $(x_1', x_2') = (1/2, 1)$, $b(y) = y$, $c(\theta_m, y) = y^2 - y + \frac{1}{2}$, $c(\theta_{m'}, y) = (2y)^2 - y + \frac{1}{4}$, a bit of algebra shows that m strictly prefers (b) to (a), while m' strictly prefers (a) to (b).

Corollary 1. Existence and Pareto Optimality of Interior Stable Allocations.
There is an interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$*, all interior stable allocations are Pareto optimal, and any stable allocation where a player is partnered is an interior stable allocation.*

Proof. Since $b(y)$ and $c(\theta, y)$ are continuous, $b(y)$ is strictly increasing in y (and, thus, locally nonsatiated), X is compact with nonempty interior, and agreeability and intolerability hold, both Propositions 3 and 4 apply. □

Remark. The class of benefit functions, cost functions, and endowed types satisfying Assumption 4 is nonempty – e.g., Example 1, where $b(y) = y$ and $c(\theta, y) = {}^{y^2}\!/\!_{\theta} + {}^{1}\!/\!_{8}$, as well as Examples 4 and 5 below. Other examples may be readily constructed.

5 Results for the Effort Game

In this section, we state and prove our results for the Effort Game. We defer the proofs of the major results until the end of each subsection in order to discuss the results.

Benefits, Efforts, and Matches

Let $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ be an interior stable allocation. Let $z_i^\star$ denote player i's effort in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., $z_i^\star$ is the first component of $\boldsymbol{x}^{i^\star} = (x_1^{i^\star}, x_2^{i^\star})$ when i is a man and $z_i^\star$ is the second component of $\boldsymbol{x}^{i^\star}$ when i is a woman. Consider players who are partnered by $\phi^\star$. Each partnered player i produces a *strictly* positive benefit $b_i^\star = b(z_i^\star)$.[15] We rank partnered players by the benefits they produce from greatest to least and place them into groups of equal benefit.[16] For the men, we label these groups $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$, where $G_1^{\mathbb{M}}$ contains the men who produce the highest benefit, $G_2^{\mathbb{M}}$ contains the men who produce the second highest benefit, and so on. For the women, we label the analogous groups $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{W}}}^{\mathbb{W}}$. We have that $J_{\mathbb{M}}$ and $J_{\mathbb{W}}$ are positive and finite, since at least one man and woman are partnered and since $\mathcal{M}$ and $\mathcal{W}$ are finite. To simplify notation, we suppress the dependence of $\{z_i^\star\}_{i\in\mathcal{N}}$, $\{b_i^\star\}_{i\in\{j\in\mathcal{N}|\phi^\star(j)\neq j\}}$, $\{G_l^{\mathbb{M}}\}_{l=1}^{J_{\mathbb{M}}}$, $\{G_l^{\mathbb{W}}\}_{l=1}^{J_{\mathbb{W}}}$, $J_{\mathbb{M}}$, and $J_{\mathbb{W}}$ on $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. The next proposition shows how players match across benefit groups.

Proposition 5. Benefits and Matches.
Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *be an interior stable allocation and let* $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ *and* $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{W}}}^{\mathbb{W}}$ *be the associated benefit groups. Then, (i) there are an equal number of male and female benefit groups, i.e.,* $J_{\mathbb{M}} = J_{\mathbb{W}}$*, and (ii) a man* m *is in the* l*-th benefit group of men if and only if his*

[15]That $b_i^\star > 0$ follows from (i) $b(0) \geq 0$, (ii) $b(y)$ is strictly increasing in y, and (iii) $z_i^\star > 0$ since the allocation is interior.
[16]We omit single players because their efforts produce benefits for no one.

match $\phi^\star(m)$ is in the l-th benefit group of women, i.e., $m \in G_l^{\mathbb{M}} \iff \phi^\star(m) \in G_l^{\mathbb{W}}$ for all $l \in \{1, \ldots, J_{\mathbb{M}}\}$. An analogous result holds for women.

Proposition 5 gives that, in any interior stable allocation, matching is **assortative in benefit**, i.e., only players from benefit groups with the same index match with each other. An implication is that players match on the basis of how happy they make each other – those who make their partners happiest match, followed by those who make their partners less happy, and so on. We can see this in the stable allocation we found in Example 1.

Example 1 (Continued). Assortative in Benefit.

In the stable allocation we found, man 1 and woman 4 each exert effort 1, while man 2 and woman 3 each exert effort $1/2$. It follows that, $b_1^\star = b_4^\star = 1$ and $b_2^\star = b_3^\star = \frac{1}{2}$. Thus, there are two (singleton) benefit groups on each side: $G_1^{\mathbb{M}} = \{1\}$, $G_2^{\mathbb{M}} = \{2\}$, $G_1^{\mathbb{W}} = \{4\}$, and $G_2^{\mathbb{W}} = \{3\}$. Since man 1 is matched to woman 4 and man 2 is matched to woman 3, we have that matching is assortative in benefit. $\triangle$

The intuition behind Proposition 5 is that players "compete" for the best possible match. To illustrate, suppose that there is an interior stable allocation where two partnered men m and m' both produce the same benefit, but m' has match w' who produces more benefit than the match of m. Then, m can increase his effort by an arbitrarily small amount and give w' a higher benefit than she is currently receiving. Since w' desires the highest benefit possible, she'll agree to match with m instead of m'. Man m is willing to increase his effort slightly because, in exchange for an arbitrarily small change in cost, he gains a partner with strictly higher benefit, and so does strictly better. Thus, m and w' block. Hence, a necessary condition of stability is that all men of the same benefit are matched to women of the same benefit.[17]

Since the benefit function is strictly increasing, players in higher benefit groups exert more effort than those in lower benefit groups. In particular, men in $G_1^{\mathbb{M}}$ exert the most effort among men, men in $G_2^{\mathbb{M}}$ exert the second most effort among men, and, in general, men in $G_l^{\mathbb{M}}$ exert the l-th most effort among men. Analogously, women in $G_l^{\mathbb{M}}$ exert the l-th most effort among women. Thus, a corollary of Proposition 5 is the following.

Corollary 2. Efforts and Matches.

Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation. Then, a man m exerts the l-th most effort among men if and only if his match $\phi^\star(m)$ exerts the l-th most effort among women for all $l \in \{1, \ldots, J_{\mathbb{M}}\}$. An analogous result holds for women.

[17] While Proposition 5 doesn't hold when X is discrete, a related result does hold: in any stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, the match of any man in $G_j^{\mathbb{M}}$ produces a weakly higher benefit than the match of any man in $G_{j+1}^{\mathbb{M}}$; likewise for women. That is, matching is "weakly" assortative in benefit.

Proof. Obvious and omitted. □

The corollary gives that, in any interior stable allocation, the hardest working players match, as do the second hardest working players, and so forth. We can see this in the stable allocations we found in Example 1: man 1 and woman 4 are matched and they both exert effort 1, while man 2 and woman 3 are matched and they both exert effort $1/2$.

We'll prove Proposition 5 by induction. First, we'll use a competition argument to establish that men in the first benefit group match with women in the first benefit group and vice-versa. Subsequently, we'll use induction to show that the analogous result holds for the second benefit groups, the third benefit groups, and so on.

Proof of Proposition 5. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{W}}}^{\mathbb{W}}$ be the associated benefit groups, let $\{z_i^\star\}_{i\in\mathcal{N}}$ be the associated efforts, and let $\{b_i^\star\}_{i\in\{j|\phi^\star(j)\neq j\}}$ be the associated benefits. Recall that $b_i^\star > 0$ for each partnered player.

Let $\underline{J} = \min\{J_{\mathbb{M}}, J_{\mathbb{W}}\}$ and $\overline{J} = \max\{J_{\mathbb{M}}, J_{\mathbb{W}}\}$. Let $j \leq \underline{J}$. We'll show that $m \in G_l^{\mathbb{M}} \iff \phi^\star(m) \in G_l^{\mathbb{W}}$ for all $l \in \{1, \ldots, j\}$. We refer to this as the "induction" result since we prove it by induction on j: we first establish that it holds when $j = 1$; subsequently, we show that if it holds at $j - 1$, then it also holds at j when $j > 1$.

The first part of the proposition follows from the induction result. Simply, if $\underline{J} = J_{\mathbb{M}} < J_{\mathbb{W}} = \overline{J}$, then the women in group $J_{\mathbb{M}} + 1$ are matched to men *not* in groups 1 to $J_{\mathbb{M}}$, a contradiction as these groups contain all partnered men. Analogously, $\underline{J} = J_{\mathbb{W}} < J_{\mathbb{M}} = \overline{J}$ leads to a contradiction. Thus, we have $\underline{J} = J_{\mathbb{M}} = J_{\mathbb{W}} = \overline{J}$. The second part of the proposition also follows from the induction result, take $j = J_{\mathbb{M}}$.

Let $j = 1$. There are two cases: (i) $\overline{J} = 1$ and (ii) $\overline{J} > 1$. If case (i), then we have $\underline{J} = 1$ and the induction result is trivially true. Thus, we proceed under case (ii).

We establish that $m \in G_1^{\mathbb{M}}$ implies $\phi^\star(m) \in G_1^{\mathbb{W}}$. We do this by contradiction. Let $m \in G_1^{\mathbb{M}}$ and suppose that $w = \phi^\star(m)$ is in $G_{l'}^{\mathbb{W}}$ with $l' > 1$. Let w' be an arbitrary woman in $G_1^{\mathbb{W}}$ and let $m' = \phi^\star(w')$. We have (a) that m' may or may not be in $G_1^{\mathbb{M}}$ and (b) that $b_{w'}^\star - b_w^\star > 0$. We'll establish that m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Man m and woman w' block with agreement $(x_1, z_{w'}^\star)$ if there is an $x_1 \in [0, \beta]$ such that

$$u_{w'}(m, x_1, z_{w'}^\star) > u_{w'}(\phi^\star, \bar{\boldsymbol{x}}^\star) \tag{5.1}$$

$$u_m(w', x_1, z_{w'}^\star) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star). \tag{5.2}$$

Since $z_{w'}^\star$ is the same on both sides of equation (5.1), the equation reduces to $b(x_1) > b_{m'}^\star$. Since w' gives m benefit $b_{w'}^\star$ at $(x_1, z_{w'}^\star)$, equation (5.2) reduces to $b_{w'}^\star - b_w^\star > c(\theta_m, x_1) -$

$c(\theta_m, z^\star_m)$. To summarize, m and w' block if there is an x_1 in $[0, \beta]$ such that

$$b(x_1) > b^\star_{m'} \tag{5.3}$$

$$b^\star_{w'} - b^\star_w > c(\theta_m, x_1) - c(\theta_m, z^\star_m). \tag{5.4}$$

If $m' \notin G^{\mathbb{M}}_1$, then $b(z^\star_m) > b^\star_{m'}$. Thus, $x_1 = z^\star_m$ satisfies equations (5.3) and (5.4) as $b^\star_{w'} - b^\star_w > 0$. Since $z^\star_m \in (0, \beta)$, m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

If $m' \in G^{\mathbb{M}}_1$, consider $x_1 = z^\star_m + \delta$, with $\delta > 0$. Since $b(z^\star_m) = b^\star_{m'}$ and $b(y)$ is strictly increasing in y, we have that $z^\star_m + \delta$ satisfies equation (5.3). Since $c(\theta, y)$ is continuous in y and $b^\star_{w'} - b^\star_w > 0$, we also have that $z^\star_m + \delta$ satisfies equation (5.4) for δ sufficiently small. Since $z^\star_m \in (0, \beta)$, we can shrink δ such that $z^\star_m + \delta \in [0, \beta]$. Thus, m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

An analogous argument gives that $w \in G^{\mathbb{M}}_1$ implies $\phi^\star(w) \in G^{\mathbb{M}}_1$. Since $\phi^\star(m) = w$ if and only if $\phi^\star(w) = m$, we have $m \in G^{\mathbb{M}}_l \iff \phi^\star(m) \in G^{\mathbb{W}}_l$ for all $l \in \{1\}$, i.e., the induction result is true when $j = 1$.

Let $1 < j \leq \underline{J}$ and assume that $m \in G^{\mathbb{M}}_l \iff \phi^\star(m) \in G^{\mathbb{W}}_l$ for all $l \in \{1, \ldots, j-1\}$. By the induction hypothesis, it suffices to show that $m \in G^{\mathbb{M}}_j \iff \phi^\star(m) \in G^{\mathbb{W}}_j$ to prove the induction result. Again, there are two cases: (i) $j = \overline{J}$ or (ii) $j < \overline{J}$. If case (i), then we're done. Simply, when $m \in G^{\mathbb{M}}_j$, the induction hypothesis gives he is *not* matched to a woman in a lower-indexed group. Since there is no higher index group of women, we necessarily have that $\phi^\star(m) \in G^{\mathbb{W}}_j$. Analogously, if $w \in G^{\mathbb{W}}_j$, then $\phi^\star(m) \in G^{\mathbb{M}}_j$. It follows that $m \in G^{\mathbb{M}}_j \iff \phi^\star(m) \in G^{\mathbb{W}}_j$. Thus, we proceed under case (ii).

We establish that $m \in G^{\mathbb{M}}_j$ implies $\phi^\star(m) \in G^{\mathbb{W}}_j$. As before, we do this by contradiction. Let $m \in G^{\mathbb{M}}_j$ and suppose that $w = \phi^\star(m)$ is in $G^{w}_{l'}$ with $l' \neq j$. The induction hypothesis gives that $l' > j$. Let w' be an arbitrary woman in $G^{\mathbb{W}}_j$ and let $m' = \phi^\star(w')$. Again, m' may or may not be in $G^{\mathbb{M}}_j$; but, the induction hypothesis gives that $m' \in G^{\mathbb{M}}_{l'}$ with $l' \geq j$. We'll (again) establish that m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Repeating the same argument as above gives that m and w' block with agreement $(x_1, z^\star_{w'})$ if there is an $x_1 \in [0, \beta]$ such that

$$b(x_1) > b^\star_{m'} \tag{5.5}$$

$$b^\star_{w'} - b^\star_w > c(\theta_m, x_1) - c(\theta_m, z^\star_m). \tag{5.6}$$

If $m' \notin G^{\mathbb{M}}_j$, then $x_1 = z^\star_m$ satisfies equations (5.5) and (5.6) is in $[0, \beta]$. If $m' \in G^{\mathbb{M}}_j$, consider $x_1 = z^\star_m + \delta$, with $\delta > 0$. Since m and m' are in the same benefit group and $b(y)$ is strictly increasing, we have that $z^\star_m + \delta$ satisfies equation (5.5). Since $c(\theta, y)$ is continuous in y and

$b^\star_{w'} - b^\star_w > 0$, we have that equation (5.6) holds for δ sufficiently small. Since $z^\star_m \in (0, \beta)$, we can shrink δ such that $z^\star_m + \delta \in [0, \beta]$. In either case, m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

An analogous argument gives that $w \in G^{\mathbb{W}}_j$ implies $\phi^\star(w) \in G^{\mathbb{M}}_j$. It follows that $m \in G^{\mathbb{M}}_j \iff \phi^\star(m) \in G^{\mathbb{W}}_j$. □

Remark. Since the proof didn't use the fact that the benefit function is the same for all players – it only used the fact that m and m' are in the same benefit group in the blocking step – the proposition continues to hold when the benefit a player produces depends on his or her identity/type and effort.

Remark. The proposition also holds when the benefit function is non-monotone and the cost function is strictly increasing. The proof goes through if we modify the blocking step by keeping the effort of man m constant and slightly decreasing the effort of woman w'. Then w' gains from a reduction in cost and m gains from a higher benefit match. However, the proposition doesn't hold if both the benefit and cost functions are non-monotone since then a small change in effort may not make a make a player strictly better off.

Remark. Additionally, the proposition holds when payoffs are continuous, strictly increasing in the benefit each player receives, strictly decreasing in the cost each player incurs, and are *not* additively separable. The proof goes through because, in the blocking step, w' still does strictly better from the additional effort m exerts and m still does strictly better since he gains a discrete jump in his payoff.

Benefits, Efforts, Matches, and Types

To relate types to benefits and efforts, we make the following assumption.

Assumption 5. Cost and Type.
Higher type players have lower incremental costs of effort, i.e., $c(\theta, y') - c(\theta, y)$ is strictly decreasing in θ for all $y < y'$.

That is, $c(\cdot)$ is submodular. This is the case, for instance, when the marginal cost of effort is strictly decreasing in type. The next proposition shows how benefit and type are related.

Proposition 6. Benefits and Types.
Let Assumption 5 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $G^{\mathbb{M}}_1, \ldots, G^{\mathbb{M}}_{J_{\mathbb{M}}}$ and $G^{\mathbb{W}}_1, \ldots, G^{\mathbb{W}}_{J_{\mathbb{M}}}$ be the associated benefit groups. Then, higher type men produce higher benefits, i.e., if two men m and m' are partnered, then $\theta_m < \theta_{m'}$ implies $m \in G^{\mathbb{M}}_l$ and $m' \in G^{\mathbb{M}}_j$ with $j \leq l$. An analogous result holds for women.

Notice that this result applies to every interior stable allocation. The intuition is that higher types can "outcompete" lower types because their lower marginal costs allow them to profitably offer slightly higher benefits. Thus, the competition for matches drives them to offer higher benefits. To sketch the argument, suppose there is an interior stable allocation where a lower type man m produces a greater benefit than a higher type man m', i.e., where $b^\star_{m'} < b^\star_m$. Let $z^\star_m$ and $z^\star_{m'}$ be the efforts of m and m'. Also, let w be the match of m, let w' be the match of m', let $z^\star_w$ and $z^\star_{w'}$ be the efforts of w and w', and let $b^\star_w$ and $b^\star_{w'}$ be their benefits. Since the benefit function is the same for both men, w' does strictly better with m whenever he exerts slightly more effort than m'. Thus, stability requires that m doesn't do better w' at such an effort, implying $b^\star_w - c(\theta_m, z^\star_m) \geq b^\star_{w'} - c(\theta_m, z^\star_{m'})$. Since m' has a strictly lower marginal cost than m and since $z^\star_{m'} < z^\star_m$, we have $b^\star_w - b^\star_{w'} > c(\theta_{m'}, z^\star_m) - c(\theta_{m'}, z^\star_{m'})$. That is, m' does strictly better by offering w agreement $(z^\star_m, z^\star_w)$ then he does in the stable allocation. Hence, he's willing to outcompete m by offering a slightly higher benefit to win w, implying that m' and w block. It follows that a necessary condition of stability is that higher type men produce higher benefits.

An implication of the proposition is that players in higher benefit groups have (weakly) higher types. In particular, for the men, the contraposition of Proposition 6 implies that

$$\min\{\theta_m | m \in G^{\mathbb{M}}_l\} \geq \max\{\theta_m | m \in G^{\mathbb{M}}_{l+1}\},$$

for any $l \in \{1, \ldots, J_{\mathbb{M}} - 1\}$. That is, the lowest type man in the l-th benefit group has a type at least as high as the highest type man in the $l+1$ benefit group. An analogous result holds for women.

Two corollaries of Proposition 6 follow. The first concerns the relationship between a type and effort, and the second concerns the relationship between type and the benefit produced by a player's match.

Corollary 3. Efforts and Types.
Let Assumption 5 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $\{z^\star_i\}_{i\in\mathcal{N}}$ be the associated efforts. Then, higher type men exert more effort, i.e., if two men m and m' are partnered, then $\theta_m < \theta_{m'}$ implies that $z^\star_m \leq z^\star_{m'}$. An analogous result holds for women.

Proof. This result follows directly from Proposition 6 as $b(y)$ is strictly increasing. □

Corollary 4. Benefits, Matches, and Types.
Let Assumption 5 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $G^{\mathbb{M}}_1, \ldots, G^{\mathbb{M}}_{J_{\mathbb{M}}}$ and $G^{\mathbb{W}}_1, \ldots, G^{\mathbb{W}}_{J_{\mathbb{M}}}$ be the associated benefit groups. Then, higher type men are matched to women in higher benefit groups, i.e., if two men m and m' are partnered, then $\theta_m < \theta_{m'}$ implies $\phi^\star(m) \in G^{\mathbb{W}}_l$ and $\phi^\star(m') \in G^{\mathbb{W}}_j$ with $j \leq l$. An analogous result holds for women.

Proof. Apply Proposition 5 to Proposition 6. □

Corollaries 3 and 4 show that, in any interior stable allocation, players with lower marginal costs exert higher efforts and receive higher benefits than players with higher marginal costs. Since Example 1 satisfies Assumption 5 (as the marginal cost $2y/\theta$ is strictly decreasing in θ), Proposition 6 and Corollaries 3 and 4 give that higher types are in higher benefit groups, exert more effort, and obtain matches who produce larger benefits. In fact, we can see this in the stable allocation we found in Example 1.

Example 1 (Continued). Benefits, Efforts, Matches, and Types.

Recall that man 1 and woman 4 are the high types, while man 2 and woman 3 are the low types. First, since $G_1^{\mathbb{M}} = \{1\}$, $G_2^{\mathbb{M}} = \{2\}$, $G_1^{\mathbb{W}} = \{4\}$, and $G_2^{\mathbb{W}} = \{3\}$, we have that higher types are in higher benefit groups. Second, since man 1 and woman 4 exert effort 1, while man 2 and woman 3 exert effort $1/2$, we have that higher types exert more effort. Third, since $b_1^\star = b_4^\star = 1$, $b_2^\star = b_3^\star = \frac{1}{2}$, man 1 and woman 4 are matched, and man 2 and woman 3 are matched, we have that higher type players get matches who produce higher benefits. △

We'll prove Proposition 6 by formalizing the intuition given above and by leveraging the following (almost obvious) lemma.

Lemma 8. A Useful Property of Continuous Functions.
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $f(y) \leq v$ for all $y \in (s, t]$, with $s < t$. Then $f(s) \leq v$.

Proof. This is almost obvious. Let $\{y_j\}_{j=1}^{\infty}$ be a sequence converging to s with $y_j \in (s, t]$. Then $f(s) = \lim_{j\to\infty} f(y_j) \leq \lim_{j\to\infty} v = v$ where the inequality follows from the bound. □

Proof of Proposition 6. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{W}}$ be the associated benefit groups, let $\{z_i^\star\}_{i\in\mathcal{N}}$ be the associated efforts, and let $\{b_i^\star\}_{i\in\{j|\phi^\star(j)\neq j\}}$ be the associated benefits. Let m and m' be two partnered men with $\theta_m < \theta_{m'}$. Then, $m \in G_l^{\mathbb{M}}$ for some l and $m' \in G_j^{\mathbb{M}}$ for some j. We need to establish that $j \leq l$. If $l = J_{\mathbb{M}}$, then this is trivially true, so we take $l < J_{\mathbb{M}}$. We'll establish the result by contradiction. Suppose that $l < j$, i.e., $b_m^\star > b_{m'}^\star$, which implies $z_m^\star > z_{m'}^\star$. Let $w = \phi^\star(m)$ and $w' = \phi^\star(m')$. Proposition 5 gives that $b_w^\star > b_{w'}^\star$, which implies $z_w^\star > z_{w'}^\star$. We'll show that m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. But first, we need two preliminary facts.

Fact one: m does weakly better in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ than he does by matching with w' at agreement $(z_{m'}^\star, z_{w'}^\star)$, i.e.,

$$b_w^\star - b_{w'}^\star \geq c(\theta_m, z_m^\star) - c(\theta_m, z_{m'}^\star). \tag{5.7}$$

Consider a possible match of m and w' with agreement $(x_1, z_{w'}^\star)$, where $x_1 \in [0, \beta]$. For $x_1 > z_{m'}^\star$, we have that $u_{w'}(m, x_1, z_{w'}^\star) > u_{w'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$ since $b(x_1) > b_{m'}^\star = b(z_{m'}^\star)$. Thus, the stability of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ implies that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_m(w', x_1, z_{w'}^\star)$. Simplifying gives that $b_w^\star - b_{w'}^\star \geq c(\theta_m, z_m^\star) - c(\theta_m, x_1)$ for $x_1 > z_{m'}^\star$. Since $c(\theta, y)$ is continuous in y, the desired result follows from Lemma 8: take $f(y) = -c(\theta_m, y)$, take $s = z^\star_m$, take $t = \beta$, and take $v = b^\star_w - b^\star_{w'} - c(\theta_m, z^\star_m)$.

Fact two: m' does strictly better by matching with w at agreement $(z^\star_m, z^\star_w)$ than he does in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e.,

$$b^\star_w - b^\star_{w'} > c(\theta_{m'}, z^\star_m) - c(\theta_{m'}, z^\star_{m'}). \tag{5.8}$$

Since $z^\star_{m'} < z^\star_m$, Assumption 5 gives that

$$c(\theta_{m'}, z^\star_m) - c(\theta_{m'}, z^\star_{m'}) < c(\theta_m, z^\star_m) - c(\theta_m, z^\star_{m'}).$$

The desired result follows from applying this to equation (5.7).

Now we can show that m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. For the pair to block with agreement $(x_1, z^\star_w)$, we need $u_w(m', x_1, z^\star_w) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{m'}(w, x_1, z^\star_w) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$, with $x_1 \in [0, \beta]$. Simplifying gives that m' and w block if there is an $x_1 \in [0, \beta]$ such that

$$b(x_1) > b(z^\star_m) \tag{5.9}$$

$$b^\star_w - b^\star_{w'} > c(\theta_{m'}, x_1) - c(\theta_{m'}, z^\star_{m'}). \tag{5.10}$$

Consider $x_1 = z^\star_m + \delta$, where $\delta > 0$. Since $b(y)$ is strictly increasing, we have that $z^\star_m + \delta$ satisfies equation (5.9). Since equation (5.8) holds, the cost function is continuous in y, and $b^\star_w - b^\star_{w'} > 0$, we can pick δ sufficiently small such that $z^\star_m + \delta$ satisfies (5.10). Since $z^\star_m$ is interior, we can make δ small enough so that $z^\star_m + \delta \in [0, \beta]$. Thus, m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. □

Remark. Since we never used the fact that men and women have the same benefit function, the proposition generalizes to the case where each side has its own benefit function. For the exact same reason, Propositions 7 and 8 also continue to hold when each side has its own benefit function.

Payoffs and Types

Corollary 4 tells us that higher types receive higher benefits from their matches. At the same time, Corollary 3 tells us higher types exert more effort, implying they may incur higher costs. Thus, it's unclear whether higher types actually do better than lower types. To resolve this quandary, it suffices to make the following assumption.

Assumption 6. Common Cost of Minimum Effort.
There is a common cost of minimum effort, i.e., $c(\theta, 0) = d$ for all types θ, where $d \geq 0$.[18]

[18]Strictly speaking, Assumption 4 requires that d is greater than $b(0)$: suppose that man m is matched to woman w and at agreement $(0, 0)$, then $u_m(w, 0, 0) = b(0) - d$, so intolerability requires that $b(0) - d < 0$. That said, the exact size of d is immaterial to what we do next.

An immediate implication of this assumption is the following.

Lemma 9. Higher Types Have Lower Costs.
When Assumptions 5 and 6 hold, cost is strictly decreasing in type, i.e., $c(\theta, y) > c(\theta', y)$ *when* $\theta < \theta'$ *and* $y > 0$*.*

Proof. This is almost obvious. By Assumption 5, we have $c(\theta', y) - c(\theta', 0) < c(\theta, y) - c(\theta, 0)$ for all $y > 0$. Plugging in d for $c(\theta, 0)$ and $c(\theta', 0)$, and canceling gives $c(\theta, y) > c(\theta', y)$. □

This result is plays a key role in obtaining the next proposition, which examines the relationship between type and payoff.

Proposition 7. Payoffs and Types.
Let Assumptions 5 and 6 hold, and let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *be an interior stable allocation. Then, higher type men obtain higher payoffs, i.e., for two men* m *and* m'*,* $\theta_m < \theta_{m'}$ *implies that* $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \leq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$*. Additionally, men of the same type have equal payoffs, i.e.,* $\theta_m = \theta_{m'}$ *implies that* $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$*. Analogous results hold for women.*

Notice that this result applies to any interior stable allocation and that this proposition doesn't require that m and m' be partnered. The intuition is that higher types can "imitate" and outcompete lower types whenever lower types do strictly better. To elaborate, suppose there is an interior stable allocation where a lower type man m makes strictly more than a higher type man m'. Then, m' can do strictly better by offering w, the match of m, a slightly higher benefit than she's currently receiving. This is because m' has a strictly lower cost than m, so he makes strictly more than m when he receives the same benefit as m and exerts the same effort as m. Thus, m' is willing to outcompete m by offering a slightly higher benefit to win w, implying that m' and w block. It follows that a necessary condition of stability is that higher partnered types make at least as much as lower partnered types. An analogous imitation argument establishes that men with the same type have equal payoffs. The proof formalizes these arguments.

The next two examples illustrate the first and second parts of Proposition 7 respectively.

Example 1 (Continued). Payoffs and Types.

Since inspection shows that Assumption 6 holds, the first part of Proposition 7 gives that higher types earn more. Indeed, we see this in the stable allocation we previously found. The high types, man 1 and woman 4, each earn $3/8$ $(= 1 - 1/8 - 1/2 \cdot 1^2)$, while the low types, man 2 and woman 3, each earn $1/8$ $(= 1/2 - 1/8 - (1/2)^2)$. △

Example 4. Payoffs and Types for a Variant of Example 1.

Suppose $X = [0, 2]^2$. Let $\mathcal{M} = \{1, 2\}$ and let $\mathcal{W} = \{3, 4\}$. Let $b(y) = y$ and let $c(\theta, y) = \frac{1}{\theta} y^2$. Let $\theta_1 = \theta_2 = \theta_4 = 2$ and let $\theta_3 = 1/2$. It's readily verified that one stable allocation is for (i) man 1 and woman 3 to match with agreement ($^1/_2, ^1/_2$) and for (ii) man 2 and woman 4 to match with agreement $(1.0634, 0.9404)$. Since inspection shows that Assumptions 5 and 6 hold, the second part of Proposition 7 gives that both men earn the same in this stable allocation. This prediction is born out: a bit of computation gives that both men earn 0.375, even though they're in different benefit groups. △

We'll prove Proposition 7 in two steps. First, we'll give a lemma that establishes higher types are partnered whenever lower types are partnered. Subsequently, we'll use this lemma and the argument we sketched above to prove the proposition.

Lemma 10. Higher Types are Partnered when Lower Types are Partnered.
Let Assumptions 5 and 6 hold, and $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation. Consider two men m and m' with $\theta_m < \theta_{m'}$. Then, if m is partnered, then so too is m', i.e., $\phi^\star(m) \neq m \implies \phi^\star(m') \neq m'$. An analogous result holds for women.

Proof. We'll prove the lemma via contradiction. Suppose that man m is partnered, while man m' isn't. Let $w = \phi^\star(m)$. We'll establish that man m' and woman w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Let $b_m^\star$ and $z_m^\star$ be the benefit and effort of m in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, and let $b_w^\star$ and $z_w^\star$ be the benefit and effort of w in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

We first establish that

$$b_w^\star - c(\theta_{m'}, z_m^\star) > 0. \tag{5.11}$$

By individual rationality, we have that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = b_w^\star - c(\theta_m, z_m^\star) \geq 0$. Since higher type men have strictly lower costs per Lemma 9, we have

$$b_w^\star - c(\theta_{m'}, z_m^\star) > b_w^\star - c(\theta_m, z_m^\star) \geq 0.$$

Now we establish the contradiction. Consider man m' and woman w with agreement $(x_1, z_w^\star)$. Woman w is willing to match with m' if $u_w(m', x_1, z_w^\star) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$, which reduces to $b(x_1) > b(z_m^\star)$. And m' is willing to match with w if $u_{m'}(w, x_1, z_w^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, which reduces to $b_w^\star - c(\theta_{m'}, x_1) > 0$. Thus, man m' and woman w block if there is an $x_1 \in [0, \beta]$ such that

$$b(x_1) > b(z_m^\star) \tag{5.12}$$

$$b_{w'}^\star - c(\theta_{m'}, x_1) > 0. \tag{5.13}$$

Consider $x_1 = z_m^\star + \delta$, where $\delta > 0$. Since $b(y)$ strictly increasing, we have that $z_m^\star + \delta$ satisfies equation (5.12). Since equation (5.11) holds and $c(\theta, y)$ is continuous in y, we may pick δ sufficiently small such that $z_m^\star + \delta$ satisfies equation (5.13). Since $z_m^\star$ is interior, we may make δ small so that $z_m^\star + \delta \in [0, \beta]$. Thus, m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. □

Proof of Proposition 7. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, let $\{z_i^\star\}_{i \in \mathcal{N}}$ be the associated efforts, and let $\{b_i^\star\}_{i \in \{j | \phi^\star(j) \neq j\}}$ be the associated benefits. Let m and m' be two different men. We'll first establish that $\theta_m < \theta_{m'}$ implies that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \leq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Subsequently, we'll establish that $\theta_m = \theta_{m'}$ implies that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Let $\theta_m < \theta_{m'}$. There are four cases to consider: (i) m and m' are both single, (ii) m is single and m' is partnered, (iii) m is partnered and m' is single, and (iv) both m and m' are partnered. If case (i), then both m and m' earn zero. If case (ii), then m earns zero and m' earns at least zero as $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is individually rational. Since Lemma 10 gives that case (iii) is impossible, we only need to establish that m makes no more than m' in case (iv).

Consider case (iv). We argue by contradiction. Suppose that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, and let $w = \phi^\star(m)$ and $w' = \phi^\star(m')$. We'll show that m' and w block. However, we first need a preliminary result:

$$b_w^\star - c(\theta_{m'}, z_m^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star). \tag{5.14}$$

This follows directly from the contraction hypothesis and Lemma 9: $b_w^\star - c(\theta_{m'}, z_m^\star) > b_w^\star - c(\theta_m, z_m^\star) = u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Now we establish the contradiction. Man m' and woman w block with agreement $(x_1, z_w^\star)$ if there is an $x_1 \in [0, \beta]$ such that $u_w(m', x_1, z_w^\star) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{m'}(w, x_1, z_w^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e.,

$$b(x_1) > b(z_m^\star) \tag{5.15}$$

$$b_w^\star - c(\theta_{m'}, x_1) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star). \tag{5.16}$$

Consider $x_1 = z_m^\star + \delta$, with $\delta > 0$. Since $b(y)$ strictly increasing, we have that $z_m^\star + \delta$ satisfies equation (5.15). Since equation (5.14) holds and $c(\theta, y)$ is continuous in y, we may pick δ sufficiently small such that $z_m^\star + \delta$ satisfies equation (5.16). Since $z_m^\star$ is interior, we may make δ small enough so that $z_m^\star + \delta \in [0, \beta]$. Thus, m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

We now establish that $\theta_m = \theta_{m'}$ implies $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. We argue via contradiction. Suppose (without loss) that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, we'll again show that m' and w block. As above m' and w block with agreement $(x_1, z_w^\star)$ if there is an $x_1 \in [0, \beta]$ with $u_w(m', x_1, z_w^\star) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{m'}(w, x_1, z_w^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e.,

$$b(x_1) > b(z_m^\star) \tag{5.17}$$

$$b_w^\star - b_{w'}^\star > c(\theta_{m'}, x_1) - c(\theta_{m'}, z_{m'}^\star). \tag{5.18}$$

Since

$$u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = b^\star_w - c(\theta_m, z^\star_m) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star) = b^\star_{w'} - c(\theta_{m'}, z^\star_{m'})$$

and $\theta_m = \theta_{m'}$, we have

$$b^\star_w - b^\star_{w'} > c(\theta_{m'}, z^\star_m) - c(\theta_{m'}, z^\star_{m'}).$$

Thus, at $x_1 = z^\star_m + \delta$, where $\delta > 0$, we have that both equations (5.17) and (5.18) are true for δ sufficiently small since $b(y)$ is strictly increasing and $c(\theta, y)$ is continuous in y. Since $z^\star_m$ is interior, we may take δ sufficiently small such that $z^\star_m + \delta \in [0, \beta]$. Thus, m' and w block, a contradiction. □

Remark. If both men m and m' are partnered in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, we can give a stronger result: $\theta_m < \theta_{m'}$ implies $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Simply change the contradiction hypothesis in case (iv) from $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$ to $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and observe that equation (5.14) still holds as Lemma 9 gives cost is strictly decreasing in type.

Remark. If players have different fixed costs, then the first conclusion of Proposition 7 doesn't hold. To see this, consider a variant of Example 1 where man 1 and woman 4's fixed cost is $^9/_{16}$ instead of $^1/_4$. It's readily verified that the stable allocation we found in Example 1 is also a stable allocation in this variant. However, man 1 and woman 4 now earn $^1/_{16}$, while man 2 and woman 3 earn $^1/_8$. Thus, the lower types make more.

Remark. It's obvious from the second part of the proof that Assumptions 5 and 6 aren't necessary for the second conclusion of Proposition 7.

Matches and Types

Corollary 4 suggests that higher type men match with higher type women. The next proposition shows that this happens in at least one stable allocation. We follow Legros and Newman [14] and say that an allocation $(\phi, \bar{\boldsymbol{x}})$ **exhibits positive assortative matching in types** if, for any two partnered men m and m' with $\theta_m < \theta_{m'}$, we have that m' matches with a higher type woman than m does, i.e., $\theta_w \leq \theta_{w'}$, where $w = \phi(m)$ and $w' = \phi(m')$.[19]

Proposition 8. Matches and Types.
Let Assumption 5 hold, then there is an interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ that exhibits positive assortative matching in types.[20]

[19] Defining positive assortative matching is terms of men is without loss. If $(\phi, \bar{\boldsymbol{x}})$ exhibits positive assortative matching and if $\theta_w < \theta_{w'}$ for two women w and w', then it's readily verified that $\theta_{\phi^\star(w)} \leq \theta_{\phi^\star(w')}$.

[20] We give a stronger result in the Supplement: *every* interior stable allocation exhibits positive assortative matching in type when Assumption 5 holds, $b(y)$ and $c(\theta, y)$ are continuously differentiable, and $^{\partial b(y)}/_{\partial y} > 0$ and $^{\partial c(\theta,y)}/_{\partial y} > 0$ for all $y \in [0, \beta]$. We omit a detailed discussion because the conditions are strong the result adds little economic insight.

That is, there is at least one interior stable allocation where higher ability players match with each other. Not all interior stable allocations exhibit assortative matching because players may match in arbitrary ways within their benefit groups. For instance, if there are two men m and m' in $G_l^{\mathbb{M}}$, with $\theta_m < \theta_{m'}$, and two women w and w' in $G_l^{\mathbb{W}}$, with $\theta_w < \theta_{w'}$, it's possible that m is matched to w' and m' is matched to w. That said, all men and women in the same-indexed benefit groups exert the same effort. Thus, we can rematch these players so that higher types are matched to higher types without affecting the stability of the allocation. Hence, we obtain Proposition 8.

Proof of Proposition 8. Starting with an interior stable allocation $(\phi', \bar{\boldsymbol{x}}')$, we construct another interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ which exhibits positive assortative matching in types. We'll proceed in three steps. First, we'll describe the construction. Second, we'll establish $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior stable allocation. Third, we'll establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exhibits positive assortative matching in types. Throughout, let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{W}}$ be the benefit groups associated with $(\phi', \bar{\boldsymbol{x}}')$, and let z_i' and b_i' denote player i's effort and benefit in $(\phi', \bar{\boldsymbol{x}}')$. Also, let $z_i^\star$ and $b_i^\star$ denote player i's effort and benefit in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

First, we construct $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ from $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \ldots, \boldsymbol{x}^{N})$ as follows.

1. Set $\phi^\star$ such that single players remain single, i.e., for each $i \in \mathcal{N}$ with $\phi'(i) = i$, set $\phi^\star(i) = i$.
2. For each $l \in \{1, \ldots, J_{\mathbb{M}}\}$:
   (a) List the men in $G_l^{\mathbb{M}}$ in descending order of their types and label them m_1, m_2, …, $m_{|G_l^{\mathbb{M}}|}$; break any ties randomly. So m_1 is the highest type man in $G_l^{\mathbb{M}}$, m_2 is the second highest type man, and so on. Likewise, list the women in $G_l^{\mathbb{W}}$ in descending order of their types and label them w_1, …, $w_{|G_l^{\mathbb{M}}|}$. (Recall that $|G_l^{\mathbb{M}}| = |G_l^{\mathbb{W}}|$ by Proposition 5.)
   (b) Set $\phi^\star$ such that the j-th man and j-th woman on each list are matched, i.e., such that $\phi^\star(m_j) = w_j$ and $\phi^\star(w_j) = m_j$ for $j \in \{1, \ldots, |G_l^{\mathbb{M}}|\}$.
3. Set $\bar{\boldsymbol{x}}^\star = \bar{\boldsymbol{x}}'$.

Step (3) ensures that all players keep their efforts, i.e., $z_i^\star = z_i'$ for all $i \in \mathcal{N}$.

Second, we establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior stable allocation. To do this, we need two preliminary facts.

Fact one: $\boldsymbol{x}^{m'} = \boldsymbol{x}^{w'}$ for all $m \in G_l^{\mathbb{M}}$ and all $w \in G_l^{\mathbb{W}}$ for each $l \in \{1, \ldots, J_{\mathbb{M}}\}$. Since $b(y)$ is strictly increasing, (i) all men in $G_l^{\mathbb{M}}$ exert the same effort, label it $\tilde{x}_1$, and (ii) all women in $G_l^{\mathbb{W}}$ exert the same effort, label it $\tilde{x}_2$. Thus, if $m \in G_l^{\mathbb{M}}$, the Proposition 5 gives $\phi'(m) \in G_l^{\mathbb{W}}$, so $\boldsymbol{x}^{m'} = (\tilde{x}_1, \tilde{x}_2)$. Likewise, if $w \in G_l^{\mathbb{W}}$, then $\phi'(w) \in G_l^{\mathbb{M}}$, so $\boldsymbol{x}^{w'} = (\tilde{x}_1, \tilde{x}_2)$. The fact follows.

Fact two: $u_i(\phi^\star(i), \boldsymbol{x}^{i^\star}) = u_i(\phi'(i), \boldsymbol{x}^{i'})$ for all $i \in \mathcal{N}$. Step (3) of the construction gives that $z_i^\star = z_i'$ for all $i \in \mathcal{N}$. Thus, we have that i's costs are unchanged. Since $\phi^\star(i)$ and $\phi'(i)$ are both in the same benefit group in $(\phi', \bar{\boldsymbol{x}}')$ by step (2) and we keep their efforts constant via step (3), we also have that the benefit i receives is unchanged. Thus, i earns the same payoff in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $(\phi', \bar{\boldsymbol{x}}')$.

Since $\phi^\star$ is a matching by step (2) and $\bar{\boldsymbol{x}}^\star$ is on the interior of X^N by step (3), to show that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior and feasible allocation, we only need to show that $\boldsymbol{x}^{i^\star} = \boldsymbol{x}^{\phi^\star(i)^\star}$ for all $i \in \mathcal{N}$. Without loss, suppose i is a man; the argument is analogous if i is a woman. If i is single, this follows from step (1). If i is partnered, let l_i be such that $i \in G_{l_i}^{\mathbb{M}}$. We have

$$\boldsymbol{x}^{i^\star} = \boldsymbol{x}^{i'} = \boldsymbol{x}^{\phi^\star(i)'} = \boldsymbol{x}^{\phi^\star(i)^\star},$$

where the first and third equalities are due to step (3), and the second equality is because $\phi^\star(i) \in G_{l_i}^{\mathbb{W}}$ by step (2) and the first preliminary fact.

Since $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is a feasible and interior allocation, the second preliminary fact gives that $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi', \bar{\boldsymbol{x}}')$. Thus, Lemma 7 implies that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable.

Third, we establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exhibits positive assortative matching in types. Let m and m' be partnered men such that $\theta_m < \theta_{m'}$. Proposition 6 gives that there are two cases: (i) $b_m^\star = b_{m'}^\star$ or (ii) $b_m^\star < b_{m'}^\star$. Let $w = \phi^\star(m)$ and $w' = \phi^\star(m')$. If case (i), then in step (2), m' occupies an earlier position in the list than m. Thus, m' is matched to a higher type woman than m, i.e., $\theta_w \leq \theta_{w'}$. If case (ii), we establish that $\theta_w \leq \theta_{w'}$ via contradiction. Suppose $\theta_w > \theta_{w'}$, then Proposition 6 gives that $b_w^\star \geq b_{w'}^\star$. Yet, $b_m^\star < b_{m'}^\star$ implies that $b_w^\star < b_{w'}^\star$ by Proposition 5, a contradiction. □

Remark. It's clear from the proof that every interior stable allocation is payoff equivalent to an interior stable allocation that exhibits positive assortative matching in types. Thus, Proposition 3 of Legros and Newman [14] implies that the Pareto frontier of each couple satisfies generalized increasing differences, which is their sufficient condition on the Pareto frontier for positive assortative matching in types. In light of this, one can think of our result as giving natural conditions on the primitive payoffs that imply Legros and Newman's general sufficient condition.

Comparative Statics of Effort and Welfare

In this subsection, we focus on the "symmetric stable allocation" and examine how increases in players' types influence their efforts, benefits, and welfare. We do this to examine how decreases in people's opportunity costs of effort, perhaps due to reductions in work/family responsibilities and more sleep, affect their decisions and outcomes. We make the following assumption.

Assumption 7. Universal Strict Agreeability and Universal Intolerability.
There are an equal number of men and women, i.e., $M = N/2$. For all θ and θ' in Θ, we've:
(i) $b(x_2) - c(\theta, x_1) > 0$ and $b(x_1) - c(\theta', x_2) > 0$ for some $(x_1, x_2) \in X$.
(ii) $b(x_2) - c(\theta, x_1) < 0$ or $b(x_1) - c(\theta', x_2) < 0$ for each $(x_1, x_2) \in \partial X$.

Part (i) guarantees any type of man and any type of woman can find an agreement that makes them strictly better off than if they're single. Part (ii) guarantees that intolerability always holds. Thus, Assumption 7 implies Assumption 4.

Let $\{\theta_i\}_{i\in\mathcal{N}}$ denote the endowment of players' types.

Definition. We say that the type endowment $\{\theta_i\}_{i\in\mathcal{N}}$ is *symmetric* when:
(i) No two men have the same type and no two women have the same type, i.e., $\theta_m \neq \theta_{m'}$ for any two men m and m' and $\theta_w \neq \theta'_w$ for any two women w and w'.
(ii) Men and women are endowed with the same types, i.e., $\cup_{i\in\mathcal{M}}\{\theta_i\} = \cup_{i\in\mathcal{W}}\{\theta_i\}$.

For instance, the type endowment in Example 1 is symmetric: the endowment is $\{\theta_1, \theta_2, \theta_3, \theta_4\} = \{2, 1, 1, 2\}$, and we have $\{\theta_i\}_{i\in\mathcal{M}} = \{\theta_1, \theta_2\} = \{2, 1\}$ and $\{\theta_i\}_{i\in\mathcal{W}} = \{\theta_3, \theta_4\} = \{1, 2\}$.

Next, we describe the construction of a "symmetric stable allocation."

Construction. Symmetric Stable Allocation.
Let Assumption 7 hold and let the type endowment $\{\theta_i\}_{i\in\mathcal{N}}$ be symmetric. Label the men m_1, m_2, …, m_M in descending order of their types; so, m_1 is the highest type man, m_2 is the second highest type man, and so on. Likewise, label the women w_1, w_2, …, w_M in descending order of their types.

We construct a **symmetric stable allocation** $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = (\phi^\dagger, \boldsymbol{x}^{1^\dagger}, \ldots, \boldsymbol{x}^{N^\dagger})$ as follows. Set $\phi^\dagger$ such that men and women with the same rank-order type are matched, i.e., such that $\phi^\dagger(m_l) = w_l$ and $\phi^\dagger(w_l) = m_l$ for each $l \in \{1, \ldots, M\}$. (Thus, every player is partnered.) Set $\bar{\boldsymbol{x}}^\dagger$ such that $\boldsymbol{x}^{m_l^\dagger} = \boldsymbol{x}^{w_l^\dagger} = (x_l^\dagger, x_l^\dagger)$ where

$$x_l^\dagger = \max_{\leq}\{\arg\max_{y\in[0,\beta]} b(y) - c(\theta_{m_l}, y)\}, \tag{5.19}$$

for each $l \in \{1, \ldots, M\}$, i.e., m_l and w_l exert the same effort and this effort is the largest solution of $\max_{y\in[0,\beta]} b(y) - c(\theta_{m_l}, y)$. ○

The next lemma gives a few properties of the symmetric stable allocation. It shows, for instance, that the symmetric stable allocation is actually stable. Also, we say an allocation $(\phi', \bar{\boldsymbol{x}}')$ is **welfare maximizing** if it maximizes $T(\phi, \bar{\boldsymbol{x}})$ on the set of feasible allocations.

Lemma 11. Properties of the Symmetric Stable Allocation.
Let Assumption 7 hold and let the endowment of types $\{\theta_i\}_{i\in\mathcal{N}}$ *be symmetric. Then (i) a symmetric stable allocation* $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ *exists and is unique (i.e., the procedure described above only produces one allocation), (ii)* $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ *is an interior Pareto stable allocation, and (iii)* $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ *is welfare maximizing. In addition, for each player i, we have* $\theta_i = \theta_{\phi^\dagger(i)}$ *and*

$$u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = \max_{y\in[0,\beta]} b(y) - c(\theta_i, y). \tag{5.20}$$

One may verify, for instance, that the stable allocation given in Example 1 is a symmetric stable allocation. The lemma's proof exploits the symmetry of the type endowment and the construction of the symmetric stable allocation to show that payoffs are given by equation (5.20). Everything else follows from this and Assumption 7 because players' payoffs are additively separable in benefit and cost.

We develop comparative statics for the symmetric stable allocation because it's focal: it's Pareto optimal, it's welfare maximizing, and it treats equals equally in that matched players have the same type, exert the same effort, and have the same payoffs. The next proposition gives our comparative statics results. We write $u_i(\phi, \bar{\boldsymbol{x}}, \theta_i)$ to emphasize the dependence of i's payoff in the allocation $(\phi, \bar{\boldsymbol{x}})$ on her type θ_i.

Proposition 9. Comparative Statics of the Symmetric Stable Allocation.
Let Assumptions 5, 6, and 7 hold. Let $\{\theta_i\}_{i\in\mathcal{N}}$ *and* $\{\theta'_i\}_{i\in\mathcal{N}}$ *be symmetric endowments of types such that* $\theta'_i \geq \theta_i$ *for each player i. Let* $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ *be the symmetric stable allocation when types are* $\{\theta_i\}_{i\in\mathcal{N}}$*, let* $(\phi', \bar{\boldsymbol{x}}')$ *be the symmetric stable allocation when types are* $\{\theta'_i\}_{i\in\mathcal{N}}$*, and let* $\{z^\dagger_i\}_{i\in\mathcal{N}}$ *and* $\{z'_i\}_{i\in\mathcal{N}}$ *be players' efforts in* $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ *and* $(\phi', \bar{\boldsymbol{x}}')$ *respectively. Then, for each player i, as types increase from* $\{\theta_i\}_{i\in\mathcal{N}}$ *to* $\{\theta'_i\}_{i\in\mathcal{N}}$*,*
(i) The effort i exerts increases, i.e., $z^\dagger_i \leq z'_i$*.*
(ii) The effort exerted by i's match increases, i.e., $z^\dagger_{\phi^\dagger(i)} \leq z'_{\phi'(i)}$*.*
(iii) The payoff of i increases, i.e., $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger, \theta_i) \leq u_i(\phi', \bar{\boldsymbol{x}}', \theta'_i)$*.*

Since $b(y)$ is strictly increasing, (i) implies that the benefit player i produces increases and (ii) implies that the benefit i receives from his or her match increases. Thus, if a player's opportunity cost of effort increases, we have that his or her effort, benefit, and payoff decline. The intuition for the proposition is that cost is submodular and strictly decreasing in type. Thus, as types increase, equation (5.19) ensures that players' efforts increase and equation (5.20) ensures that their payoffs increase. We illustrate the conclusion of Proposition 9 with the following version of Example 1.

Example 5. Effort and Welfare Comparative Statics for a Variant of Example 1.

Let $\mathcal{M} = \{1, 2\}$ and $\mathcal{W} = \{3, 4\}$. Let $X = [0, 2]^2$. Let $\Theta = \{1, 2, 2.1\}$. Let $b(y) = y$ and $c(\theta, y) = \frac{1}{\theta} y^2 + {}^1\!/\!_8$. It's readily verified that Assumptions 5, 6, and 7 hold for any assignment of types to players. Let $\theta_1 = \theta_4 = 2$ and $\theta_2 = \theta_3 = 1$ be the initial endowment of types. And let $\theta'_1 = \theta'_3 = 2$ and $\theta'_2 = \theta'_4 = 2.1$ be the new endowment of types. Since both endowments are trivially symmetric and since $\theta'_i \geq \theta_i$ for each player i, Proposition 9 gives that all players exert more effort, produce and receive greater benefits, and are better off under the new endowment. We can see this directly.

The symmetric stable allocation under $\{\theta_i\}_{i \in \mathcal{N}}$ is $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = (\phi^\dagger, \boldsymbol{x}^{1^\dagger}, \boldsymbol{x}^{2^\dagger}, \boldsymbol{x}^{3^\dagger}, \boldsymbol{x}^{4^\dagger})$, where $\phi^\dagger(1) = 4$ and $\boldsymbol{x}^{1^\dagger} = \boldsymbol{x}^{4^\dagger} = (1, 1)$, and $\phi^\dagger(2) = 3$ and $\boldsymbol{x}^{2^\dagger} = \boldsymbol{x}^{3^\dagger} = ({}^1\!/\!_2, {}^1\!/\!_2)$. In this allocation, 1 and 4 each earn ${}^3\!/\!_8$, while 2 and 3 each earn ${}^1\!/\!_8$. The symmetric stable allocation under $\{\theta'_i\}_{i \in \mathcal{N}}$ is $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \boldsymbol{x}^{2'}, \boldsymbol{x}^{3'}, \boldsymbol{x}^{4'})$, where $\phi'(1) = 3$ and $\boldsymbol{x}^{1'} = \boldsymbol{x}^{3'} = (1, 1)$, and $\phi'(2) = 4$ and $\boldsymbol{x}^{2'} = \boldsymbol{x}^{4'} = (1.05, 1.05)$. In this allocation, 1 and 3 each earn ${}^3\!/\!_8$, while 2 and 4 each earn ${}^2\!/\!_5$. Thus, players exert more effort and earn more. Since $b(y) = y$, the benefits they produce and receive also increase. $\triangle$

We'll prove Lemma 11 by first establishing that payoffs are given by (5.20). We'll then leverage this fact, the Assumption, and the method of construction to prove the stated results. Subsequently, we'll prove Proposition 9 by exploiting the submodularity and type-monotonicity of the cost function.

Proof of Lemma 11. We'll prove the lemma in five steps. First, we'll argue existence and uniqueness of the symmetric stable allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Second, we'll establish that players' payoffs are given by equation (5.20). Third, we'll leverage this equation and Assumption 7 to show that the symmetric stable allocation is actually stable. Fourth, we'll show that the symmetric stable allocation is interior and is Pareto optimal. Fifth, we'll establish that the symmetric stable allocation maximizes welfare.

We first establish existence and uniqueness of $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Since the benefit function and the cost function are continuous in y, $\arg\max_{y \in [0, \beta]} b(y) - c(\theta_{m_l}, y)$ is non-empty and compact by standard arguments. Thus, the it has has a unique maximal element in the standard order, so $x_l^\dagger$ of equation (5.19) exists and is unique for each $l \in \{1, \ldots, M\}$. It follows that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ exists and, since $\phi^\dagger$ is uniquely determined by the construction because no two players of the same gender have the same type, that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is unique. It's an allocation since $\phi^\dagger$ is a matching by construction and $\bar{\boldsymbol{x}}^\dagger \in A(\phi^\dagger)$ by construction.

Recall that the men and women are labeled $m_1, \ldots, m_M$ and $w_1, \ldots, w_M$ in descending order of their types. Since the type endowment is symmetric, we have $\theta_{m_1} = \theta_{w_1}$, $\theta_{m_2} = \theta_{w_2}$, $\ldots$, and $\theta_{m_M} = \theta_{w_M}$. That is, man m_1 and woman w_1 have the same type, man m_2 and woman w_2 have the same type, and so on.

Second, we establish that equation (5.20) holds. Suppose that i's label is w_j, then m's label is m_j by construction of $\phi^\dagger$. By the previous paragraph, $\theta_i = \theta_{w_j} = \theta_{m_j} = \theta_m$, or more succinctly, $\theta_i = \theta_{\phi^\dagger(i)}$. Thus,

$$b(x_j^\dagger) - c(\theta_i, x_j^\dagger) = b(x_j^\dagger) - c(\theta_{m_j}, x_j^\dagger) = \max_{y\in[0,\beta]} b(y) - c(\theta_{m_j}, y) = \max_{y\in[0,\beta]} b(y) - c(\theta_i, y),$$

where the second equality is due to optimality. Hence,

$$u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = b(x_j^\dagger) - c(\theta_i, x_j^\dagger) = \max_{y\in[0,\beta]} b(y) - c(\theta_i, y).$$

Since an analogous argument applies if i is a man, equation (5.20) holds.

Third, we establish that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is stable. By construction, $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is feasible because (i) $\phi^\dagger$ is a matching and (ii) $\bar{\boldsymbol{x}}^\dagger \in X^N$ since $x_l^\dagger \in [0, \beta]$ for all $l \in \{1, \ldots, M\}$. It's also individually rational. To see this, consider man m. Suppose that m's label is m_j, then m is matched to woman w_j by $\phi^\dagger$ and we have $\theta_m = \theta_{m_j} = \theta_{w_j}$. Consider the maximum of the sum of m and w_j's payoffs,

$$\begin{aligned}&\max_{(x_1,x_2)\in X} b(x_2) - c(\theta_{m_j}, x_1) + b(x_1) - c(\theta_{w_j}, x_2)\\ &\quad = \max_{x_1\in[0,\beta]} b(x_1) - c(\theta_{m_j}, x_1) + \max_{x_2\in[0,\beta]} b(x_2) - c(\theta_{w_j}, x_2) = 2\max_{y\in[0,\beta]} b(y) - c(\theta_{m_j}, y).\end{aligned}$$

(This max exists by standard arguments.) Since $\theta_{m_j} = \theta_{w_j} \in \Theta$, Assumption 7 implies that there is an $(x_1', x_2') \in X$ such that $b(x_2') - c(\theta_{m_j}, x_1') > 0$ and $b(x_1') - c(\theta_{w_j}, x_2') > 0$. Hence, $\max_{(x_1,x_2)\in X} b(x_2) - c(\theta_{m_j}, x_1) + b(x_1) - c(\theta_{w_j}, x_2) > 0$, so $\max_{y\in[0,\beta]} b(y) - c(\theta_{m_j}, y) > 0$, which implies that $u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > 0$ by equation (5.20). Since the argument is analogous for a woman, we have individual rationality.

We argue that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ cannot be blocked by contradiction. Suppose that a man m and a woman w block $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Then, there is an $(x_1', x_2') \in X$ such that $u_m(w, x_1', x_2') > u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ and $u_w(m, x_1', x_2') > u_w(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. This implies that

$$b(x_2') - c(\theta_m, x_1') + b(x_1') - c(\theta_w, x_2') > u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) + u_w(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger).$$

Yet, equation (5.20) gives

$$\begin{aligned}u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) + u_w(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) &= \max_{x_1\in[0,\beta]} b(x_1) - c(\theta_m, x_1) + \max_{x_2\in[0,\beta]} b(x_2) - c(\theta_w, x_2)\\ &= \max_{(x_1,x_2)\in X} b(x_2) - c(\theta_m, x_1) + b(x_1) - c(\theta_w, x_2).\end{aligned}$$

Thus, we have

$$b(x_2') - c(\theta_m, x_1') + b(x_1') - c(\theta_w, x_2') > \max_{(x_1,x_2)\in X} b(x_2) - c(\theta_m, x_1) + b(x_1) - c(\theta_w, x_2),$$

a contradiction as (x_1', x_2') is in X. It follows that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is stable.

Fourth, we show that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is interior and is Pareto optimal. Since Assumption 7 implies Assumption 4, Corollary 1 gives the desired result as all players are partnered in $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$.

Fifth, we establish that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is welfare maximizing. By equation (5.20),

$$T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = \sum_{i\in\mathcal{N}} \max_{z_i\in[0,\beta]} b(z_i) - c(\theta_i, z_i).$$

Let $(\phi', \bar{\boldsymbol{x}}')$ be a feasible allocation, let $S = \{i \in \mathcal{N} | \phi'(i) \neq i\}$ be the set of partnered players in $(\phi', \bar{\boldsymbol{x}}')$, and let $\{z_i'\}_{i\in\mathcal{N}}$ gives players' efforts in $(\phi', \bar{\boldsymbol{x}}')$. We have

$$T(\phi', \bar{\boldsymbol{x}}') = \sum_{i\in S} b(z'_{\phi'(i)}) - c(\theta_i, z_i') = \sum_{i\in S} b(z_i') - c(\theta_i, z_i').$$

The first equality is due to the fact single players get zero and the second equality follows from rearranging the sum since $b(z_i')$ appears in the term of $\phi'(i)$. Hence,

$$T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) - T(\phi', \bar{\boldsymbol{x}}') = \sum_{i\in S}\left(\max_{z_i\in[0,\beta]}\{b(z_i) - c(\theta_i, z_i)\} - (b(z_i') - c(\theta_i, z_i'))\right) + \sum_{i\in\mathcal{N}\setminus S} \max_{y\in[0,\beta]} b(y) - c(\theta_i, y).$$

The first summand is weakly positive by optimality since $z_i' \in [0, \beta]$ for each $i \in \mathcal{N}$ by the feasibility of $(\phi', \bar{\boldsymbol{x}}')$, and the second summand is positive by Assumption 7. Thus, $T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) - T(\phi', \bar{\boldsymbol{x}}') \geq 0$. $\square$

Remark. In general, one should not expect welfare maximizing allocations to be stable or stable allocations to be welfare maximizing. See the Supplement for details.

Proof of Proposition 9. Let $\{\theta_i\}_{i\in\mathcal{N}}$ and $\{\theta_i'\}_{i\in\mathcal{N}}$ be symmetric endowments of types with $\theta_i' \geq \theta_i$ for each player i. Let $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ be the symmetric stable allocation when types are $\{\theta_i\}_{i\in\mathcal{N}}$, let $(\phi', \bar{\boldsymbol{x}}')$ be the symmetric stable allocation when types are $\{\theta_i'\}_{i\in\mathcal{N}}$, and let $\{z_i^\dagger\}_{i\in\mathcal{N}}$ and $\{z_i'\}_{i\in\mathcal{N}}$ be players' efforts in $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ and $(\phi', \bar{\boldsymbol{x}}')$ respectively.

We need a preliminary result. Let $h(\theta) = \max_{\leq}\{\arg\max_{y\in[0,\beta]} b(y) - c(\theta, y)\}$. Since $-c(\theta, y)$ is supermodular by Assumption 5, Topkis' Monotonicity Theorem (Theorem 2.8.1 of [25]) gives that $h(\theta)$ is non-decreasing.

Consider woman w whose type increases from θ_w to θ_w'. In $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$, she and her match

$\phi^\dagger(w)$ both exert effort $h(\theta_w)$: by construction they exert effort $h(\theta_{\phi^\dagger(w)})$ and $\theta_w = \theta_{\phi^\dagger(w)}$ by Lemma 11. In $(\phi', \bar{\boldsymbol{x}}')$, w and her match $\phi'(w)$ both exert effort $h(\theta'_w)$. Since $h(\theta)$ is non-decreasing, we have that w's effort increases and that the effort of her match increases as well, even though the identity of her match may change. That is, $z^\dagger_w \leq z'_w$ and $z^\dagger_{\phi^\dagger(w)} \leq z'_{\phi'(w)}$. Since an analogous argument holds for each man, parts (i) and (ii) of the proposition follow.

It remains to show that $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger, \theta_i) \leq u_i(\phi', \bar{\boldsymbol{x}}', \theta'_i)$ for each player i. Write

$$u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger, \theta_i) = \max_{y \in [0,\beta]} b(y) - c(\theta_i, y) \leq \max_{y \in [0,\beta]} b(y) - c(\theta'_i, y) = u_i(\phi', \bar{\boldsymbol{x}}', \theta'_i),$$

where the equalities are due to Lemma 11 and the weak inequality is due to the fact that $b(y) - c(\theta_i, y) \leq b(y) - c(\theta'_i, y)$ since cost decreases with type by Lemma 9. □

Remark. Assumption 7 and symmetry of the type endowment are essential to the conclusions of Proposition 9. Without them, (i) the symmetric stable allocation may fail to be stable and (ii) some player may be made strictly worse off by an increase in types; we give an example of (ii) in the Supplement.

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