Robustness to Parametric Assumptions in Missing Data Models

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Suppose we have a random sample from a population of interest. For each sampled unit we observe the covariate $X$, which we assume is discrete with support $\{x_1, \ldots, x_K\}$. For some units, we also observe the variable $Y$. Let $D = 1$ if we observe $Y$, and $D = 0$ otherwise. We are interested in the population mean of $Y$, $\theta = E[Y] = \sum_{k=1}^{K} p_k \mu_k$, where $\mu_k = E[Y \mid X = x_k]$ and $p_k = \Pr(X = x_k)$.

We assume that $Y$ is missing at random (MAR): $Y \perp D \mid X$. Suppose also that the propensity score $\epsilon_k = \Pr(D = 1 \mid X = x_k)$ is bounded away from zero (a support condition). Then in large samples, there will be at least some $\epsilon_k$ observed for each possible value of $X$, so that $E[Y \mid X = x_k, D = 1]$ is identified. Since $\mu_k = E[Y \mid X = x_k, D = 1]$ under MAR, we have

$$\theta = \sum_{k=1}^{K} p_k E[Y \mid X = x_k, D = 1].$$

Let $M_k$ equal the number of sampled units with $X = x_k$ (i.e., in cell $k$), and let $\hat{\mu}_k = \frac{\sum_{j=1}^{K} M_j}{M_k} \hat{M}_k$. The poststratification estimator for $\theta$ is

$$\hat{\theta}_{PS} = \sum_{k=1}^{K} \hat{\mu}_k \bar{Y}_k,$$

where $\bar{Y}_k$ is the average of $Y$ across those units with $D = 1$ and $X = x_k$ (i.e., the complete-case $k$ cell mean or the sample analog of $E[Y \mid X = x_k, D = 1]$).

When $M_k$ is large for all $k = 1, \ldots, K$ the poststratification estimator $\hat{\theta}_{PS}$ works well in practice and attains the semiparametric variance bound for $\theta$ derived by Jinyong Hahn (1998). Unfortunately in many applications it is common for $K$ to be large and $M_k$ to be small (at least for some values of $k$). In such settings the problem of empty cells, where $Y$ is unavailable for all sampled units with $X = x_k$, may be severe (Paul R. Rosenbaum 1987).

In settings with small cells, there may be substantial gains from imposing restrictions on the means $\mu_k$, but there is also a danger of misspecification. We explore ways to increase the robustness of parametric imputation estimators. First, we develop a simple empirical Bayes estimator, which combines parametric and unadjusted estimates of $\mu_k$ in a data-driven way. Second, we consider ways to use knowledge of the propensity score to help guard against misspecification of $\mu_k$, using a double robust estimator and an empirical Bayes approach. This does not contradict the efficiency bound analysis of Hahn (1998), which is relevant for settings where $M_k$ is large for all $k$.

I. Sampling framework and estimators

Following Joshua Angrist and Jinyong Hahn (2004) we consider a stratified random sampling scheme. Let $N$ be the total sample size with $M_k$ chosen such that $M_k/N = p_k$ for all $k$ (i.e., we assume that $p_k$, which characterizes the marginal distribution of $X$, is known). Within each cell, the probability of observing the outcome $Y$ is $\epsilon_k$, so that the number of observed outcomes is $n_k \sim \text{Binomial}(\epsilon_k, M_k)$.

Conditional on $n_k$, the observed outcomes $Y_{k1}, \ldots, Y_{kn_k}$ are i.i.d. (and independent across cells) with mean $\mu_k$ and variance $\sigma_k^2$.

The poststratification estimator for $\theta$, modi-
fied to take into account that the $p_k$ are known, is
\[ \hat{\theta}_{PS} = \sum_{k=1}^{K} p_k \bar{Y}_k, \]
where $\bar{Y}_k = n_k^{-1} \sum_{i=1}^{n_k} Y_{ki}$. The nonparametric imputation estimator of Hahn (1998), and the estimated propensity score weighting estimator of Keisuke Hirano, Guido W. Imbens and Geert Ridder (2003) (modified appropriately for the missing data problem considered here), are both equal to $\hat{\theta}_{PS}$ in the discrete covariate case. This estimator may perform poorly if some cells are empty (i.e., $n_k = 0$), then the estimator must be modified, for example by dropping empty cells or combining cells in some way.

An alternative is to posit a restricted model for the cell means:
\[ \mu_k = x'_k \beta, \]
where $\beta$ is a low-dimensional parameter. (We could also easily handle specifications of the form $\mu_k = t(x_k) \beta$ for a known function $t$.) Then
\[ \mathbb{E}[Y_k | n_1, \ldots, n_K] = x'_k \beta, \]
and, conditional on $(n_1, \ldots, n_K)$, the $(\bar{Y}_1, \ldots, \bar{Y}_K)$ will be mutually independent. We could estimate $\beta$ by a weighted least squares (WLS) regression of the $\bar{Y}_k$ on $x_k$, with weights proportional to $n_k$. (This is equivalent to an ordinary least squares (OLS) regression of all the observed $Y_{ki}$ on $X_{ki} = x_k$.) Then the parametric imputation estimator is
\[ \hat{\theta}_{PI} = \sum_{k=1}^{K} p_k (x'_k \hat{\beta}). \]

The parametric estimator would typically do better when the assumption on the means (1) holds, and could be used even if some cells are empty. However, if (1) does not hold, then $\hat{\theta}_{PI}$ may be severely biased. Our goal is to develop estimators that improve upon the poststratification estimator when cell sizes are small, but are not as sensitive to misspecification as the parametric imputation estimator.

Following Carl N. Morris (1983) and Gary Chamberlain (2009), we consider an empirical Bayes approach. (See also David S. Lee and David Card (2008) for a closely related approach.) In the following statements, we implicitly condition on $n_1, \ldots, n_K$. Suppose that
\[ \bar{Y}_k | \mu_k \sim N(\mu_k, v_k), \quad k = 1, \ldots, K, \]
where $v_k = \sigma_k^2 / n_k$, and
\[ \mu_k \sim N(x'_k \beta, \tau^2), \quad k = 1, \ldots, K. \]
This reduces to (1) when $\tau^2 = 0$. Under this setup the marginal distribution of the cell averages $\bar{Y}_1, \ldots, \bar{Y}_K$ is
\[ \bar{Y}_k \sim N(x'_k \beta, v_k + \tau^2). \]
Let $\gamma_k = v_k / (v_k + \tau^2)$. The posterior for $\mu_k$, treating $\beta, v_k$, and $\tau^2$ as known, is
\[ \mu_k \ind \sim N(\mu_k^*, v_k(1 - \gamma_k)), \]
where
\[ \mu_k^* = (1 - \gamma_k) \bar{Y}_k + \gamma_k (x'_k \beta). \]
This suggests the (infeasible) estimator
\[ \hat{\theta}_{EB0} = \sum_{k=1}^{K} p_k \left[ (1 - \gamma_k) \bar{Y}_k + \gamma_k (x'_k \beta) \right]. \]

To construct a feasible version, let $\hat{\beta}$ be the least squares estimator as in the imputation estimator. Let $\hat{\tau}^2$ be the pseudo maximum likelihood estimator of $\tau$ in (2), taking as given the regression estimates\(^1\) $\hat{\beta}$ and the following estimates of the $v_k$:
\[ \hat{v}_k = \frac{\hat{\sigma}_k^2}{n_k}, \]
where the $\hat{\sigma}_k^2$ are the within-cell sample variances of the $y_{ki}$. We then form
\[ \hat{\gamma}_k = \frac{\hat{v}_k}{\hat{v}_k + \hat{\tau}_k^2}, \]

\(^1\)We could estimate $\beta$ and $\tau$ jointly by pseudo maximum likelihood, but for our extensions below, this form is somewhat more convenient. Another alternative is to carry out full Bayesian hierarchical inference.
Ⅱ. Double robustness

James Robins and coauthors have proposed an alternative approach to robustifying estimators based on parametric mean restrictions. In the double robust (DR) approach, the empirical researcher posits a model for the means, and a model for the propensity score (in our notation, the \( e_k \)). A DR estimator is one that is consistent for the parameters of interest provided at least one of the two parametric restrictions are satisfied. Various DR estimators have been proposed, including James M. Robins, Andrea Rotnitzky and Lue Ping Zhao (1994), Keisuke Hirano and Guido W. Imbens (2001), Heejung Bang and James M. Robins (2005), Jeffrey M. Wooldridge (2007), Weihua Cao, Anastasios A. Tsiatis and Marie Davidian (2009), and Bryan S. Graham, Christine Campos de Xavier Pinto and Daniel Egel (2010).

Suppose we have two possible parametric restrictions:

**ASSUMPTION DR1:** \( \mu_k = x'_k \beta \) for all \( k \).

**ASSUMPTION DR2:** \( e_k = G(x_k) \) for all \( k \), where \( G \) is a known function.

Bang and Robins (2005) show that a DR estimator can be constructed by augmenting a regression with the inverse of the (parametric) propensity score. In our setup, this can be implemented through the following weighted linear projection problem: choose \( \alpha_1, \alpha_2 \) to solve

\[
\min_{\alpha_1, \alpha_2} \sum_{k=1}^{K} p_k E \left[ \left( \bar{Y}_k - x'_k \alpha_1 - G^{-1}(x_k) \alpha_2 \right)^2 \right].
\]

The results in Bang and Robins (2005) imply the following result, which we prove for completeness:

**PROPOSITION 1:** If DR1, DR2, or both hold, then

\[
\theta = \sum_{k=1}^{K} p_k \left[ x'_k \alpha_1^* + \alpha_2^* \right].
\]

**PROOF:**

The minimization problem (3) is equivalent to the problem

\[
\min_{\alpha_1, \alpha_2} \sum_{k=1}^{K} p_k e_k \left( \mu_k - x'_k \alpha_1 - G^{-1}(x_k) \alpha_2 \right)^2.
\]

First, suppose DR1 holds. Then clearly (4) is equivalent to

\[
\sum_{k=1}^{K} p_k \left[ x'_k \alpha_1^* + G^{-1}(x_k) \alpha_2^* \right] = \sum_{k=1}^{K} p_k \left[ x'_k \beta \right] = \sum_{k=1}^{K} p_k \mu_k = \theta.
\]

Next, suppose DR2 holds. The first order conditions for (4) implies

\[
\sum_{k=1}^{K} p_k \frac{e_k}{G(x_k)} \left( \mu_k - x'_k \alpha_1^* - G^{-1}(x_k) \alpha_2^* \right) = 0.
\]
Hence if \( e_k = G(x_k) \) for all \( k \),

\[
\sum_{k=1}^{K} p_k \mu_k = \sum_{k=1}^{K} p_k \left[ x_k' \alpha_1^* + G^{-1}(x_k) \alpha_2^* \right].
\]

To construct a feasible version of this estimator, let

\[
\hat{e}_k = \frac{n_k}{M_k}.
\]

Then \( p_k \hat{e}_k \propto n_k \), so we could solve

\[
\min_{\alpha_1, \alpha_2} \sum_{k=1}^{K} n_k \left( \bar{y}_k - x_k' \hat{\alpha}_1 - G^{-1}(x_k) \hat{\alpha}_2 \right)^2.
\]

This is WLS of \( \bar{y}_k \) on \( (x_k', G^{-1}(x_k))' \), with weights proportional to \( n_k \), and is equivalent to OLS of the observed \( y_{k_i} \) on \( (X_{k_i}', G^{-1}(X_{k_i}))' \).

Let \( \hat{\alpha}_1 \) and \( \hat{\alpha}_2 \) denote these estimates. The Bang and Robins DR estimator is

\[
\hat{\theta}_{DR} = \sum_{k=1}^{K} p_k \left[ x_k' \hat{\alpha}_1 + G^{-1}(x_k) \hat{\alpha}_2 \right].
\]

An empirical Bayes extension can be based on the marginal model

\[
\bar{y}_k \sim N(x_k' \alpha_1 + G^{-1}(x_k) \alpha_2, \sigma^2).
\]

We can form the empirical Bayes estimate exactly as before, after augmenting the regressor vector with the term \( G^{-1}(x_k) \). Note however that under Assumption DR2, we will not necessarily have \( \tau^2 = 0 \). This suggests that it may be useful to consider alternative estimators for \( \tau^2 \), which shrink the estimate to zero when the data indicate that the propensity score restriction is close to being satisfied. We defer such extensions to future work.

III. Monte Carlo study

We carry out a simple simulation study to compare the various estimators. Suppose the covariate cells are

\[
\{x_1, \ldots, x_K\} = \{-J, \ldots, 0, \ldots, J\},
\]

so that \( K = 2J + 1 \), and \( M_k = M \) for all \( k \), so that \( p_k = 1/K \). We specify \( \mu_k = x_k \beta \), which implies that \( \theta = 0 \). The propensity score is

\[
e_k = \begin{cases} 
0.75 & \text{if } x_k < 0 \\
(0.5K - 0.5J)/(K - J) & \text{if } x_k \geq 0
\end{cases}
\]

This gives an overall probability of \( 1/2 \) of observing the outcome. The outcomes \( y_{ki} \) are independently drawn from a normal distribution with mean \( \mu_k \) and variance \( \sigma^2 \) (the variance is constant across cells).

Under this model, the variance bound for estimating \( \theta \) is

\[
VB = \sum_{k=1}^{K} \frac{\sigma^2}{p_k \hat{e}_k}.
\]

(See Theorem 5 of Xiaohong Chen, Han Hong and Alessandro Tarozzi (2004) and Section 5.2 of Guido W. Imbens and Jeffrey M. Wooldridge (2009).)

We consider six designs with \( J \) chosen such that \( K = 5, 15, 25, 75, 125, \) and \( 375; N = 3000 \) across all designs such that the common cell sizes \( M = 600, 200, 120, 40, 24 \) and \( 8 \). We choose \( \sigma^2 \) based on \( K \) and \( M \) to set \( VB = 30 \). This implies that an efficient estimator should have a standard deviation of \( 0.1 \) in large samples. We also choose \( \beta \) so that the variance of \( \mu_k \) is equal to \( 30 \) in each design. For each design we perform 1000 Monte Carlo replications.

We apply the estimators developed above under two parametric specifications (where required). In the first, \( \mu_k \) is correctly assumed to be linear in \( x_k \). In the second, \( \mu_k \) is erroneously assumed to be constant over \( x_k \). To conserve space we only report the latter sets of results in detail.

These results are reported in Table 1. Each row of the Table corresponds to an estimator, with columns denoting the different designs. The entries show mean bias for each estimator/design as well as its standard deviation across Monte Carlo replications (in parentheses).

The sampling distribution of the post-stratification estimator \( \hat{\theta}_{PS} \) is well approximated by conventional asymptotic approximations for designs where the number cells \( K \) is small and cell size \( M \) is reasonably large. However, when \( K = 375 \) (so that \( M = 8 \)), the presence of empty cells induces substantial bias and inflates
variance.

Not surprisingly, the parametric imputation, double robust, and empirical Bayes estimators all perform well when they incorporate a correctly-specified conditional mean model (results not shown). When the conditional mean model is incorrect, as in Table 1, their properties diverge. The parametric imputation estimator is biased when it is based on an incorrectly specified conditional mean model. The double robust estimator exhibits low bias. Although in our experiments this estimator is also based on an incorrect conditional mean model, it does utilize the true propensity score. It sampling distribution, however, is relatively more dispersed than that of the post stratification estimator. The empirical Bayes estimator moderately outperforms the parametric imputation estimator across all designs. However for \( K \) large / \( M \) small it also exhibits substantial bias. Incorporating the true propensity score into the marginal model eliminates this bias. Importantly, the sampling distribution of this estimator is less dispersed than that of the double robust estimator with a standard deviation 15 to 20 percent smaller.

IV. Conclusion

In many applications the number of discrete covariate cells is large relative to the sample size. In such situations many cells may contain few, or even no, observations of the outcome of interest \( Y \). Using a parametric model to impute cell means is one approach to solving this empty cell problem. We have outlined an alternative approach to estimating cell means and associated population average parameters. In cells with many observed outcomes our approach is nonparametric; in cells with few such observations it is essentially parametric; while in intermediate cases we combine a nonparametric and parametric estimate of the cell mean. Incorporating the propensity score into our parametric imputation model appears to help guard against misspecification.

In further work it would be useful to explore other ways of choosing the \( \gamma_k \) and to formally characterize the large sample properties of our estimator. Of particular interest are asymptotic sequences which allow \( K \) to grow with \( N \), as first suggested by Angrist and Hahn (2004).

REFERENCES


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<th>K</th>
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Table 1—Monte Carlo Results for Incorrectly Specified Conditional Mean Model


