Combining Two Consistent Estimators*

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Proposed Running Head: Combining Two Consistent Estimators

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Abstract

This paper shows how a weighted average of a forward and reverse Jackknife IV estimator yields an estimator that is robust against heteroscedasticity and many instruments. This estimator, called the HFUL (Heteroscedasticty robust Fuller) estimator was introduced by Hausman et al. (2012) but without the derivation. Combining consistent estimators is a theme that is associated with Jerry Hausman and, therefore, we present this derivation in this volume. We also show how changing the denominator of the HFUL objective function yields a new estimator.

1 Introduction

One idea that is associated with Jerry Hausman is the idea of combining two estimators. For example, Hausman (1978) takes the difference between an efficient estimator and a robust estimator and derives the famous Hausman test. Another example is Hahn and Hausman (2002), who use the difference between a forward and reverse Two Stage Least Squares estimator. In this paper, we give the derivation of a recent estimator, HFUL (Heteroscedasticty robust Fuller). In particular, we show that this estimator is a weighted average of a forward and reverse Jackknife IV estimator (JIVE). This estimator was introduced by Hausman et al. (2012) but without the derivation. Jackknife IV estimators were proposed by Phillips and Hale (1977), Blomquist and Dahlberg (1999), Angrist and Imbens and Krueger (1999), and Ackerberg and Deveraux (2003). In their paper "The Case Against JIVE," Davidson and MacKinnon (2006) show that JIVE performs poorly compared to LIML, due to the large dispersion of JIVE. However, Chao, Swanson, Hausman, Newey, and Woutersen.(2012) show that JIVE is consistent under heteroscedasticity and the many instruments asymptotic (unlike LIML). This paper shows has a forward and reverse JIVE can be combined to construct an estimator with less dispersion.

We also show how changing the denominator of the HFUL objective function yields a new estimator. In particular, we use the denominator implied by JIV1 in the terminology of Angrist and Imbens and Krueger (1999). In the special case that all instruments are dummies, this new estimator is the estimator proposed by Bekker and van der Ploeg (2005).

2 Deriving a Heteroscedasticty Robust Estimators

2.1 The Model and Previous Estimators

The model we consider is given by

$$\begin{aligned} \underset{n\times 1}{y} &= \underset{n\times G}{X}\underset{G\times 1}{\delta_0} + \underset{n\times 1}{\varepsilon}, \\ X &= \Upsilon + U, \end{aligned}$$

where n is the number of observations, G the number of right-hand side variables, Υ is a matrix of observations on the reduced form, and U is the matrix of reduced form disturbances. For the asymptotic approximations, the elements of Υ will be implicitly allowed to depend on n, although we suppress dependence of Υ on n for notational convenience. Estimation of δ_0 will be based on a $n \times K$ matrix Z of instrumental variable observations. We will assume that $Z_1, ..., Z_n$ are nonrandom and that observations (ε_i, U_i) are independent across i and have mean zero.

This model allows for Υ to be a linear combination of Z, i.e. $\Upsilon = Z\pi$ for some $K \times G$ matrix π. Furthermore, columns of X may be exogenous, with the corresponding column of U being zero. The model also allows for Z to approximate the reduced form. For example, let X_i, Υ_i, and Z_i denote the i^{th} row (observation) for X, Υ, and Z respectively. We could have $\Upsilon_i = f_0(w_i)$ be an unknown function of a vector w_i of underlying instruments and $Z_i = (p_{1K}(w_i), ..., p_{KK}(w_i))'$ for approximating functions $p_{kK}(w)$, such as power series or splines. In this case linear combinations of Z_i may approximate the unknown reduced form, e.g. as in Donald and Newey (2001).

To describe previous estimators let $P = Z(Z'Z)^{-1}Z'$. The LIML estimator $\tilde{\delta}$ is given by

$$\tilde{\delta}^* = \arg\min_{\delta} \hat{Q}^*(\delta), \hat{Q}^*(\delta) = \frac{(y - X\delta)'P(y - X\delta)}{(y - X\delta)'(y - X\delta)}.$$

FULL is obtained as

$$\breve{\delta}^* = (X'PX - \breve{\alpha}^* X'X)^{-1}(X'Py - \breve{\alpha}^* X'y).$$

for $\breve{\alpha}^* = [\tilde{\alpha}^* - (1 - \tilde{\alpha}^*)C/T]/[1 - (1 - \tilde{\alpha}^*)C/T]$ and $\tilde{\alpha}^* = Q(\tilde{\delta}^*)$. Under homoscedasticity, FULL has moments of all orders, is approximately mean unbiased for $C = 1$, and is second order admissible for $C \geq 4$ under standard large sample asymptotics. Both LIML and FULL are members of a class of estimators of the form

$$\hat{\delta}^* = (X'PX - \hat{\alpha}^* X'X)^{-1}(X'Py - \hat{\alpha}^* X'y).$$

For example, LIML has this form for $\hat{\alpha}^* = \tilde{\alpha}^*$, FULL for $\hat{\alpha}^* = \breve{\alpha}^*$, and two stage least squares (2SLS) for $\hat{\alpha}^* = 0$.

We can use Fisher consistency to characterize the problem with heteroskedasticity and many instruments. Fisher consistency means that the derivative of the objective function at the truth converges to zero when normalized correctly. This condition is necessary for consistency. For expository purposes, consider first 2SLS, having objective function $(y - X\delta)'P(y - X\delta)$. The derivative of this objective function, times $-1/2n$, is equal to $X'P\varepsilon/n$. Like means, quadratic forms converge to their expectations under appropriate conditions. By virtue of independence and $E[\varepsilon_i] = 0$ we have

$$E[X_i P_{ij} \varepsilon_j] = E[X_i] P_{ij} E[\varepsilon_j] = 0, i \neq j.$$

Thus,

$$\begin{aligned}\frac{1}{n}X'P\varepsilon &= \frac{1}{n}E[X'P\varepsilon]+o_p(1)=\frac{1}{n}\sum_{i,j=1}^{n}E[X_iP_{ij}\varepsilon_j]+o_p(1)\\ &= \frac{1}{n}\sum_{i=1}^{n}E[X_iP_{ii}\varepsilon_i]+o_p(1)=\frac{1}{n}\sum_{i=1}^{n}E[U_i\varepsilon_i]P_{ii}+o_p(1),\end{aligned}$$

so the Fisher consistency condition is $\sum_{i=1}^{n}E[U_i\varepsilon_i]P_{ii}/n \longrightarrow 0$. Because of many instruments

$$P_{ii} \nrightarrow 0,$$

so 2SLS is not consistent, Bekker (1994), even with homoscedasticity where $E[U_i\varepsilon_i]$ is constant over i.

For LIML, with objective function $\hat{Q}^*(\delta)$ given above, we have

$$(-\varepsilon'\varepsilon/2n)\partial\hat{Q}^*(\delta_0)/\partial\delta=\frac{1}{n}\left(X-\frac{X'\varepsilon}{\varepsilon'\varepsilon}\varepsilon\right)'P\varepsilon=\frac{1}{n}\left(X-\hat{\gamma}\varepsilon\right)'P\varepsilon, \hat{\gamma}=X'\varepsilon/\varepsilon'\varepsilon.$$

Let $\sigma_i^2=E[\varepsilon_i^2]$, $\gamma_i=E[X_i\varepsilon_i]/\sigma_i^2=E[U_i\varepsilon_i]/\sigma_i^2$, and $\gamma_n=\sum_i E[X_i\varepsilon_i]/\sum_i\sigma_i^2=\sum_i\sigma_i^2\gamma_i/\sum_i\sigma_i^2$. By standard arguments $X'\varepsilon/\varepsilon'\varepsilon-\gamma_n \xrightarrow{p} 0$ and $\varepsilon'P\varepsilon/n$ is bounded in probability. Then similarly to 2SLS,

$$\begin{aligned}\frac{1}{n}\left(X-\hat{\gamma}\varepsilon\right)'P\varepsilon &= \frac{1}{n}\left(X-\gamma_n\varepsilon\right)'P\varepsilon+o_p(1)=\frac{1}{n}E[\left(X-\gamma_n\varepsilon\right)'P\varepsilon]+o_p(1)\\ &= \sum_{i=1}^{n}E[\left(X_i-\gamma_n\varepsilon_i\right)P_{ii}\varepsilon_i]/n+o_p(1)=\sum_{i=1}^{n}(\gamma_i-\gamma_n)P_{ii}\sigma_i^2/n+o_p(1).\end{aligned}$$

Thus, for LIML the Fisher consistency condition is

$$\frac{1}{n}\sum_{i=1}^{n}(\gamma_i-\gamma_n)P_{ii}\sigma_i^2 \longrightarrow 0.$$

There are two interesting cases where this condition holds.

A) γ_i does not vary with i : In this case, $\gamma_i=\gamma_n$ so that the term on the left is identically zero. Thus, homoscedasticity in the coefficient $\gamma_i=E[X_i\varepsilon_i]/E[\varepsilon_i^2]$ of the regression of X_i on ε_i leads to Fisher consistency of LIML.

B) P_{ii} does not vary with i : In this case $\sum_i(\gamma_i-\gamma_n)P_{ii}\sigma_i^2=P_{11}\sum_i(\gamma_i-\gamma_n)\sigma_i^2=0$. When the instruments are dummy variables, sometimes referred to as grouping instruments, this condition is satisfied if all of the columns of Z have the same number of ones, i.e. the group sizes are equal.

Bekker and van der Ploeg (2005) showed that this condition gives consistency of LIML for grouping instruments.

In the general heteroskedastic case where γ_i and $P_{ii}\sigma_i^2$ are correlated across i, the Fisher consistency condition will not be satisfied and so LIML will not be consistent. Furthermore, due to weak instruments the bias may be large even when $\sum_i(\gamma_i - \gamma_n)P_{ii}\sigma_i^2/n$ is small. Analogous arguments can also be used to show that, with heteroskedasticity, FULL and LIML are inconsistent under many instruments. Bekker and van der Ploeg (2005) and Hausman et al. (2012) had pointed out that LIML can be inconsistent with heteroskedasticity but this appears to be the first characterization of Fisher consistency of LIML.

The lack of consistency of these estimators under many instruments and heteroskedasticity can be attributed to the presence of the $i = j$ terms in the double sums in their first order conditions. One way to make the estimators robust to heteroskedasticity is to remove these terms. A version $\bar{\delta}$ of 2SLS without the $i = j$ terms solves the normal equations

$$0 = X'P(y - X\bar{\delta}) - \sum_{i=1}^{n} P_{ii}X_i(y_i - X_i'\bar{\delta}) = \sum_{i \neq j} X_i P_{ij}(y_j - X_j'\bar{\delta}).$$

Solving for $\bar{\delta}$ gives

$$\begin{aligned} \bar{\delta} &= \left(\sum_{i \neq j} X_i P_{ij} X_j'\right)^{-1} \sum_{i \neq j} X_i P_{ij} y_j \\ &= \left(X'PX - \sum_{i=1}^{n} P_{ii} X_i X_i'\right)^{-1} \left(X'Py - \sum_{i=1}^{n} P_{ii} X_i y_i\right). \end{aligned}$$

This is the second JIV estimator (JIV2) of Angrist, Imbens, and Krueger (1999). Because the normal equations remove the $i = j$ terms this estimator is Fisher consistent. It was pointed out by Ackerberg and Deveraux (2003) and Chao et al. (2012) that it is consistent under many weak instruments and heteroskedasticity.

Under homoscedasticity and many weak instruments this estimator turns out to not be efficient. Also, Davidson and MacKinnon (2005) argued that it has inferior small sample properties to LIML under homoscedasticity. Using the weighted average of forward and reverse JIVE overcomes these problems.

2.2 Combining Forward and Reverse JIVE

The heteroskedasticity robust LIML estimator (HLIM) is obtained by dropping $i = j$ terms from the numerator of the LIML objective function,

$$\tilde{\delta} = \arg\min_{\delta} \hat{Q}(\delta), \hat{Q}(\delta) = \frac{\sum_{i \neq j}(y_i - X_i'\delta)P_{ij}(y_j - X_j'\delta)}{(y - X\delta)'(y - X\delta)}.$$

Like the jackknife IV estimator, $\tilde{\delta}$ will be consistent under heteroskedasticity because the $i = j$ terms have been removed from the normal equations. Here we will show consistency, asymptotic normality, and consistency of an asymptotic variance estimator.

Like LIML, this estimator is invariant to normalization. Let $\bar{X} = [y, X]$. Then $\tilde{d} = (1, -\tilde{\delta}')'$ solves

$$\min_{d:d_1=1} \frac{d'\left(\sum_{i \neq j} \bar{X}_i P_{ij} \bar{X}_j'\right) d}{d'\bar{X}'\bar{X}d}.$$

Another normalization, such as imposing that another d is equal to 1 would produce the same estimator, up to the normalization.

Also, computation of this estimator is straightforward. Similarly to LIML, $\tilde{\alpha} = \hat{Q}(\tilde{\delta})$ is the smallest eigenvalue of $(\bar{X}'\bar{X})^{-1} \sum_{i \neq j} \bar{X}_i P_{ij} \bar{X}_j'$. Also, first order conditions for $\tilde{\delta}$ are

$$0 = \sum_{i \neq j} X_i P_{ij} \left(y_j - X_j'\tilde{\delta}\right) - \tilde{\alpha} \sum_i X_i (y_i - X_i'\tilde{\delta}).$$

Solving gives

$$\tilde{\delta} = \left(\sum_{i \neq j} X_i P_{ij} X_j' - \tilde{\alpha} X'X\right)^{-1} \left(\sum_{i \neq j} X_i P_{ij} y_j - \tilde{\alpha} X'y\right).$$

This estimator has a similar form to LIML except that the $i = j$ terms have been deleted from the double sums.

It is interesting to note that LIML and HLIM coincide when P_{ii} is constant. In that case,

$$\hat{Q}^*(\delta) = \hat{Q}(\delta) + \frac{\sum_i (y_i - X_i'\delta)P_{ii}(y_i - X_i'\delta)}{(y - X\delta)'(y - X\delta)} = \hat{Q}(\delta) + P_{11},$$

so that the LIML objective function equals the HLIM objective function plus a constant. This explains why constant P_{ii} will lead to LIML being consistent under heteroskedasticity.

By replacing $\tilde{\alpha}$ with some other value $\hat{\alpha}$ we can form a k-class version of a jackknife estimator, having the form

$$\hat{\delta} = \left(\sum_{i \neq j} X_i P_{ij} X_j' - \hat{\alpha} X'X\right)^{-1} \left(\sum_{i \neq j} X_i P_{ij} y_j - \hat{\alpha} X'y\right)$$

The JIV estimator is obtained by setting $\hat{\alpha} = 0$. A heteroskedasticity consistent version of FULL is obtained by replacing $\tilde{\alpha}$ with $\hat{\alpha} = [\tilde{\alpha} - (1-\tilde{\alpha})C/T]/[1-(1-\tilde{\alpha})C/T]$ for some constant C. The small sample properties of this estimator are unknown, but we expect its performance relative to HLIM to be similar to that of FULL relative to LIML. As pointed out by Hahn, Hausman, and Kuersteiner (2004), FULL has much smaller dispersion than LIML with weak instruments, so we expect the same for HFUL. Monte Carlo results given in Hausman (2012) confirm these properties.

An asymptotic variance estimator is useful for constructing large sample confidence intervals and tests. To describe it, let $\hat{\varepsilon}_i = y_i - X_i'\hat{\delta}$, $\hat{\gamma} = X'\hat{\varepsilon}/\hat{\varepsilon}'\hat{\varepsilon}, \hat{X} = X - \hat{\varepsilon}\hat{\gamma}'$,

$$\hat{H} = \sum_{i \neq j} X_i P_{ij} X_j' - \hat{\alpha} X'X, \hat{\Sigma} = \sum_{i,j=1}^{n} \sum_{k \notin \{i,j\}} \hat{X}_i P_{ik} \hat{\varepsilon}_k^2 P_{kj} \hat{X}_j' + \sum_{i \neq j} P_{ij}^2 \hat{X}_i \hat{\varepsilon}_i \hat{\varepsilon}_j \hat{X}_j'.$$

The variance estimator is

$$\hat{V} = \hat{H}^{-1} \hat{\Sigma} \hat{H}^{-1}.$$

We can interpret the HLIM estimator $\tilde{\delta}$ as a combination of forward and reverse jackknife IV (JIV) estimators. For simplicity, we give this interpretation in the scalar δ case. Let $\tilde{\varepsilon}_i = y_i - X_i'\tilde{\delta}$ and $\tilde{\gamma} = \sum_i X_i \tilde{\varepsilon}_i / \sum_i \tilde{\varepsilon}_i$. First-order conditions for $\tilde{\delta}$ are

$$0 = -\frac{\partial \hat{Q}(\tilde{\delta})}{\partial \delta} \sum_i \tilde{\varepsilon}_i^2/2 = \sum_{i \neq j} (X_i - \tilde{\gamma}\tilde{\varepsilon}_i) P_{ij} (y_j - X_j'\tilde{\delta}) = \sum_{i \neq j} [(1 + \tilde{\gamma}\tilde{\delta}) X_i - \tilde{\gamma} y_i] P_{ij} (y_j - X_j'\tilde{\delta}).$$

The forward JIV estimator $\bar{\delta}$ is

$$\bar{\delta} = \left(\sum_{i \neq j} X_i P_{ij} X_j \right)^{-1} \sum_{i \neq j} X_i P_{ij} y_j.$$

The reverse JIV is obtained as follows. Dividing the structural equation by δ_0 gives

$$X_i = y_i/\delta_0 - \varepsilon_i/\delta_0.$$

Applying JIV to this equation to estimate $1/\delta_0$ and then inverting gives the reverse JIV

$$\bar{\delta}^r = \left(\sum_{i \neq j} y_i P_{ij} X_j \right)^{-1} \sum_{i \neq j} y_i P_{ij} y_j.$$

Then collecting terms in the first-order conditions for HLIM gives

$$
\begin{aligned}
0 &= (1+\tilde{\gamma}\tilde{\delta})\sum_{i\neq j} X_i P_{ij}(y_j - X_j'\tilde{\delta}) - \hat{\gamma}\sum_{i\neq j} y_i P_{ij}(y_j - X_j'\tilde{\delta}) \\
&= (1+\hat{\gamma}\tilde{\delta})\sum_{i\neq j} X_i P_{ij} X_j(\bar{\delta} - \tilde{\delta}) - \hat{\gamma}\sum_{i\neq j} y_i P_{ij} X_j(\bar{\delta}^r - \tilde{\delta}).
\end{aligned}
$$

Dividing through by $\sum_{i\neq j} X_i P_{ij} X_j$ gives

$$
0 = (1+\tilde{\gamma}\tilde{\delta})(\bar{\delta} - \tilde{\delta}) - \hat{\gamma}\bar{\delta}(\bar{\delta}^r - \tilde{\delta}).
$$

If we replace $\hat{\gamma}$ by some other estimator $\bar{\gamma}$ and the $\hat{\gamma}\bar{\delta}$ coefficient following the minus sign by $\bar{\gamma}\tilde{\delta}$ we obtain a linearized version of this equation that can be solved for $\hat{\delta}$ to obtain

$$
\dot{\delta} = \frac{\bar{\delta}}{1 - \bar{\gamma}(\bar{\delta} - \bar{\delta}_r)}.
$$

This estimator will be asymptotically equivalent to the HLIM and the HFUL estimator. This result is analogous to that of Hahn and Hausman (2002) that under homoscedasticity LIML is an optimal combination of forward and reverse bias corrected two stage least squares estimators.

2.3 Adjusting the Denominator of HLIM and HFUL

Bekker and van der Ploeg (2005, BP) considered estimators that are consistent with dummy instruments and group heteroskedasticity. A particular interesting estimator which BP have proposed is what they refer to as the MM estimator. It turns out that this MM estimator is a special case of a type of jackknifed LIML estimator, where the numerator quadratic form corresponds to the objective function of JIV1 instead of that of JIV2 (as in HLIM). More specifically, consider an estimator which minimizes the following modified variance ratio

$$
\begin{aligned}
Q_{JLIM}(\delta) &= \frac{(y - X\delta)'(P - D_P)(I_n - D_P)^{-1}(y - X\delta)}{(y - X\delta)' M D_P (I_n - D_P)^{-1}(y - X\delta)} \\
&= \frac{Q_{JIV1}(\delta)}{(y - X\delta)' M D_P (I_n - D_P)^{-1}(y - X\delta)}, \qquad (1)
\end{aligned}
$$

where $M = I_n - P$ and $D_P = diag(P_{11}, ..., P_{nn})$. Note that the numerator of (1) is simply the objective function of JIV1, since minimizing $Q_{JIV1}(\delta)$ with respect to δ leads to the estimator

$$
\begin{aligned}
\widehat{\delta}_{JIV1} &= \left(X'(P - D_P)(I_n - D_P)^{-1} X\right)^{-1} X'(P - D_P)(I_n - D_P)^{-1} y \\
&= \left(\sum_{i\neq j} X_i P_{ij}(1 - P_{jj})^{-1} X_j'\right)^{-1} \sum_{i\neq j} X_i P_{ij}(1 - P_{jj})^{-1} y_j,
\end{aligned}
$$

which is the jackknife IV estimator originally proposed by Phillips and Hale (1977). See Chao, Swanson, Hausman, Newey, and Woutersen (2012) for further discussion.

It is also possible to rewrite the objective function (1) in an alternative form which will be more convenient for the purpose of establishing a correspondence with the results of BP. To proceed, note first that, by elementary algebraic manipulations, it is easy to show that

$$(P - D_P)(I_n - D_P)^{-1} = P - MD_P(I_n - D_P)^{-1}.$$

Hence, we can rewrite (1) as

$$\begin{aligned} Q_{JLIM}(\delta) &= \frac{(y - X\delta)'\left[P - MD_P(I_n - D_P)^{-1}\right](y - X\delta)}{(y - X\delta)' MD_P(I_n - D_P)^{-1}(y - X\delta)} \\ &= \frac{(y - X\delta)' P(y - X\delta)}{(y - X\delta)' MD_P(I_n - D_P)^{-1}(y - X\delta)} - 1 \\ &= Q^*_{JLIM}(\delta) - 1 \end{aligned}$$

so that the estimator which minimizes $Q_{JLIM}(\delta)$ is clearly the same as the one which minimizes

$$Q^*_{JLIM}(\delta) = \frac{(y - X\delta)' P(y - X\delta)}{(y - X\delta)' MD_P(I_n - D_P)^{-1}(y - X\delta)}. \quad (2)$$

To show that the estimator obtained from (2) specializes to the MM estimator of BP with dummy-variable instruments and across-group heteroskedasticity, consider the grouped data IV regression model studied in their paper, which takes the form

$$\begin{aligned} y_{ij} &= x'_{ij}\delta + \varepsilon_{ij}, \\ x_{ij} &= \pi_j + v_{ij} \end{aligned}$$

for $i = 1, ..., n_j$ and $j = 1, ..., m$. Stacking first the observations within each group, we obtain (for $j = 1, ..., m$)

$$y_j = \begin{pmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{n_j,j} \end{pmatrix}, X_j = \begin{pmatrix} x'_{1j} \\ x'_{2j} \\ \vdots \\ x'_{n_j,j} \end{pmatrix}, \ \varepsilon_j = \begin{pmatrix} \varepsilon_{1j} \\ \varepsilon_{2j} \\ \vdots \\ \varepsilon_{n_j,j} \end{pmatrix}, \ V_j = \begin{pmatrix} v'_{1j} \\ v'_{2j} \\ \vdots \\ v'_{n_j,j} \end{pmatrix},$$

and also let

$$Z_j = \iota_{n_j} e'_{j,m}$$

where $\iota_{n_j} = (1, 1, ..., 1)'$ is an $(n_j \times 1)$ vector of ones and $e_{j,m}$ is the j^{th} column of a $m \times m$ identity matrix. Using these notations, it is easily seen that the IV model studied in BP can be written in our notations as

$$\begin{aligned} y &= X\delta + \varepsilon, && (3) \\ X &= Z\Pi + V, && (4) \end{aligned}$$

where $y = (y_1', y_2', ..., y_m')'$, $X = (X_1', X_2', ..., X_m')'$, $Z = (Z_1', Z_2', ..., Z_m')'$, $\Pi = (\pi_1, \pi_2, ..., \pi_m)'$, $\varepsilon = (\varepsilon_1', \varepsilon_2', ..., \varepsilon_m')'$, and $V = (V_1', V_2', ..., V_m')'$ with the components of these vectors and matrices being as defined above. Moreover, with dummy-variable instruments as considered in BP, we have

$$\begin{aligned} P &= Z\left(Z'Z\right)^{-1}Z' \\ &= \begin{pmatrix} n_1^{-1}\iota_{n_1}\iota_{n_1}' & 0 & \cdots & 0 \\ 0 & n_2^{-1}\iota_{n_2}\iota_{n_2}' & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n_m^{-1}\iota_{n_m}\iota_{n_m}' \end{pmatrix} = \begin{pmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & P_m \end{pmatrix}, \end{aligned}$$

$$D_P = \begin{pmatrix} n_1^{-1}I_{n_1} & 0 & \cdots & 0 \\ 0 & n_2^{-1}I_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & n_m^{-1}I_{n_m} \end{pmatrix}$$

and

$$\begin{aligned} M &= I_n - P \\ &= \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_m \end{pmatrix}, \end{aligned}$$

where $n = \sum_{j=1}^{m} n_j$. Hence, in this setting, (2) specializes to the MM objective function given in equation (13) of BP, viz

$$\begin{aligned}
Q^*_{JLIM}(\delta) &= \frac{(y-X\delta)' P (y-X\delta)}{(y-X\delta)' M D_P (I_n - D_P)^{-1} (y-X\delta)} \\
&= \frac{\sum_{j=1}^{m} (y_j - X_j\delta)' P_j (y_j - X_j\delta)}{\sum_{j=1}^{m} (y_j - X_j\delta)' \left[n_j^{-1}/\left(1-n_j^{-1}\right)\right] M_j (y_j - X_j\delta)} \\
&= \frac{\sum_{j=1}^{m} (y_j - X_j\delta)' P_j (y_j - X_j\delta)}{\sum_{j=1}^{m} (y_j - X_j\delta)' (n_j - 1)^{-1} M_j (y_j - X_j\delta)} \\
&= \frac{n\sum_{j=1}^{m} \widetilde{\delta}' w_j \overline{Y}_j \overline{Y}_j' \widetilde{\delta}}{\sum_{j=1}^{m} \widetilde{\delta}' S_j \widetilde{\delta}} \\
&= n Q_{MM}(\delta) \qquad (5)
\end{aligned}$$

where $\widetilde{\delta} = (1, -\delta')'$, $w_j = n_j / \sum_{j=1}^{m} n_j = n_j/n$,

$$\begin{aligned}
\overline{Y}_j &= \frac{1}{n_j}\sum_{i=1}^{n_j} \left(y_{ij}, x_{ij}'\right)', \\
S_j &= \frac{1}{n_j - 1}\sum_{i=1}^{n_j} \left(\begin{pmatrix} y_{ij} \\ x_{ij} \end{pmatrix} - \overline{Y}_j\right)\left(\begin{pmatrix} y_{ij} \\ x_{ij} \end{pmatrix} - \overline{Y}_j\right)'
\end{aligned}$$

as defined in their paper. Note a slight difference in our notations and that of BP in that we set

$$M_j = I_{n_j} - P_j = I_{n_j} - n_j^{-1}\iota_{n_j}\iota_{n_j}'$$

here, whereas

$$M_j = w_j \overline{Y}_j \overline{Y}_j'$$

in the notations of BP.

Now, given that the numerator of (1) corresponds to the objective function of JIV1, we can show, following arguments similar to that used for proving Theorems 1 and 2, that the estimator which minimizes $Q_{JLIM}(\delta)$ is consistent and asymptotically normal under many instrument and many weak instrument asymptotics. Indeed, in the special case with dummy instruments and across-group heteroskedasticity, BP have already shown that the MM estimator is consistent and asymptotically normal under group-asymptotics which takes $m \to \infty$ holding each n_j fixed, for $j = 1, ..., m$. However, it should be noted that the MM estimator will not in general be consistent under group asymptotics in the presence of more general instruments which are not group indicators.

To analyze more fully situations where the MM estimator may or may not be consistent, we consider the following slight generalization of the setup studied in BP, where

$$\begin{aligned} y_{ij} &= x_{ij}'\delta + \varepsilon_{ij}, \\ \underset{G\times 1}{x_{ij}} &= \underset{1\times 1}{z_{ij}} \underset{G\times 1}{\pi_j} + \underset{G\times 1}{v_{ij}}, \end{aligned}$$

for $j = 1, ..., m$ and $i = 1, ..., n_j$, so that we allow for instruments which are possibly not group indicators. Now, from the previous discussion, it is apparent that the MM estimator can be equivalently obtained by minimizing the alternative objective function

$$Q_{MM}^{*}(\delta) = \frac{\sum_{j=1}^{m} (y_j - X_j\delta)' \left[P_j - (n_j - 1)^{-1} M_j\right] (y_j - X_j\delta)}{\sum_{j=1}^{m} (y_j - X_j\delta)' (n_j - 1)^{-1} M_j (y_j - X_j\delta)},$$

where $P_j = z_{\cdot j}\left(z_{\cdot j}' z_{\cdot j}\right)^{-1} z_{\cdot j}'$ with $z_{\cdot j} = \left(z_{1j}, z_{2j}, .., z_{n_j,j}\right)'$ and $M_j = I_{n_j} - P_j$. All other notations are as defined previously. Now, in large sample, this objective function will be close to

$$\overline{Q}_{MM}(\delta) = \frac{\sum_{j=1}^{m} E\left\{(y_j - X_j\delta)' \left[P_j - (n_j - 1)^{-1} M_j\right] (y_j - X_j\delta)\right\}}{\sum_{j=1}^{m} E\left\{(y_j - X_j\delta)' (n_j - 1)^{-1} M_j (y_j - X_j\delta)\right\}}.$$

To gain some insights into conditions under which the limiting objective function $\overline{Q}_{MM}(\delta)$ may be minimized at $\delta = \delta_0$, we define $P_{j,ii}$ to be the i^{th} diagonal element of P_j and let

$$\begin{aligned} \sigma^2(i,j) &= E\left[\varepsilon_{ij}^2\right], \ \vartheta(i,j) = E\left[x_{ij}\varepsilon_{ij}\right], \varpi(i,j) = \frac{1}{n_j - 1}\left(1 - P_{j,ii}\right), \\ W(i,j) &= \frac{P_{j,ii} - \varpi(i,j)}{\varpi(i,j)}, \ \pi(i,j) = \frac{\varpi(i,j)}{m}; \end{aligned}$$

Note that, by straightforward calculations, we obtain

$$
\begin{aligned}
& \left.\frac{\partial}{\partial\delta}\overline{Q}_{MM}(\delta)\right|_{\delta=\delta_0} \\
= \;& \frac{-2\sum_{j=1}^{m} E\left(X_j'\left[P_j-(n_j-1)^{-1}M_j\right]\varepsilon_j\right)}{\sum_{j=1}^{m}(n_j-1)^{-1}E\left(\varepsilon_j' M_j\varepsilon_j\right)} \\
& +2\frac{\sum_{j=1}^{m} E\left(\varepsilon_j'\left[P_j-(n_j-1)^{-1}M_j\right]\varepsilon_j\right)}{\left[\sum_{j=1}^{m}(n_j-1)^{-1}E\left(\varepsilon_j' M_j\varepsilon_j\right)\right]^2}\sum_{j=1}^{m}\left(\frac{1}{n_j-1}\right)E\left(X_j' M_j\varepsilon_j\right) \\
= \;& \frac{-2}{\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sigma^2(i,j)\,\pi(i,j)}\left\{\sum_{j=1}^{m}\sum_{i=1}^{n_j}W(i,j)\left(\vartheta(i,j)-\sigma^2(i,j)\frac{\sum_{j=1}^{m}\sum_{i=1}^{n_j}\vartheta(i,j)\,\pi(i,j)}{\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sigma^2(i,j)\,\pi(i,j)}\right)\pi(i,j)\right\} \\
= \;& -2\left(\widehat{E}\left[\sigma^2(i,j)\right]\right)^{-1}\widehat{E}\left[W(i,j)\left\{\vartheta(i,j)-\frac{\widehat{E}\left[\vartheta(i,j)\right]}{\widehat{E}\left[\sigma^2(i,j)\right]}\sigma^2(i,j)\right\}\right] \\
= \;& -2\left(\widehat{E}\left[\sigma^2(i,j)\right]\right)^{-1}\widehat{E}\left[W(i,j)\,\psi(i,j)\right] \qquad (6)
\end{aligned}
$$

where, in the expressions above, we have taken

$$
\begin{aligned}
\widehat{E}\left[\vartheta(i,j)\right] &= \sum_{j=1}^{m}\sum_{i=1}^{n_j}\vartheta(i,j)\,\pi(i,j)\,,\ \widehat{E}\left[\sigma^2(i,j)\right]=\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sigma^2(i,j)\,\pi(i,j)\,, \\
\psi(i,j) &= \vartheta(i,j)-\frac{\widehat{E}\left[\vartheta(i,j)\right]}{\widehat{E}\left[\sigma^2(i,j)\right]}\sigma^2(i,j)\,,\ \widehat{E}\left[W(i,j)\,\psi(i,j)\right]=\sum_{j=1}^{m}\sum_{i=1}^{n_j}W(i,j)\,\psi(i,j)\,\pi(i,j)\,.
\end{aligned}
$$

Now, we can interpret $W(i,j)$, $\sigma^2(i,j)$, and $\psi(i,j)$ as functions of the discrete random variables (indices) i and j, which have joint probability mass distribution (pmf) given by $\pi(i,j)$. Moreover, interpret

$$
\pi(j)=\sum_{i=1}^{n_j}\pi(i,j)=\sum_{i=1}^{n_j}\frac{\varpi(i,j)}{m}=\frac{1}{m}\sum_{i=1}^{n_j}\frac{1}{n_j-1}\left(1-P_{j,ii}\right)=\frac{1}{m}
$$

as the marginal pmf of j and define

$$
\pi(i\mid j)=\frac{\pi(i,j)}{\pi(j)}=\varpi(i,j)
$$

to be the conditional pmf of i given j. Furthermore, observe that

$$
\begin{aligned}
\widehat{E}\left[W(i,j)\ \mid j\right] &= \sum_{i=1}^{n_j}W(i,j)\,\pi(i\mid\ j)=\sum_{i=1}^{n_j}\left[P_{j,ii}-(n_j-1)^{-1}\left(1-P_{j,ii}\right)\right]=0, \\
\widehat{E}\left[W(i,j)\right] &= \widehat{E}\left[\widehat{E}\left[W(i,j)\ \mid j\right]\right]=0\ \ \text{(by law of iterated expectations),}
\end{aligned}
$$

so that, in particular,

$$\widehat{E}\left[W\left(i,j\right)\psi\left(i,j\right)\right]=\widehat{Cov}\left(W\left(i,j\right),\psi\left(i,j\right)\right)+\widehat{E}\left[W\left(i,j\right)\right]\widehat{E}\left[\psi\left(i,j\right)\right]=\widehat{Cov}\left(W\left(i,j\right),\psi\left(i,j\right)\right).$$

Hence, we can rewrite (6) as

$$\left.\frac{\partial}{\partial\delta}\overline{Q}_{MM}\left(\delta\right)\right|_{\delta=\delta_0}=-2\left(\widehat{E}\left[\sigma^2\left(i,j\right)\right]\right)^{-1}\widehat{Cov}\left(W\left(i,j\right),\psi\left(i,j\right)\right) \qquad (7)$$

from which it follows that $\delta=\delta_0$ is a critical point of $\overline{Q}_{MM}\left(\delta\right)$ if and only if

$$\widehat{Cov}\left(W\left(i,j\right),\psi\left(i,j\right)\right)=0,$$

since

$$\widehat{E}\left[\sigma^2\left(i,j\right)\right]=\sum_{j=1}^{m}\sum_{i=1}^{n_j}\sigma^2\left(i,j\right)\pi\left(i,j\right)\leq C\sum_{j=1}^{m}\sum_{i=1}^{n_j}\pi\left(i,j\right)=C<\infty,$$

holds under a condition that the second moments of $\{\varepsilon_{ij}\}$ are uniformly bounded.

Next, consider the situation where error variance is homoskedastic within-group, but there may be heteroskedasticity across groups, i.e., for each j, $\sigma^2\left(i,j\right)=\sigma^2\left(j\right)$ amd $\vartheta\left(i,j\right)=\vartheta\left(j\right)$ so that $\psi\left(i,j\right)=\psi\left(j\right)$. Here, by the law of iterated expectations

$$\widehat{Cov}\left(W\left(i,j\right),\psi\left(i,j\right)\right)=\widehat{E}\left[W\left(i,j\right)\psi\left(j\right)\right]=\widehat{E}_j\left[\psi\left(j\right)\widehat{E}\left[W\left(i,j\right)\mid j\right]\right]=0,$$

so that $\delta=\delta_0$ is a critical point of $\overline{Q}_{MM}\left(\delta\right)$ in this case. This suggests that, in the absence of within-group heteroskedasticity, the MM estimator will be consistent even in situations where the available instruments are not group indicators. On the other hand, suppose that instruments are group indicators as assumed in BP; then, $P_{j,ii}=1/n_j$ for $i=1,...,n_j$; and it follows that

$$W\left(i,j\right)=P_{j,ii}-\left(n_j-1\right)^{-1}\left(1-P_{j,ii}\right)=\frac{1}{n_j}-\frac{1}{n_j-1}\left(1-\frac{1}{n_j}\right)=0$$

for $i=1,..,n_j$ and for each j, so that, trivially,

$$\widehat{Cov}\left(W\left(i,j\right),\psi\left(i,j\right)\right)=0.$$

Note that this is true even with within-group heteroskedasticity. Finally, given the possibility of within-group heteroskedasticity and instruments which are not group indicators; we obtain in general

$$
\begin{aligned}
& \widehat{Cov}\left(W\left(i,j\right),\psi\left(i,j\right)\right) \\
= \; & \widehat{E}\left[W\left(i,j\right)\psi\left(i,j\right)\right] \\
= \; & \sum_{j=1}^{m}\sum_{i=1}^{n_j}\left(\frac{P_{j,ii}-\varpi\left(i,j\right)}{\varpi\left(i,j\right)}\right)\psi\left(i,j\right)\pi\left(i,j\right) \\
= \; & \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n_j}\left[P_{j,ii}-\left(n_j-1\right)^{-1}\left(1-P_{j,ii}\right)\right]\psi\left(i,j\right) \\
= \; & \frac{1}{m}\sum_{j=1}^{m}\left(\frac{n_j}{n_j-1}\right)\sum_{i=1}^{n_j}\left\{P_{j,ii}-\left(\frac{1}{n_j}\right)\right\}\psi\left(i,j\right)\neq 0
\end{aligned}
$$

Hence, $\delta=\delta_0$ may not be a critical point of $\overline{Q}_{MM}\left(\delta\right)$ in more general settings; and it can be shown that, unlike HLIM, the MM estimator will not be consistent when both non-dummy instruments and within-group heteroskedasticity are present, although a generalization of the MM estimator obtained by minimizing (1) will be.

It is also of interest to compare the asymptotic distribution of the MM estimator to that of HLIM in the setting studied by BP, i.e., in the setting with across-group heteroskedasticity (but within-group homoscedasticity) and dummy-variable instruments. Under group asymptotics, it can be shown, following the same argument as that used to prove Theorem 2, that

$$V_{MM}^{-1/2}\left(\widehat{\delta}_{MM}-\delta_0\right)\xrightarrow{d}N\left(0,I_G\right),\text{ as } m\to\infty,$$

where

$$V_{MM}=H_{MM}^{-1}\left(\overline{\Omega}_{MM}+\Psi_{MM}\right)H_{MM}^{-1} \tag{8}$$

and where

$$
\begin{aligned}
H_{MM} &= \frac{1}{m}\sum_{j=1}^{m}n_j\pi_j\pi_j',\ \overline{\Omega}_{MM}=\frac{1}{m}\sum_{j=1}^{m}n_j\sigma_j^2\pi_j\pi_j', \\
\Psi_{MM} &= \frac{1}{m}\sum_{j=1}^{m}\sum_{1\leq\ i\neq k\leq\ n_j}\left(\frac{1}{n_j-1}\right)^2\left\{E\left[\varepsilon_{ij}^2\right]E\left[\overline{V}_{kj}\overline{V}_{kj}'\right]+E\left[\overline{V}_{ij}\varepsilon_{ij}\right]E\left[\varepsilon_{kj}\overline{V}_{kj}'\right]\right\} \\
&= \frac{1}{m}\sum_{j=1}^{m}\left(\frac{n_j}{n_j-1}\right)\Xi_{MM,j},
\end{aligned}
$$

Here, $\overline{V}_{ij}=v_{ij}-\left(\overline{\sigma}_{21}/\overline{\sigma}^2\right)\varepsilon_{ij}$, with $\overline{\sigma}_{21}=m^{-1}\sum_{j=1}^{m}E\left[v_{ij}\varepsilon_{ij}\right]$ and $\overline{\sigma}^2=m^{-1}\sum_{j=1}^{m}E\left[\varepsilon_{ij}^2\right]$; and $\Xi_{MM,j}=E\left[\varepsilon_{ij}^2\right]E\left[\overline{V}_{kj}\overline{V}_{kj}'\right]+E\left[\overline{V}_{ij}\varepsilon_{ij}\right]E\left[\varepsilon_{kj}\overline{V}_{kj}'\right]$, where the dependence of $\Xi_{MM,j}$ on the index

j only is due to the fact that we have within-group homoscedasticity. On the other hand, for HLIM, we obtain

$$V_{HLIM}^{-1/2}\left(\widehat{\delta}_{HLIM}-\delta_0\right)\xrightarrow{d}N\left(0,I_G\right),\text{ as }m\to\infty,$$

where

$$V_{HLIM}=H_{HLIM}^{-1}\left(\overline{\Omega}_{HLIM}+\Psi_{HLIM}\right)H_{HLIM}^{-1}\tag{9}$$

and where

$$\begin{aligned}
H_{HLIM} &= \frac{1}{m}\sum_{j=1}^{m}\left(n_j-1\right)\pi_j\pi_j',\ \overline{\Omega}_{HLIM}=\frac{1}{m}\sum_{j=1}^{m}\frac{\left(n_j-1\right)^2}{n_j}\sigma_j^2\pi_j\pi_j',\\
\Psi_{HLIM} &= \frac{1}{m}\sum_{j=1}^{m}\sum_{1\le\ i\ne k\le\ n_j}\left(\frac{1}{n_j}\right)^2\left\{E\left[\varepsilon_{ij}^2\right]E\left[\widetilde{V}_{kj}\widetilde{V}_{kj}'\right]+E\left[\widetilde{V}_{ij}\varepsilon_{ij}\right]E\left[\varepsilon_{kj}\widetilde{V}_{kj}'\right]\right\}\\
&= \frac{1}{m}\sum_{j=1}^{m}\left(\frac{n_j-1}{n_j}\right)\Xi_{HLIM,j}
\end{aligned}$$

and where $\widetilde{V}_{ij}=v_{ij}-\left(\overline{\overline{\sigma}}_{21}/\overline{\overline{\sigma}}^2\right)\varepsilon_{ij}$, with $\overline{\overline{\sigma}}_{21}=\left(\sum_{j=1}^m n_j\right)^{-1}\sum_{j=1}^m n_jE\left[v_{ij}\varepsilon_{ij}\right]$ and $\overline{\overline{\sigma}}^2=\left(\sum_{j=1}^m n_j\right)^{-1}\sum_{j=1}^m n_jE\left[\varepsilon_{ij}^2\right]$; and $\Xi_{HLIM,j}=E\left[\varepsilon_{ij}^2\right]E\left[\widetilde{V}_{kj}\widetilde{V}_{kj}'\right]+E\left[\widetilde{V}_{ij}\varepsilon_{ij}\right]E\left[\varepsilon_{kj}\widetilde{V}_{kj}'\right]$.

Comparing (8) with (9) for the case where $n_j=n_*$ for all j, i.e., for the case where the size of the sample is the same for all groups; we see that

$$\begin{aligned}
\overline{\overline{\sigma}}_{21} &= \left(\sum_{j=1}^m n_j\right)^{-1}\sum_{j=1}^{m}n_jE\left[v_{ij}\varepsilon_{j,i}\right]=\frac{1}{mn_*}\sum_{j=1}^{m}n_*\sigma_{21,j}=\frac{1}{m}\sum_{j=1}^{m}\sigma_{21,j}=\overline{\sigma}_{21},\\
\overline{\overline{\sigma}}^2 &= \left(\sum_{j=1}^m n_j\right)^{-1}\sum_{j=1}^{m}n_jE\left[\varepsilon_{j,i}^2\right]=\frac{1}{mn_*}\sum_{j=1}^{m}n_*\sigma_j^2=\frac{1}{m}\sum_{j=1}^{m}\sigma_j^2=\overline{\sigma}_{21},
\end{aligned}$$

so that

$$\widetilde{V}_{ij}=v_{ij}-\left(\overline{\overline{\sigma}}_{21}/\overline{\overline{\sigma}}^2\right)\varepsilon_{ij}=v_{ij}-\left(\overline{\sigma}_{21}/\overline{\sigma}^2\right)\varepsilon_{ij}=\overline{V}_{ij},$$

and, thus,

$$\begin{aligned}
\Xi_{MM,j} &= E\left[\varepsilon_{ij}^2\right]E\left[\overline{V}_{kj}\overline{V}_{kj}'\right]+E\left[\overline{V}_{ij}\varepsilon_{ij}\right]E\left[\varepsilon_{kj}\overline{V}_{kj}'\right]\\
&= E\left[\varepsilon_{ij}^2\right]E\left[\widetilde{V}_{kj}\widetilde{V}_{kj}'\right]+E\left[\widetilde{V}_{ij}\varepsilon_{ij}\right]E\left[\varepsilon_{kj}\widetilde{V}_{kj}'\right]\\
&= \Xi_{HLIM,j}\\
&= \Xi_j\ \ (say),
\end{aligned}$$

for $j = 1, ..., m$ and $i, k = 1, ..., n_j$. It follows that in this case

$$\begin{aligned}
& V_{MM} \\
= \; & H_{MM}^{-1}\left(\overline{\Omega}_{MM} + \Psi_{MM}\right) H_{MM}^{-1} \\
= \; & \left(\frac{1}{m}\sum_{j=1}^{m} n_* \pi_j \pi_j'\right)^{-1} \left[\frac{1}{m}\sum_{j=1}^{m} n_* \sigma_j^2 \pi_j \pi_j' + \frac{1}{m}\sum_{j=1}^{m}\left(\frac{n_*}{n_* - 1}\right)\Xi_j\right]\left(\frac{1}{m}\sum_{j=1}^{m} n_* \pi_j \pi_j'\right)^{-1} \\
= \; & \left(\frac{1}{m}\sum_{j=1}^{m} \pi_j \pi_j'\right)^{-1} \left[\frac{1}{mn_*}\sum_{j=1}^{m} \sigma_j^2 \pi_j \pi_j' + \frac{1}{mn_*}\sum_{j=1}^{m}\left(\frac{1}{n_* - 1}\right)\Xi_j\right]\left(\frac{1}{m}\sum_{j=1}^{m} \pi_j \pi_j'\right)^{-1} \\
= \; & \left(\frac{1}{m}\sum_{j=1}^{m} (n_* - 1)\, \pi_j \pi_j'\right)^{-1} \left[\frac{1}{m}\sum_{j=1}^{m} \frac{(n_* - 1)^2}{n_*} \sigma_j^2 \pi_j \pi_j' + \frac{1}{m}\sum_{j=1}^{m}\left(\frac{n_* - 1}{n_*}\right)\Xi_j\right]\left(\frac{1}{m}\sum_{j=1}^{m} (n_* - 1)\, \pi_j \pi_j'\right)^{-1} \\
= \; & H_{HLIM}^{-1}\left(\overline{\Omega}_{HLIM} + \Psi_{HLIM}\right) H_{HLIM}^{-1} \\
= \; & V_{HLIM}
\end{aligned}$$

Hence, in the special case where the number of observations are the same in each group, the asymptotic covariance matrix of the MM estimator is equivalent to that of HLIM. However, in the more general case where n_j varies with j, the covariance matrices are not equal, but it does not appear that there is a uniform ranking of the two estimators in terms of asymptotic efficiency since the relative "size" of the covariance matrices in this case will depend on the values of the underlying parameters. This is analogous to the result obtained in Chao, Swanson, Hausman, Newey, and Woutersen (2012), where JIV1 and JIV2 are not found to dominate each other in terms of asymptotic efficiency under many-instrument and many-weak-instrument asymptotics.

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