Making Efficient Public Good Decisions using an Augmented Ausubel Auction.

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Abstract

We provide the appropriate generalization of Ausubel's 2004 ascending bid auction to environments where the goods are non-rival and non-excludable. Like its private good counterpart, we show that the public good Ausubel auction encourages truthful revelation of preferences, is privacy preserving, and yields an equilibrium allocation that is outcome equivalent to the public good Vickrey sealed-bid auction. We then generalize this auction to fit a broader set of public good environments.

1 Introduction

Despite almost fifty years of research on incentive design, it remains unclear how, or even whether, a practical institution can be designed to overcome free-riding incentives in a public good environment. The public good provision problem becomes easier if consumers truthfully reveal their preferences to a decision maker. However, consumers' interests are rarely aligned with those of the decision maker. This difference creates incentives for the misrepresentation of preferences, often referred to as the "preference revelation"

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problem. Despite proposed solutions, existing methods tend to posses undesirable properties which either limit their applicability or hinder their behavioral success. In this paper, we look to recent contributions in dynamic private good auctions and experimental economics to give us insight into a new, and hopefully improved, way of overcoming the preference revelation problem in a public good setting.

Early success in preference revelation mechanisms came in a private good setting starting with the path breaking paper of Vickrey (1961). Among the results in the paper, Vickrey established a multi-unit *sealed bid* auction for efficiently allocating multiple units of a homogeneous good to the bidders. This auction was particularly nice because it encouraged truthful revelation of preferences as a dominant strategy and is relatively easy to study analytically. His mechanism was later, and independently, generalized by Clarke (1971) and Groves (1973) to accommodate public goods and is now commonly called the VCG mechanism. Loeb (1977) surveyed these three papers as well as Tideman and Tullock (1975) and Groves and Loeb (1976) and demonstrated how a multi-unit sealed bid Vickrey auction can be re-defined to make efficient decisions in public good settings.[1]

Despite the nice theoretical properties of the VCG mechanism, Rothkompf (2008) and others have pointed out that VCG mechanisms are far from being practical. In fact, he lists thirteen reasons for why VCG mechanisms may be difficult to implement. Among these reasons, a lack of privacy which may allow for dishonest practices by the seller. For example, in a private good case, the incentive for introducing a 'shill' bidder to increase prices. This property may make buyers hesitant about revealing their whole valuation schedule to an auctioneer. Privacy preservation is also a major concern in environments with public goods where consumers would prefer not reporting their entire valuation schedule to the government (especially when they may hold some belief that this action could influence how much they are taxed). A potential solution to this privacy preservation critique is to look for a *dynamic* public good auction that can make efficient public good decisions.

The idea that a dynamic auction may be better suited at privacy preservation is motivated by the private good auction literature. It is, perhaps, best illustrated in the case with a single private good comparing the second

[1]While this class of mechanisms (often referred to as Vickrey-Clarke-Groves (VCG) mechanisms) has been well studied, in this paper, we focus on the specific formulation of the public good Vickrey mechanism as defined by Loeb.

price sealed bid auction with a Japanese auction. In the second price sealed bid auction bidders submit their maximum willingness to pay to the auctioneer, the highest bidder wins and pays the second highest price. In the Japanese auction, or “button auction,” an auctioneer slowly raises the price and bidders signal each round whether they want to continue and are able to drop out if the price becomes to high, the last bidder still in the auction wins and pays the price the second to last bidder dropped out at. However, unlike the second price auction, in the Japanese auction the winner never reveals his maximum willingness to pay. The advantage, therefore, of looking at a dynamic auction is that the auction typically stops before bidders can reveal their whole valuation schedule. In this spirit, Ausubel (2004) introduced a dynamic auction procedure that retains the nice properties of the private good sealed-bid multi unit Vickrey auction (and is in fact outcome equivalent to the private good Vickrey auction) while, in general, preserving the privacy of some of the bidders.[2] In addition to being privacy preserving, this auction has been more “behaviorally” successful than its sealed bid counterpart when tested in the laboratory. Kagel and Levine (2001) find that bidders in the private good Ausubel auction do, in general, bid truthfully when compared with bidders in uniform price sealed bid auctions. In a follow up paper, Kagel, Kinross, and Levin (2001) compare the private good, multi-unit Vickrey auction against the private good Ausubel auction with different information feedback treatments. They find that the Ausubel auction with information feedback (i.e., knowledge of when bidders drop out of the auction) outperforms the sealed bid multi-unit Vickrey auction. The authors attribute this success to the observation that subjects in the Ausubel auction seem to benefit from the relative transparency of mechanism.

In this paper, we show that Ausubel's 2004 auction can be naturally redefined to fit a public good context such that truthful revelation of valuations, depending on the information feedback conditions, is a very robust refinement of a Nash equilibrium; second, the equilibrium of the induced auction and yields an outcome equivalent to the public good sealed bid Vickrey auction; third, we show that, in general, the public good Ausubel auction, hereafter the Ascending Quantity Ausubel Auction (AQ-AA), preserves the valuation privacy of *all* of the participants.[3] This relationship with the private good

[2]Bergemann and Välimäki (2007) is a dynamic treatment of the VCG mechanism in an infinite horizon environment.

[3]This is a departure from the private good case which, typically, preserves the privacy of only the bidder with the highest marginal valuations for last unit sold.

case is summarized in the following table.

		Type of Good		
		Private		*Public*
	Sealed-Bid	Vickrey (1961)	$\longleftrightarrow$	Loeb (1977)
		$\updownarrow$		$\updownarrow$
Format		Outcome Equivalent		Outcome Equivalent
		$\updownarrow$		$\updownarrow$
	Dynamic	Ausubel (2004)	$\longleftrightarrow$	AQ-AA (This Paper)

Finally, we discuss generalizing the public good Ausubel auction to include so called "public bads" and compare the outcome to outcomes yielded by a family of Groves mechanisms.

2 The Public Good Economy

The setting consists of $N \geq 2$ bidders that participate in an auction to determine the production level of some good x that is both non-rival and non-excludable. This good can be produced according to a non-decreasing *marginal cost* schedule $c = (c^1, ..., c^{\bar{x}}) \in \mathbb{R}_+^{\bar{x}}$ such that $c^1 \leq c^2 \leq \cdots \leq c^{\bar{x}}$, where $\bar{x}$ is the maximum production level. The set of all production levels is therefore $X = \{0, 1, ..., \bar{x}\}$, where for each unit x, the total production cost is

$$c(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sum_{t=1}^{x} c^t & \text{otherwise.} \end{cases}$$

On the bidder side, for each positive unit m produced, there is some *positive* reservation price $\theta_i^m \in \{0, \epsilon, 2\epsilon, ...\}$ that indexes each bidder i's maximum willingness to pay for that unit, where $\epsilon > 0$ is the minimum increment (i.e., pennies). Reservation prices for i are weakly decreasing in the public good, i.e., $\theta_i^1 \geq \theta_i^2 \geq \cdots \geq \theta_i^{\bar{x}}$. We denote an arbitrary profile of $\bar{x}$ reservation prices for bidder i by $\theta_i = (\theta_i^1, ..., \theta_i^{\bar{x}})$. Thus, the set of all such profiles for bidder i is denoted

$$\Theta_i = \{\theta_i = (\theta_i^1, ..., \theta_i^{\bar{x}}) |\ \theta_i^m \geq 0\ \forall m \text{ and }\ \theta_i^1 \geq \theta_i^2 \geq \cdots \geq \theta_i^{\bar{x}}\},$$

where $\Theta = \times_{i=1}^{N}\Theta_i$ is the set of bidder profiles. Types are independently drawn before the auction begins (i.e., at stage 0) according to the distribution $f_i(\cdot)$, where each bidder observes only his own type profile. The the joint probability mass function (PMF) is denoted by $f(\cdot)$. It will also be convenient, for some of our results, to bound the set of bidder profiles, where for any non-negative number k (which is a multiple of ϵ), let

$$\Theta_i(k) = \{(\theta_i^1(k), ..., \theta_i^{\bar{x}}(k))|\theta_i^m(k) \in \{0, \epsilon, 2\epsilon, ..., k\} \ \forall m, \ \theta_i^1(k) \geq ... \geq \theta_i^{\bar{x}}(k)\}$$

is the set of weakly decreasing *bounded* type profiles on the ϵ- grid $[0, k]$, where $\theta_i(k)$ is an arbitrary element.

Bidders care only about the level of the public good $x \in X$ produced and the amount of money $\tau_i \in \mathbb{R}_+$they have to pay toward production. We assume these preferences can be represented by a quasi-linear payoff function $u_i : X \times\Theta \times \mathbb{R}_+ \rightarrow \mathbb{R}$ which is equal to his private value, $v_i(x, \theta_i)$, for the quantity x of the public good produced less the total payment τ_i – i.e., $u_i(x, \tau_i, \theta) = v_i(x, \theta_i) - \tau_i$. The value $v_i(x, \theta_i)$ is assumed to be equal to i's total surplus,

$$v_i(x, \theta_i) = \begin{cases} 0 & \text{if } x = 0 \\ \sum_{j=1}^{x} \theta_i^j & \text{otherwise.} \end{cases}$$

An allocation determines the level of the public good produced and a total payment for each bidder, i.e., $(x, \tau_1, ..., \tau_N) \in X \times \mathbb{R}_+^N$. For this paper, we are interested in allocations with desirable social welfare properties such as Pareto optimal and ones that make efficient public good decisions but leave either budget surpluses or deficits. The following definitions formalize these two concepts.

Definition 1: The public good decision x is *efficient* if and only if it maximizes social surplus (i.e., $\hat{x} \in \arg\max_{x \in X} [\sum_{i=1}^{N} v_i(x, \theta_i) - c(x)]$).

Definition 2: A public good allocation is *Pareto optimal* if: the amount of the public good maximizes social surplus; and the payments collected from the bidders add up to the total cost of producing $\hat{x}$ (i.e., $\sum_{i=1}^{N} \hat{\tau}_i = c(\hat{x})$).

Thus, an institution can make an efficient public good decision, but if the taxes collected do not equal the cost of production the allocation is not Pareto optimal. We focus on auctions that satisfy the first definition.

Finally, some auxiliary assumptions used at various points in the paper include:

- **Bounded Type Assumption (BTA):** No bidder type prefers to unilaterally finance production of the public good – i.e., $k = c^1 - \epsilon$ is the maximum reservation price any bidder could have for any positive unit.

- **Bounded Marginal Cost Assumption (BMCA):** The marginal cost for the last unit $c^{\bar{x}} \leq (N-1)(c^1 - \epsilon)$.

- **Full Support Assumption (FSA):** For each i, the probability mass function $f_i(\cdot)$ from which bidder types are drawn has support equal to $\Theta_j(c^1 - \epsilon)$.

In the next section, we re-examine Loeb's treatment of the public good Vickrey auction. This will set the stage for introducing the Public good Ausubel auction.

3 The Public Good Vickrey Auction

Loeb (1977) naturally redefined a sealed bid private good auction (the multi-unit Vickrey auction) to apply to a *constant* marginal cost public good environment – i.e., for a public good environment with a cost function defined according to

$$c(x) = cx.$$

In this section, we generalize Loeb's description of the public good Vickrey auction to incorporate 'potentially' increasing marginal cost environments and describe how this public good Vickrey auction takes bids submitted by bidders and maps them into an allocation. While our description of the action space in the Vickrey auction differs from the one in Loeb, it is straightforward to check that the two descriptions are equivalent. The difference in description makes for a transparent comparison to the public good Ausubel auction later in the paper; additionally, the description followed in this paper is perhaps more natural when thinking about the Vickrey mechanism in an auction context. In the next section we describe how the actions taken by bidders when they participate in the auction are used to determine an outcome. In the subsequent section, we add bidder preferences to the analysis thereby allowing us to examine the Vickrey auction as a game.

3.1 Rules of the Public Good Vickrey Mechanism:

A sealed bid auction in a public good context needs to map the bids of the participating bidders into an allocation $(x, \tau_1, ..., \tau_N)$ to distribute to all of the bidders. The sealed bid Vickrey mechanism has each bidder submit a bid in the form of a demand schedule to the auctioneer, hereafter the government, who produces the level of the public good that maximizes social surplus and determines a tax payment for each bidder based on the demand schedules reported by all of the other bidders.[4]

Specifically, for each element $x \in X$, each bidder i sends a bid b_i^x to the government. For each bidder i, the set of all possible bids is $B_i = \Theta_i$, where $b_i \in B_i$ implies $b_i^1 \geq b_i^2 \geq \cdots \geq b_i^{\bar{x}}$. Let $b = (b_1, ..., b_N) \in B = \times_i B_i$ be an arbitrary profile of bids (one for each bidder) and $b_{-i} \in B_{-i} = \times_{j \neq i} B_j$ be an arbitrary collection of bids by bidders other than i. These bids can be viewed as each bidder's reported maximum willingness to pay of each of the possible units of the public good. The government wants to produce the level of public good that maximizes social surplus. It therefore takes the bidder's bids as proxies for their actual valuation schedule and uses these schedules to determine whether the reported value for a unit exceeds the marginal cost – i.e., for each i, the government constructs a reported valuation function $\tilde{v}_i : X \times B_i \rightarrow \mathbb{R}$, where

$$\tilde{v}_i(x, b_i) = \begin{cases} 0 & \text{if } x = 0 \\ \sum_{j=1}^{x} b_i^j & \text{otherwise.} \end{cases}$$

The government chooses the level of the public good that maximizes reported social surplus (i.e., $\hat{x}(b) \in \arg\max_{x \in X} \left[\sum_{i=1}^{N} \tilde{v}_i(x, b_i) - c(x)\right]$).

Given this decision rule of the government and the decreasing bid/ increasing marginal cost environment, a unit is efficient to produce if and only if the total bids for that unit $\sum_i b_i^x$ exceed the marginal cost. Thus, by bidding b_i^x for the x-th unit, bidder i is essentially "supplying" b_i^x units of the marginal cost to the other bidders. Alternatively, for any bidder i, the government can calculate how much of the per unit cost bidder i needs to supply in order for it to be reportedly efficient to produce that unit. We call this amount i's residual supply (or residual cost) function $\tilde{s}_i : X \times B_{-i} \rightarrow \mathbb{R}$, where for each $x \in X$, $\tilde{s}_i(x, b_{-i}) = \max\{0, c^x - \sum_{j \neq i} b_j^x\}$. Each bidder i

[4] As there may be multiple maximizing arguments, we shall assume in these cases that the planner always chooses the largest x.

is charged a tax equal to the reported residual supply for each positive unit that is produced and zero if no units are produced–i.e.,

$$\tau_i(x^*, b_{-i}) = \begin{cases} 0 & \text{if } x^* = 0 \\ \sum_{j=1}^{x^*} \tilde{s}_i(j, b_{-i}) & \text{otherwise.} \end{cases}$$

3.2 Truthful Revelation of Bidding Type as an Equilibrium

The next theorem shows that it is a dominant strategy for each bidder in the public good Vickrey auction to truthfully report their type profile. Loeb (1977), in a constant cost environment, defined this auction and furthermore showed that truth telling is a dominant strategy. As our environment is slightly more general than Loeb's, for completeness, we provide a proof and show that truth telling remains a dominant strategy.

Theorem 1 *In a public good Vickrey Auction with non-decreasing marginal cost, it is always a best response for each bidder to report his true type profile. Furthermore, if the BTA and BMCA are satisfied, then truth telling is a weakly dominant strategy for each bidder.*

Proof. See appendix. ■

In order to provide some intuition for some later results, it is useful to illustrate the workings of the public good Vickrey Auction through the following simple example where there are 3 bidders in the economy and the government has to decide how much of the public good to produce at a constant marginal cost of $10 per unit. Each of the bidders values the public good according to the following benefit schedule.

	Unit 1	Unit 2	Unit 3
Bidder A	14	4	2
Bidder B	4	3	2
Bidder C	3	2	1

The government makes its production decision using the Vickrey Auction.

In an (weakly) increasing marginal cost environment, the production of a unit of the public good is efficient if and only if the sum of the individual benefits for that unit exceeds the marginal cost for that unit. In the example, the efficient quantity to produce is one. As shown earlier, bidders A, B, and C each have a dominant strategy to submit their true type vectors – i.e., $b_A = (14, 4, 2)$, $b_B = (4, 3, 2)$, and $b_C = (3, 2, 1)$ respectively. Adding the bids up for each unit we have 21 for the first unit, 9 for the second unit, and 6 for the third unit. The public good Vickrey auction therefore specifies that one unit be produced. As for the Vickrey Tax, consider bidder A whose reported residual supply vector is equal to

$$(\tilde{s}_A(1, b_{-A}), \tilde{s}_A(2, b_{-A}), \tilde{s}_A(3, b_{-A})) = (3, 5, 7).$$

Since only one unit is produced, Bidder A's Vickrey Tax is equal to $\tilde{s}_A(1, b_{-A}) = 3$. Similarly, we can show the Vickrey tax of the other two bidders to each be zero. The total tax revenue is 3. Thus, in this example, the public good Vickrey mechanism runs a budget deficit.[5]

Given a bid profile, we can represent each bidder's reported demand (i.e., his reported benefit schedule), his residual supply, and his Vickrey tax graphically. The tax is interpreted as the area under residual supply up to where his demand intersect the residual supply. For example, Bidder A's reported demand, residual supply, and tax are illustrated in Figure 1.

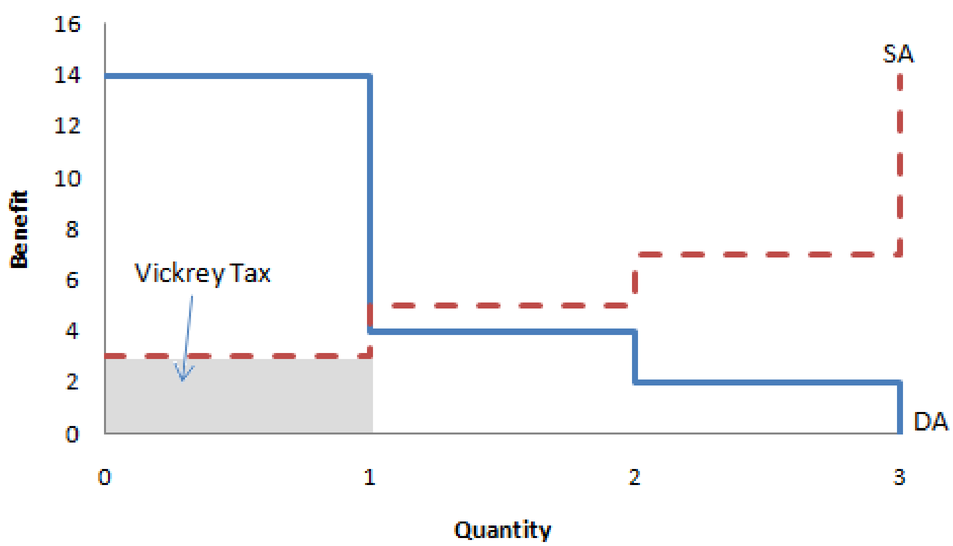

Figure 1: Revealed Bids and Tax for a Public Good Vickrey Auction

[5]The revenue generated in the public good Vickrey Auction always runs a deficit.

4 The Public Good Ausubel Auction

The Ausubel auction in the private good case is outcome equivalent to the multi-unit Vickrey auction. We now investigate whether the private good Ausubel auction can be redefined to fit the public good case and, if it can, whether the new auction remains outcome equivalent to the public good Vickrey auction.

In the private good Ausubel setting, all bidders take the auctioneer's price as given and respond with (potentially different) quantity bids. Because this price uniformly applies to all bidders in a private good setting, it can actually be thought of as a public good. In a public good setting, bidders face a uniform quantity of the public good, by definition, but have (potentially different) marginal valuations for each unit. Thus, given this dual nature of the private good/ public good provision problem just described, it is natural to re-define the private good Ausubel auction by using an ascending "quantity" auction instead of an ascending "price" auction. This is accomplished by having an auctioneer start by calling out a low quantity (instead of a price) and have individuals respond by submitting value bids (instead of quantity bids). If the bids exceed the marginal cost (for that unit), the auction continues and the auctioneer increases the quantity. This process continues until the sum of the bids no longer exceeds the marginal cost – i.e., until there is no longer excess inverse demand for the public good. The trick to making truth telling an equilibrium in such an auction is to disentangle what the bidders say and what they pay. A variation of Ausubel's "clinching rule" is a natural candidate.

4.1 Rules of the Ascending Quantity Ausubel Auction:

In this section, we outline the process by which the public good Ausubel auction maps bids into an allocation $(x, \tau_1, ..., \tau_N)$.

The auction begins at stage 1 where the initial level of the public good is set to be 1. There is a stage for each level of the public good, up to a maximum level $\bar{x}$. At each stage $t = 1, ..., \bar{x}$, the level of the public good announced is $x = t$ (i.e., the level of the public good increases in increments of 1 unit each stage) and each bidder i is required to submit a bid $b_i^t \in B_i^t$ to the auctioneer, where B_i^t is the set of feasible bids for bidder i at stage t.[6]

[6]We define this set more formally in the next section.

The profile of all bids in round t is $b^t = (b_1^t, ..., b_N^t)$, where $b^t \in B^t = \times_i B_i^t$. If $\sum_i b_i^t \geq c^t$, the auction continues ($x = t+1$), and it stops otherwise. Denote the round the auction stops by L. The quantity produced is

$$x^* = L - 1.$$

At each stage, bidders do not pay their bids, but rather some function of the bids of the other bidders. Specifically, let $\tilde{s}_i^t : X \times B_{-i}^t \to \mathbb{R}$ be i's residual supply function at stage t, where $\tilde{s}_i(t, b_{-i}^t) = \max\{0, c^t - \sum_{j \neq i} b_j^t\}$. At each t where $b_i^t \geq \tilde{s}_i(t, b_{-i}^t)$, bidder i accrues a tax equal to $\tilde{s}_i(t, b_{-i}^t)$. Since only L bids are observed in the auction, for each bidder i, let b_i denote the profile of submitted bids – i.e., $b_i = (b_i^1, ..., b_i^L)$. His total payment is therefore

$$\tau_i(L, b_{-i}) = \begin{cases} 0 & \text{if } L = 1 \\ \sum_{t=1}^{L-1} \tilde{s}_i(t, b_{-i}^t) & \text{otherwise} \end{cases}.$$

In words, a bidder accrues a positive tax in round t if his bid was "pivotal" to the auction continuing to round $t-1$ – i.e., a bidder i pays in round t if without his bid the auction would have stopped and his payment is equal to the pivotal amount (i.e. $c^t - \sum_{j \neq i} b_i^t$).

4.2 The Ascending Quantity Ausubel Auction (AQ-AA) as a Game

The rules of the Ausubel auction define how bids are mapped into an allocation. When we define what each bidder knows at each stage of the auction – i.e., the information structure, we will have a game form. Once preferences over the allocations are added we have a game. In this section we formalize the AQ-AA as a game.

The AQ-AA is a dynamic auction which takes place in stages, where T is the set of possible stages. These stages progress in ascending order – i.e., $t = 1, ..., T$. Nature moves at stage 0 and draws $\theta \in \Theta$ according to f, where each bidder observes his own type.[7] In each subsequent stage the bidders bid simultaneously. The collection of all possible bidding histories at stage t is

$$H^t = \begin{cases} \{\emptyset\} & \text{if } t = 1 \\ \{(b^1, ..., b^{t-1}) \in \times_{m=1}^{t-1} B^m\} & \text{otherwise,} \end{cases}$$

[7] For the first result in this section we allow f to be any distribution –i.e., we do not restrict attention to f that is product of its marginal distributions. Later, we restrict the f to be independent and identically distributed.

where $h^t = (b^1, ..., b^{t-1})$ denotes an arbitrary element of H^t. Some sequences of bids lead to the end of the auction. We denote the collection of possible terminal histories at stage t by

$$Z^t = \{h^{t+1} \in H^{t+1} | \sum_i b_i^m \geq c^m \text{ for } m < t \text{ and } \sum_i b_i^t < c^t\},$$

where z^t is an arbitrary element. The set of all terminal histories is $Z = \cup_t Z^t$.

Since the end of the auction determines an allocation $(x, \tau_1, ..., \tau_N)$ according to the rules which we have already specified, a bidder i's payoff function u_i is defined on terminal histories and type profiles – i.e.,

$$u_i : Z \times \Theta \to \mathbb{R}.$$

Alternatively, since strategies are complete plans of action they yield terminal histories when played out in a game. As such, we can define payoff functions on strategies rather than on terminal histories, and this is the convention we follow. However, in order to properly define the bidders' strategies we need to first specify what each bidder knows at each stage of the auction.

At stage t, each bidder i has information about the bidding history h^t. First, they know their own bid. Second, they are given information about the bids of the others according to some "feedback" rule $g_i(b^t)$. Denote the bid information i receives at stage t by H_i^t, where

$$H_i^t = \begin{cases} \{\emptyset\} & \text{if } t = 1 \\ \{((b_i^k, g_i(b^k)))_{k=1}^{t-1} | h^t \in H^t\} & \text{otherwise.} \end{cases}$$

This individualized history framework is general enough to accommodate a wide range of information conditions by varying that , such as "full bid information," "aggregate bid information," and "no bid information." These are three natural ones to investigate and are defined below.

Definition 3: Bidder i receives *full bid information* if, at each stage, $H_i^t = H^t$ – i.e., $g_i(b^k) = b_{-i}^k$.

Definition 4: A bidder receives *aggregate bid information* if, at each stage, he knows his own bid and the aggregate bid of his rivals in each previous stage,

$$H_i^t = \left\{ ((b_i^1, \sum_{j \neq i} b_j^1), ..., (b_i^{t-1}, \sum_{j \neq i} b_j^{t-1})) | \; h^t \in H^t \right\}$$

– i.e., $g_i(b^k) = \sum_{j \neq i} b_j^k$.

The final example is the case when a bidder receives no bid information. In this case, if the auction ends, then he knows it and receives a payoff. If the auction continues, then a bidder's history is just the bids that he is made in the past.

Definition 5: A bidder receives *no bid information* if, at each stage, he is only told whether the auction is going to continue or not– i.e.,

$$H_i^t = \left\{(b_i^1, 1), ..., (b_i^{t-1}, 1) | h^t \in H^t \text{ and } h^t \notin z^{t-1}\right\}$$

– i.e.,

$$g_i(b^k) = \begin{cases} 1 \text{ if } \sum b_m \geq c^m \\ 0 \text{ otherwise.} \end{cases}$$

Now, given this information structure, we define the set of actions or bids available to each bidder at any given stage. At each stage t, the set of feasible bids may depend on past bids. Write $B_i^t(h_i^t) \subseteq \{0, \epsilon, 2\epsilon, ...\}$ for the set of feasible bids for bidder i, at stage t given h_i^t. Let $B_i^t(H_i^t) = \bigcup_{h_i^t \in H_i^t} B_i^t(h_i^t)$ be the union of all such sets. At stage 1, the set of feasible bids trivially depends on the prior history, denote this set by $B_i^1(h_i^1) = B_i^1$ for each bidder i. Thus, the dynamic game Γ^e induced by the AQ-AA is a *multi-stage game of incomplete information* defined by

$$\Gamma^e = (N, T, f, Z, (\Theta_i, (H_i^t, B_i^t(\cdot))_{t=1}^T, u_i)_{i \in I}).$$

We have described what each bidder knows at each stage of the auction, it is now possible to define strategies for the game Γ^e.

Definition 6: A *pure strategy* β_i for bidder i in the game Γ^e is a collection of $T = \bar{x}$ functions (one for each stage), where each function maps a bidder's history and type into a feasible action–i.e., for each t, we have $\beta_i^t : H_i^t \times \Theta_i \rightarrow B_i^t(H_i^t)$ where $\beta_i^t(h_i^t, \theta_i) \in B_i^t(h_i^t)$.

Definition 7: A pure strategy β_i for bidder i is *"truth telling"* if for each stage t, each history $h_i^t \in H_i^t$, and each bidder type $\theta_i \in \Theta_i$, bidder i bids Θ_i^t –i.e., $\beta_i^t(h_i^t, \theta_i) = \theta_i^t$.

Finally, if the realization of types in the AQ-AA is common knowledge to all of the bidders, then the dynamic game of incomplete information reduces to a game of complete information. Specifically, for any $\hat{\theta} \in \Theta$, define each bidder i's realized utility function is $\hat{u}_i(\cdot, \hat{\theta}) = u_i(\cdot, \hat{\theta})$. The realized game of complete information $\Gamma^e(\hat{\theta})$ is defined by

$$\Gamma^e(\hat{\theta}) = (N, T, Z, ((H_i^t, B_i^t(\cdot))_{t=1}^T, \hat{u}_i(\cdot, \hat{\theta}))_{i \in I}).$$

As this is a new game, it is necessary to define a strategy in this game.

Definition 8: An *ex-post pure strategy* δ_i for bidder i in $\Gamma^e(\hat{\theta})$ is a collection of T functions (one for each stage), where each function maps a bidder's history into a feasible action–i.e., for each t

$$\delta_i^t : H_i^t \to B_i^t(H_i^t).$$

For any realization of types $\hat{\theta}$, a strategy profile β in Γ^e can be *projected* into an ex-post pure strategy profile of the realized game $\Gamma^e(\hat{\theta})$ simply by setting $\delta_i^t(h_i^t) = \beta_i^t(h_i^t, \hat{\theta}_i)$ for each t. The appropriate equilibrium concept for $\Gamma^e(\hat{\theta})$ would be subgame perfection.

In our dynamic game of incomplete information, one could define beliefs at each information set and use equilibrium concepts akin to perfect Bayesian or Sequential equilibrium. However, it is useful to look at a stronger equilibrium concept. This concept is *ex-post* perfect Nash equilibrium and is formally defined below.

Definition 9: A strategy profile β is an *ex-post perfect Nash equilibrium* of Γ^e if, for each $\theta \in \theta$, the ex-post pure strategy profile $\delta = (\delta_1, ..., \delta_N)$ is a subgame perfect Nash equilibrium of the game $\Gamma^e(\theta)$, where $\delta_i^t(h_i^t) = \beta_i^t(h_i^t, \theta_i)$ for each i and each t and h_i^t.[8]

An ex-post perfect Nash equilibrium is "regret-proof"– i.e., if a bidder were to find out the types of the other bidders, he would not want to change his strategy.

[8] In a static framework the concept is called an ex-post Nash equilibrium see, for example, Cremer and McClean (1985), Holmström and Myerson (1983), and Maskin (1992). Ausubel (2004) generalized this notion to continuation strategies at each information set.

4.3 Truthful Revelation of Bidding Type as an Equilibrium in the AQ-AA

A close inspection of the rules of the AQ-AA reveals that *if* all bidders follow truthful bidding strategies in every round, the auction continues if and only if the social value of producing the public good exceeds the marginal cost for that unit. In other words, truth telling by bidders leads to an efficient amount of the public good to be produced.

In the next theorem, we show that truth telling is an ex-post perfect Nash equilibrium of the AQ-AA.

Theorem 2 *In the full bid information AQ-AA, truth telling by each bidder is an ex-post perfect Nash equilibrium.*

Proof. See Appendix. ■

We can actually say more about the equilibrium outcome of the AQ-AA and Vickrey outcome under truth telling.

Corollary 1 *The truth telling equilibrium allocation of the AQ-AA is outcome equivalent to the truth telling equilibrium of the Public Good Vickrey Auction.*

The corollary follows from the fact that, in equilibrium, a unit is produced if and only if it is socially efficient to do so. In fact, the AQ-AA chooses the largest efficient quantity of the public good to produce. Therefore, when the auction ends, the public good level produced is the same that was produced in the truth telling equilibrium of the Vickrey Auction. Furthermore, since bidders pay their residual supply for each of the produced units, the tax payment for each consumer is equivalent in the AQ-AA and the Vickrey auction. The outcome, therefore, is equivalent to the truth telling Vickrey outcome.

Additionally, we show that truth telling in the *aggregate bid information* AQ-AA is an ex-post perfect Nash equilibrium. This is done by applying Theorem 2 to a slightly augmented 2 bidder version of the AQ-AA, where each player faces a representative agent of the $N-1$ other bidders. In the aggregate bid case this effectively reduces each bidder's problem to a 2 bidder full bid information AQ-AA. That truth telling is an ex-post perfect Nash equilibrium in the no bid information case follows almost immediately.

Corollary 2 *In both the aggregate bid information AQ-AA and the no bid information AQ-AA, truth telling by each bidder is an ex-post perfect Nash equilibrium.*

Proof. See Appendix. ■

To better illustrate the workings of the AQ-AA we re-consider the public good environment from the example in the previous section. This also allows us to provide a between-auction comparison. As shown in Theorem 2, it is an ex-post perfect Nash equilibrium for bidders A, B, and C to bid truthfully. In round 1, A, B, and C submit the bids $(b_A^1, b_B^1, b_C^1) = (14, 4, 3)$. Since $14 + 4 + 3 > 10$ the auction continues. Bidder A pays $s_A(1) = 3$ and the other two pay 0. In round 2, the quantity is increased to 2 and A, B, and C submit bids $(b_A^2, b_B^2, b_C^2) = (4, 3, 2)$. Since $4 + 3 + 2 < 10$, the auction stops. None of the bidders pay a tax in the second round and the final quantity produced is $2 - 1 = 1$. The total tax revenue is 3 (paid completely by Bidder A). Notice that this is exactly the outcome we arrived at in Example 1 using the Vickrey auction. Figure 2 illustrates Bidder A's revealed demand and residual supply. In particular, it preserves the privacy of the values for the 3rd unit (in contrast to Vickrey).

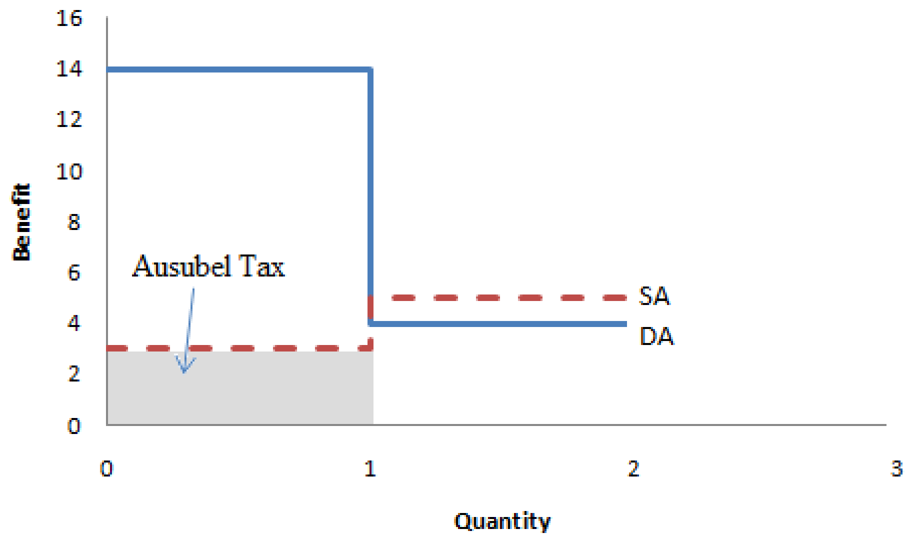

Figure 2: Revealed Bids and Tax for an Ascending Quantity Ausubel Auction

The example demonstrates that the AQ-AA preserves the privacy of all three bidders and chooses the efficient level of the public good. Moreover, the Ausubel Tax paid by each bidder is exactly the tax they would have paid under the public good Vickrey scheme demonstrating the outcome equivalence of the two procedures. Next, we formalize our notion of privacy preservation and compare the private and public good Ausubel auction in this respect.

Definition 10: We say an auction is *privacy preserving* for a consumer i if when the auction ends, i has only reported a strict subset of his type profile – in other words, the auctioneer cannot construct a complete demand schedule for the consumer.

The AQ-AA is, in general, privacy preserving for all of the participants. To see this is true, suppose the auction lasts L rounds, then, in equilibrium, each consumer i reports $b_i = (r_i^1, ..., r_i^L) \subseteq r_i$, where $b_i \subset r_i$ if $L \neq \bar{x}$. Thus, unless auction continues until the last period, the auction is privacy preserving for all consumers. This is quite a different observation than the privacy preservation property yielded in the private good case. The two settings turn out to diverge in the types marginal values that the auction protects (i.e., marginal valuations for earlier or later units), and the number of consumers for whom the auction is privacy preserving. The AQ-AA protects the marginal valuations for later units in the auction (i.e., the low marginal valuations), while, for bidders with positive demand at the end of the auction, the private good Ausubel auction protects their marginal valuations for the earlier units (i.e., the high marginal valuations). This last observation is an unfortunate property of the AQ-AA since it is the high valuations which the bidder would prefer that the government know.

In the next section, we show the conditions under which truth telling is a weakly dominant strategy as well as the technical circumstances under which bidders' truth telling strategies survive iterated elimination of weakly dominated strategies.

4.3.1 Weak Dominance and Iterated Elimination of Weakly Dominated Strategies in the AQ-AA

Unlike the Vickrey auction, truth telling in the AQ-AA under the current full bid information assumption, is *not* a weakly dominant strategy. In fact, it is not even always a best response to the bidding strategies of the other bidders. Consider a 2 bidder case where bidders A and B have the degenerate type profiles $\theta_A = (5,4)$, $\theta_B = (4,3)$ respectively that can bid any positive integer; and the public good can be produced at a marginal cost of \$6 per unit. We show truth telling is not a weakly dominant strategy by verifying that truth telling is not always a best response for Consumer A.

Consider the strategy profile $\beta = (\beta_A, \beta_B)$ in the "Full Bid Information" AQ-AA defined by:

$$\begin{aligned} \beta_A &= (5, \beta_A^2(h_A)), \text{ where } \beta_A^2(h_A) = 4 \text{ for all } h_A \\ \beta_B &= (4, \beta_B^2(h_B)), \text{ where } \beta_B^2(h_B) = \begin{cases} 0 \text{ if } b_A^1 = 5 \\ 3 \text{ otherwise.} \end{cases} \end{aligned}$$

In words, bidder A follows his truth telling strategy and B chooses an alternative, history dependent, strategy. Given B's strategy β_B, A's strategy β_A is not a best response. To see this, A's payoff from β_A (i.e., truth telling) is 3. This is since only one unit is produced and A's residual supply for the first unit is 2. If he switches to strategy

$$\tilde{\beta}_A = (4, \beta_A^2(h_A)), \text{ where } \tilde{\beta}_A^2(h_A) = 4 \text{ for all } h_A,$$

his payoff is 4. This is since 2 units of the public good are produced and A's residual supply is $s_A = (2,3)$. $\tilde{\beta}_A$ yields a higher payoff than β_A. Therefore

β_A is not a best response to β_B. Despite this negative observation about the full bid information AQ-AA, truth telling can be made a weakly dominant strategy if we restrict bids to be decreasing and the amount of information each bidder receives.

Changing the information bidders receive or feasible bids in each stage changes the game to be analyzed. Thus, appropriate changes to the game could make truth telling always a best response, or even weakly dominant. Consider the same 2 unit example as above with the "No Bid" Information condition being used, then each bidder's strategy specifies a bid for each round contingent on the auction continuing. If the bidding in each round is required to be monotonically decreasing, then each bidders strategy (in particular A's strategy) takes the form $\beta_A = (b_A^1, b_A^2)$, where $b_A^1 \geq b_A^2$. However, these are just the types of strategies used in the Vickrey auction that we saw earlier. Thus, given our result in Theorem 1, it should be straightforward to show that, under these conditions, truth telling is weakly dominant for the example as well as in general. This turns out to be true.

In the AQ-AA, if we restrict bids to be *"monotonically decreasing,"* and give *"no bid information"* to agents in each round then truth telling will be a weakly dominant strategy. The definition of "no bid information" is defined earlier in the paper. The 'monotonic bidding constraint' is just a constraint of feasible bids that requires bidders to submit weakly decreasing bids during the auction.

- **Monotonic bidding constraint (MBC):** restricts bids to be weakly decreasing –i.e., for any bid b_i^{t-1} the set of bids available to i in the t-th stage, $B_i^t(h_i^t)$, is bounded above by b_i^{t-1}.

We now formally state the result and its proof.

Theorem 3 *If MBC is imposed, then truthful revelation of bidding type in each round is always a best response to any bidding strategies of rival bidders in the "no bid information" AQ-AA. Furthermore, if the BTA and BMCA are satisfied, then truth telling is a weakly dominant strategy.*

Proof. See Appendix. ■

While weak dominance is desirable, the extra restrictions are strong. Next, we show what can be achieved with iterated elimination of weakly dominated strategies, hereafter IEWDS, using less restrictive conditions on information and feasible bidding. As in Ausubel (2004), it turns out with a mild technical assumption on the distribution of types (i.e., the FSA), we can show that truth telling for each bidder is the unique surviving strategy after IEWDS (independent of the order of elimination). The following example illustrates.

Let $X = \{0, 1, 2\}$, each unit can be produced at a constant marginal cost of $c^1 = c^2 = 2$, the minimum bid increment is $\epsilon = 1$, and $N = 3$. Furthermore, assume for each $j = A, B, C$, the FSA is satisfied, and the set of bids available at each stage t is $B_j^t(h_j^t) = \{1, 0\}$. To start, consider Bidder A when his type is $\theta_A = (1, 0)$ and he is comparing the following family of strategies:

$$\begin{aligned} \beta_A &= (b_A^1, \beta_A^2(h_A)), \text{ where } \beta_A^2(h_A) = 0 \text{ for all } h_A \\ \hat{\beta}_A &= (b_A^1, \hat{\beta}_A^2(h_A)), \text{ where } \beta_A^2(h_A) = 1 \text{ for some } h_A \\ & \qquad \text{where the auction continues} \\ & \qquad \text{to the second stage,} \end{aligned}$$

where b_A^1 is some arbitrary bid in the first round. Further suppose bidders B and C follow β_B and β_C respectively. Since β_A and $\hat{\beta}_A$ prescribe the same action in the 1st stage, if the auction ends in the 1st stage the two strategies yield the same payoff. Suppose the second stage is reached. Because bidders B and C both satisfy the FSA, we know: (1) there are truth telling strategies by these bidders that cause the auction to continue to the second stage independent of b_A^1; and (2) there are truth telling strategies that yield the following aggregate bid outcomes in the second stage: $b_B^2 + b_C^2 = \{2, 1, 0\}$.[9]

If $b_B^2 + b_C^2 = 2$, then $s_A^2 = 2 - 2 = 0$ and the second unit is produced no matter what strategy Bidder A follows – so anything is a best response (including truth telling). If $b_B^2 + b_C^2 = 1$, then $s_A^2 = 2 - 1 = 1$ and the second unit is produced if Bidder A follows $\hat{\beta}_A$, but not with β_A. However, since the second unit is only worth 0, Bidder A would prefer the second unit not be produced (which he could have ensured by telling the truth). Finally, if $b_B^2 + b_C^2 = 0$, anything Bidder A bids would be a best response. Therefore β_A weakly dominates $\hat{\beta}_A$. In a similar manner, we can do this the other types of

[9] If we only had two players this would not be possible.

Bidder A and from symmetry, we can do the same thing for bidders B and C's truth telling strategies in the second stage.

Once we eliminate all of these non-truth telling weakly dominated strategies, then nothing Bidder A does in the 1st stage can influence bidder B and C's strategy in the second stage (since they always report their type). Again looking at $\theta_A = (1, 0)$, we can show β_A weakly dominates the strategy of the form:

$$\check{\beta}_A = (0, \beta_A^2(h_A)), \text{ where } \beta_A^2(h_A) = 0 \text{ for all } h_A.$$

However, this is the last remaining non-truth telling strategy for this type. Repeating this for all of Bidder A's types and the other bidders, we eliminate all strategies other than truth telling.

Thus, in the example, we provided an order of eliminating weakly dominated strategies that left truth telling. However, this observation might be sensitive to the order of elimination– i.e., another order of elimination might discard truth telling as a strategy for some realization of type of some bidder. In the following lemma we show that the truth telling strategy is never eliminated by IEWDS.[10]

Lemma 1 *Suppose $N \geq 3$, BTA, BMCA, and the FSA are satisfied, then truthful bidding at each round is never eliminated by iterated elimination of weakly dominated strategies (IEWDS).*

Proof. See Appendix. ■

Given the lemma we know that truthful revelation of bidding type can never be eliminated by IEWDS. The next theorem demonstrates that truthful revelation of bidding type is in fact the unique outcome of IEWDS independent of the order strategies are eliminated.

Theorem 4 *Suppose $N \geq 3$, BTA, BMCA, and the FSA are satisfied, then truthful revelation of bidding type is the unique strategy to survive iterated elimination of weakly dominated strategies independent of the order of elimination.*

[10] This mirrors the second half of Ausubel's proof of Theorem 2.

Proof. See Appendix. ■

The above result required no additional restrictions on the information feedback individual bidders received.

In the next section, we outline how our intuition from the AQ-AA can be extended to include environments where bidders may become satiated or even receive "dis-utility" from the public good.

5 Extending the AQ-AA to Include Satiation and Public "Bads"

In the previous section, we were concerned with public good environments with the property that every bidder's marginal valuation for each good was *non-negative*. This obviously limits the number of applicable environments as there are many examples of goods where this property may not hold. The ability to incorporate satiation and negative evaluations in the system of preferences seems critical to developing a mechanism that is robust to a large set of situations. In this section, we discuss how one might generalize the AQ-AA to include such preference environments where some bidders may have a negative marginal valuations over some range of their type profile – i.e., the setting is exactly the same as before, except for each *positive* unit $x \in X$, there is some reservation price θ_i^x (potentially negative) that indexes bidder i's maximum willingness to pay for that unit. This number may be positive or negative. We maintain the assumption of weakly decreasing marginal valuations. It should be noted that the generalization of the AQ-AA comes with a cost. In the previous environments, each bidder only ever paid a tax. There was never any subsidies (or negative tax amounts). It turns out that in order to include public bads it becomes necessary to add subsidies into the mix. We now define the new auction, which we have dubbed the Generalized Ascending Quantity Ausubel Auction (GAQ-AA).

5.1 Rules of the Generalized Ascending Quantity Ausubel Auction:

In this section, we outline the process in which the GAQ-AA takes bidders' bids in each stage of the auction and converts them into an allocation

$(x, \tau_1, ..., \tau_N)$. The main difference between this auction and the one introduced in the last section is the potential usage of subsidies in addition to taxes at every round.

The auction begins at round 1 where the initial level of the public good is set to be 1. At each round $t = 1, ..., \bar{x}$, the level of the public good announced is $x = t$ (i.e., the level of the public good increases in increments of 1 unit per round) and each bidder i is required to submit a bid b_i^t to the auctioneer. If $\sum_i b_i^t \geq c^t$, the auction continues $(x = t + 1)$, and stops otherwise. Denote the round the auction stops by L. The quantity produced at the end of the auction is

$$x^* = L - 1.$$

Bidders do not pay their bids, but rather some function of the bids of the other bidders. Specifically, let $\tilde{s}_i : X \rightarrow \mathbb{R}$ be i's reported residual supply function, where $\tilde{s}_i(x) = c^x - \sum_{j \neq i} b_j^x$. At each t where $\sum b_i^t \geq c^t$, bidder i receives a transfer equal to $\tau_i(t) = \tilde{s}_i(t)$. Note if $\tau_i(t) < 0$, i receives a subsidy, and pays a tax otherwise. In words, bidder i pays a positive amount in round t if without his bid the auction would have stopped and receives a subsidy if without his bid the auction would not have stopped. His tax (subsidy) is then equal to the pivotal amount (non-pivotal amount) (i.e. $c^t - \sum_{j \neq i} b_i^t$) and total payment is

$$\tau_i(L) = \begin{cases} 0 & \text{if } L = 1 \\ \sum_{t=1}^{L-1} \tilde{s}_i(t) & \text{otherwise} \end{cases}.$$

Observation 1: Under similar assumptions and method of proof, Theorems 2, 3, and 4 of the AQ-AA can all be replicated for the GAQ-AA.

However, while these theorems show that all of the previous results continue to hold and that the GAQ-AA can accommodate a more general public good setting, the budget deficit under the GAQ-AA is worse than the AQ-AA since the government must now subsidize bidders.

Observation 2: The truth telling equilibrium of the GAQ-AA is outcome equivalent to the truth telling equilibrium of a specific Groves mechanism.

Given that the Vickrey mechanism belongs to the larger class of strategy-proof Groves mechanisms, it should not be surprising that the outcome of the GAQ-AA is outcome equivalent to another standard Groves mechanism.[11] While this observation might signal that a dynamic mechanism can be cooked up that is outcome equivalent to any Groves mechanism, *this is not the case.* It is reasonably easy to construct counter examples. The easiest being a Groves mechanism that violate the participation constraint such as the original Clark mechanism.

6 Conclusion

This paper has shown that the relationship between the dynamic private Ausubel auction and the static Vickrey auction can be extended to public good environments in a natural way by exploiting the dual nature of the private good/ public good problem. Once the description of the Ausubel auction has been augmented to fit the new public good environment and the appropriate assumptions/restrictions are introduced many of Ausubel's same results apply. This is encouraging. The auction seems relatively simple, transparent, and, unlike many Nash efficient public good mechanisms, scales well to increases in the number of bidders. Moreover, the strategic incentives to report their true valuation for the public good at each stage in the auction can be made relatively strong by manipulating information and bidding restrictions. We also demonstrated how one might extend these mechanisms to cover more general public good environments.

Interestingly, the AQ-AA is similar, in spirit, to a family of planning procedures initially studied by Malinvoud (1971) and Dreze and Pousin (1971). Often referred to as MDP processes, these mechanisms are individually rational and converge to Pareto optimal outcomes. In these mechanisms, at each time period, consumers report willingness to pay for a one unit increase (usually in continuous time). Similar to the AQ-AA the level of the public good adjusts up or down depending on whether reported social benefit exceeds or falls short of marginal cost. Consumers then pay their maximum willingness to pay for each unit and are also compensated by a positive share of the social surplus such that the sum of all of the consumer's shares is equal to one. The MDP process can be adopted to converge to any Pareto optimum through different divisions of the social surplus of each unit. Unfortunately, MDP

[11] For a textbook discussion of this Groves mechanism, see, for example, Varian p.427, where $c = 0$.

processes are generally vulnerable to strategic manipulation.[12] The AQ-AA, on the other hand, can be thought of as a discrete time, incentive compatible *pseudo*-MDP process. It diverges from other MDP processes in its taxation rule – i.e., its final allocation is not Pareto optimal. The divergence in outcomes results from each consumer paying his maximum willingness to pay for each unit but, unlike MDP processes, each consumer is compensated by a full share of consumer surplus for that unit. Collectively, his payment reduces to the area under his residual supply as we defined in the paper.

Despite the nice properties of the AQ-AA and GAQ-AA, these auctions have several obvious shortcomings (which are unique to public good mechanisms) that leave room for further research. First, these mechanisms do not, in general, generate enough revenue to finance the production of the public good. This type of problem is a common one in public good mechanism settings. It would, therefore, be interesting if one could generate a dynamic auction that at least always guarantee a surplus. This issue was attacked for sealed bid VCG mechanisms by Clarke (1971) and Groves and Loeb (1975), but the same techniques (an arbitrary per unit tax) applied in those papers to guarantee a budget surplus will not work for this particular context. Both the AQ-AA and the generalized AQ-AA procedures work because truthful revelation at each stage of the auction is always individually rational. In other words, when production is efficient each bidder's profit is weakly increasing. Assigning fixed shares of the cost for each unit (as in Clarke's original paper) and then apply the same procedure, would typically leave incentives for people to end the auction earlier than they otherwise would to avoid paying this tax. In other words, adding fixed cost shares may eliminate the individual rationality property from a truth-telling strategy and a loss of efficiency.[13] The second concern is that while the auction is, in general, privacy preserving for each bidder, the auction only preserves the lowest marginal valuations of the participants. One plausible solution might be to formulate a decreasing quantity rule that makes efficient decisions. If the quantity is decreasing then when the auction ends, it is the highest valuations of the bidders that are protected. Another option is to define increasing bid dynamic auction that

[12] See Laffont (1988) for an excellent textbook dicussion of the MDP processes, their properties, and how these mechanisms are vulnerable to strategic manipulation.

[13] The exception to this statement would be if the fixed cost shares were the ones prescribed in a Lindahl equilibrium. In this case, truthful revelation is always individually rational. However, knowledge of Lindahl prices requires knowledge of individual preferences which will not be known.

accomplishes the same sort of properties as the AQ-AA. Neither of these directions have been pursued by this author.

7 Appendix

Proof of Theorem 1:. Bidder i reports bid $b_i \in B_i$, which determines a value function

$$\tilde{v}_i(x, b_i) = \begin{cases} 0 & \text{if } x = 0 \\ \sum_{j=1}^{x} b_i^j & \text{otherwise.} \end{cases}$$

We can equivalently think of bidder i as reporting a value function $\tilde{v}_i(x, b_i)$. We first show that submitting his true value function, i.e., choosing $\tilde{v}_i(x, b_i) = v_i(x, \theta_i)$, is a best response to any profile of bids b_{-i}, or equivalently, any profile of value functions of the other bidders.

Suppose each bidder $j \neq i$ bids $b_j = (b_j^1, ..., b_j^{\bar{x}})$ with corresponding value function $\tilde{v}_j(x, b_j)$. In the Vickrey mechanism, the bid profile $b = (b_i, b_{-i})$ determines an allocation $(x, \tau_1, ..., \tau_N)$ by selecting an

$$x(b) \in \arg\max_{x \in X} [\sum\nolimits_m \tilde{v}_m(x, b_m) - c(x)]$$

and a tax for each bidder according to

$$\tau_m(x, b_{-m}) = \begin{cases} 0 & \text{if } x = 0 \\ \sum_{k=1}^{x} \tilde{s}_m(k, b_{-m}) & \text{otherwise} \end{cases}$$

where

$$\tilde{s}_m(k, b_{-m}) = \max\{0, c^k - \sum_{j \neq m} b_j^k\}.$$

Bidder i's payoff is $v_i(x, \theta_i) - \tau_i(x, b_{-i})$ which for $x > 0$, is $v_i(x, \theta_i) - \sum_{k=1}^{x} \tilde{s}_i(k, b_{-i})$.

Suppose x_0 units are produced when $b_i^1 = ... = b_i^{\bar{x}} = 0$ – i.e., $x_0 = x_0(0, b_{-i}) \in \arg\max_{x \in X} [\sum_{j \neq i} \tilde{v}_j(x, b_j) - c(x)]$. If the maximizer is not unique, let x_0 be the largest maximizer. There are two relevant cases:

Case 1, $x_0 = \bar{x}$: If $x_0 = \bar{x}$ then all units of the public good get produced no matter what i bids. He is therefore never pivotal, and receives the same payoff regardless of his bid –i.e., any bid (including truth telling) is a best response.

Case 2, $x_0 < \bar{x}$: Since x_0 is a maximizer, we have

$$[\sum_{j\neq i} \tilde{v}_j(x_0, b_j) - c(x_0)] - [\sum_{j\neq i} \tilde{v}_j(x_0 - 1, b_j) - c(x_0 - 1)] \geq 0,$$

which reduces to

$$\sum_{j\neq i} b_j^{x_0} - c^{x_0} \geq 0,$$

and hence

$$\tilde{s}_i(x_0, b_{-i}) = \max\{0, c^{x_0} - \sum_{j\neq i} b_j^{x_0}\} = 0.$$

Since bids are weakly decreasing and marginal costs are weakly increasing, it follows that $\tilde{s}_i(x, b_{-i})$ is weakly increasing and therefore $\tilde{s}_i(x, b_{-i}) = 0$ for all $x \leq x_0$.

Additionally, since x_0 is the largest maximizer we have

$$[\sum_{j\neq i} \tilde{v}_j(x_0, b_j) - c(x_0)] - [\sum_{j\neq i} \tilde{v}_j(x_0 + 1, b_j) - c(x_0 + 1)] > 0,$$

– i.e.,

$$c^{x_0+1} - \sum_{j\neq i} b_j^{x_0+1} > 0,$$

and hence

$$\tilde{s}_i(x_0 + 1, b_j) = c^{x_0+1} - \sum_{j\neq i} b_j^{x_0+1} > 0.$$

Since $c^x - \sum_{j\neq i} b_j^x$ is weakly increasing, then $\tilde{s}_i(x, b_{-i}) > 0$ for all $x > x_0$. Therefore x_0 is the largest x such that $\tilde{s}_i(x, b_{-i}) = 0$.

The level of the public good when i bids b_i (rather than the zero vector) is $\hat{x} = \hat{x}(b_i, b_{-i}) \in \arg\max_x [\sum_j \tilde{v}_j(x, b_j) - c(x)] \geq x_0$, since i is constrained to submit nonnegative bids. We have $\tilde{s}_i(x, b_{-i}) = 0$ for $x \leq x_0$ and $\tilde{s}_i(x, b_{-i}) > 0$ for $x > x_0$. Hence, bidder i's payoff is

$$\begin{aligned} v_i(\hat{x}, \theta_i) - \tau_i(\hat{x}, b_{-i}) &= v_i(\hat{x}, \theta_i) - \sum_{k=x_0+1}^{\hat{x}} \tilde{s}_i(k, b_{-i}) \\ &= v_i(\hat{x}, \theta_i) - \sum_{k=x_0+1}^{\hat{x}} \left(c^k - \sum_{j\neq i} b_j^k \right). \end{aligned}$$

Since $\sum_{k=x_0+1}^{\hat{x}} c^k = c(\hat{x}) - c(x_0)$ and $\sum_{k=x_0+1}^{\hat{x}} \sum_{j\neq i} b_j^k = \sum_{j\neq i} \sum_{k=x_0+1}^{\hat{x}} b_j^k = \sum_{j\neq i}[\tilde{v}_j(\hat{x}, b_j) - \tilde{v}_j(x_0, b_j)]$ we can write bidder i's payoff as

$$v_i(\hat{x}, \theta_i) - \left(c(\hat{x}) - c(x_0) - \sum_{j\neq i}[\tilde{v}_j(\hat{x}, b_j) - \tilde{v}_j(x_0, b_j)] \right).$$

Adding the constant $-c(x_0) + \sum_{j\neq i} \tilde{v}_j(x_0, b_j)$ to this expression, we have

$$v_i(\hat{x}, \theta_i) + \sum_{j\neq i} \tilde{v}_j(\hat{x}, b_j) - c(\hat{x}).$$

Note that adding a constant doesn't alter the value of $\hat{x}$ which maximizes i's payoff.

Recall x is chosen to maximize $\tilde{v}_i(x, b_i) + \sum_{j\neq i} \tilde{v}_j(x, b_j) - c(x)$. Thus, given the arbitrary bid profile b_{-i} and corresponding the $\sum_{j\neq i} \tilde{v}_j(x, b_j)$, bidder i maximizes his payoff by choosing $b_i = \theta_i$ or $\tilde{v}_i(x, b_i) = v_i(x, \theta_i)$.

If bidder i doesn't report truthfully, then there is a profile of reports by the other bidders such that bidder i is worse off than under truth telling. Hence a truthful report weakly dominates any other report.[14] ■

Claim 1 *If bidder i doesn't report truthfully, then there is a profile of reports by the other bidders such that bidder i is worse of than under truth telling. More formally, if* $\tilde{v}_i(x) \neq v_i(x)$ *then there exists some* $\sum_{j\neq i} \tilde{v}_j(x)$ *such that* $v_i(x^*) - \tau_i(x^*) > v_i(\tilde{x}) - \tau_i(\tilde{x})$, *where* $x^* = \max_{x\in X}[v_i(x) + \sum_{j\neq i} \tilde{v}_i(x) - c(x)]$ *and* $\tilde{x} \in \max_{x\in X}[v_j(x) + \sum_{j\neq i} \tilde{v}_j(x) - c(x)]$.

Proof. Suppose unit k is the first unit for which i does not report truthfully – i.e., i submits $b_i = (\theta_i^1, ..., \theta_i^{k-1}, b_i^k, ..., b_i^{\bar{x}})$, where $b_i^k \neq \theta_i^k$. Suppose $b_i^k > \theta_i^k$. We now construct a feasible profile b_{-i} that yields a lower payoff to i than he would obtain under truth telling. Specifically, choose b_j such that

$$b_j^m = \begin{cases} c^1 - \epsilon & \text{for } m = 1, ..., k-1 \\ F_j & \text{for } m = k \\ 0 & \text{for } m = k+1, ..., \bar{x} \end{cases}$$

where $F_j \in \{0, \epsilon, ..., c^1 - \epsilon\}$ is chosen such that $\sum_{j\neq i} F_j = c^{k-1} - b_i^k$. This expression, by assumption, is positive since $c^{k-1} - b_i^k \geq c^1 - b_i^k > 0$. Also,

[14] See Claim 1, in the appendix, for a proof of this last claim.

note that this profile is weakly decreasing and is therefore feasible. By the assumption, $c^m \leq (N-1)[c^1-\epsilon]$ for all m, and since $\sum_{j\neq i} b_j = (N-1)[c^1-\epsilon]$ for $m = 1, ..., k-1$, exactly $k-1$ units are produced regardless of i's bid and, given b_i and b_{-i}, the kth unit is also produced. ■

If bidder i had told the truth – i.e., $b_i = \theta_i$, then $k-1$ units would have been produced. This is by construction– i.e., $\sum_{j\neq i} b_j^k = c^k - b_i^k$ and $\theta_i^k < b_i^k$, therefore $0 \leq \theta_i^k < c^k - \sum_{j\neq i} b_j^k = \tilde{s}_i(k)$. The difference in i's payoff from k and $k-1$ units is

$$\begin{aligned} v_i(k) - \tau_i(k) - [v_i(k-1) - \tau_i(k-1)] &= \theta_i^k - [\tau_i(k) - \tau_i(k-1)] \\ &= \theta_i^k - \tilde{s}_i(k) < 0. \end{aligned}$$

Hence, bidder i is worse off as a result of not reporting truthfully. A similar argument applies for $b_i^k < \theta_i^k$.

Claim 2 *If θ_i is not bounded above, truth telling is not weakly dominant.*

Proof. Suppose $\theta_i = (c^{\bar{x}}, ..., c^{\bar{x}})$ and consider the non-truthful bid $b_i = (c^{\bar{x}} + \epsilon, ..., c^{\bar{x}} + \epsilon)$. Notice, $\bar{x}$ units get produced whether i bids θ_i or b_i regardless of the other bidders strategy profile b_{-i}. Also, for any b_{-i}, i's payoff from truth telling is identical to his payoff under b_i. Thus, truth telling does not weakly dominate b_i and is therefore not a weakly dominant strategy. ■

Proof of Thorem 2:. Suppose that the profile of truthful bidding strategies $\beta = (\beta_1, ..., \beta_N)$ is not an ex-post perfect Nash equilibrium, then there exists some realization of types $\bar{\theta} \in \Theta$ such that the projection of strategy profile β to $\Gamma^e(\bar{\theta})$, hereafter δ, is not subgame perfect. Since $\bar{x} < \infty$ and there is "full bid information," $\Gamma^e(\bar{\theta})$ is a finite horizon game with observed action. Thus, the fact that δ is not subgame perfect implies it does not satisfy the "one-stage-deviation principle" for finite horizon games.[15] This means that there is some bidder i and strategy $\tilde{\delta}_i$ that agrees with δ_i except at a single t and h^t, where $\tilde{\delta}_i$ is a better response to δ_{-i} than δ_i conditional on history h^t being reached.

Suppose the auction ends at stage L (i.e., $x^* = L-1$) if all bidders report truthfully. Given the truthful reports of the other bidders, a 1 stage

[15] See, for instance, Fudenberg and Tirole p. 109.

deviation in the AQ-AA can only result in three outcomes: Case 1, the deviation doesn't change the outcome (i.e., the auction ends at L); Case 2, the auction ends earlier in some round $E < L$; Case 3, the auction ends in round $L + 1$.

Case 1 is obviously not a profitable deviation since the auction ends at the same round and bidder i's payment is independent of his own action.

Suppose Case 2 is true, then the one stage deviation causes the auction to end earlier than L say round E ($x^* = E - 1$). Bidder i's payoff is

$$v_i(E-1, r_i) - \tau_i(E, b_{-i}).$$

The payoff from truth telling is

$$v_i(E-1, r_i) - \tau_i(E, b_{-i}) + \sum_{k=E}^{L-1} [\theta_i^k - \tilde{s}_i(k, b_{-i}^k)].$$

For each $k \in \{E, ..., L-1\}$ it must be true that $\sum_{j=1}^{N} \theta_j^k \geq c^k$, since otherwise the auction would have ended earlier when everyone was truthfully reporting. However, this implies that $\theta_i^k > c^k - \sum_{j \neq i} \theta_j^k$. Since $\theta_i^k \geq 0$, by assumption, we also have $\theta_i^k \geq \max\{0, c^k - \sum_{j=1}^{N} \theta_j^k\} = \tilde{s}_i(k, b_{-i}^k)$. Thus, for each k, we have $\theta_i^k \geq \tilde{s}_i(k, b_{-i}^k)$ which in turn implies $\sum_{k=E}^{L-1} [\theta_i^k - \tilde{s}_i(k, b_{-i}^k)] \geq 0$. Therefore ending the auction before L cannot lead to a profitable deviation.

Suppose Case 3 is true, then the one stage deviation causes the auction to end at round $L+1$ (i.e., $x^* = L$ being produced).[16] Bidder i's payoff from this deviation is

$$v_i(L-1, \theta_i) + \theta_i^L - \tau^i(L, b_{-i}) - \tilde{s}_i(L, b_{-i}^L).$$

Bidder i's payoff from truthful reporting is

$$v_i(L-1, \theta_i) - \tau_i(L, b_{-i}).$$

The single stage deviation is profitable only if and only if $\theta_i^L - \tilde{s}_i(L, b_{-i}^L) > 0$. However, when bidders were bidding truthfully the auction stopped at L. From the continuation rule of the AQ-AA, it must be that $\sum_i \theta_i^L < c^L$ or $0 \leq \theta_i^L < c^L - \sum_{j \neq i} \theta_j^L = s_i(L, b_{-i}^L)$, a contradiction.

[16] That the deviation has to come at this stage follows from the assumption of weakly decreasing marginal valuations and the fact that $\tilde{b}_i$ deviates from b_i in only one stage.

Thus, there is therefore no one-stage deviation that yields a higher payoff. This contradicts our assumption that truthful revelation was not subgame perfect in the realized game. Thus, β is ex-post perfect. ∎

Proof of Corollary 2. First, consider the aggregate bid information AQ-AA. For each i, consider the 2 bidder game between i and a representative agent for the other $N-1$ bidders. The representative agent value at each stage is the sum of the $N-1$ bidders' values whom he represents. This is a 2 bidder AQ-AA with *full bid information*. From Theorem 2, we know that truth telling is an ex-post perfect Nash equilibrium. Therefore, for every realization of types, given aggregate bid information at each stage of the auction and truth telling of the $N-1$ other agents, truth telling is a *sequentially rational*. Since this is true for all players at each stage of the auction, it is therefore true at each subgame, and we conclude that truth telling for i is an ex-post perfect Nash equilibrium of the aggregate bid information AQ-AA.

The no bid information AQ-AA follows almost directly. Consider bidder i and suppose the $N-1$ other bidders follow their truth telling strategy. If i was told the realization of types he would know the aggregate bid of the other bidders, but from above we know truth telling is sequentially rational give aggregate bid information. Since this is true for each bidder i, it follows that truth telling is an ex-post perfect Nash equilibrium of the no bid information AQ-AA. ∎

Proof of Lemma 1:. Suppose to the contrary. Consider the first round of elimination where truthful bidding, at any stage t of the auction for any type of bidder (say i), is eliminated and let $\Delta\beta_i$ be the (possibly) mixed strategy that dominates it. At this point truthful bidding for all types and all other bidders have not been eliminated. By the FSA we can construct a bid profile in truthful bidding strategies that make truthful revelation of bidding type by i pivotal at round t. Thus, deviating to $\Delta\beta_i$ will lead unprofitably to a different outcome than truthful revelation of bidding type and to a lower payoff for i. This contradicts the assumption that truthful revelation of bidding type could be eliminated by β_i since it does strictly better than β_i for this particular realization of types. ∎

Proof of Theorem 3. We start with an order of elimination that yields truthful bidding as the unique outcome of IEWDS. The proof is by induction.

(1). We compare 2 arbitrary strategies β_i and $\bar{\beta}_i$ that prescribe the same plan of action for the first $\bar{x}-1$ stages, but where $\bar{\beta}_i$ prescribes truth telling in the last period.

Start at the last period, since if the auction ends before $\bar{x}$, the two strategies yield equivalent outcomes. At round $\bar{x}$, let $\sum_{j\neq i} b_j^{\bar{x}}$ be the collected bids of the other bidders for some realization of types and histories and let $b_i^{\bar{x}} \neq \theta_i^{\bar{x}}$ be the bid prescribed by $\beta_i^{\bar{x}}(\cdot, \theta_i)$. By bidding his value in the last round, for any realization of the other bidders types, i never does worse then he could have by bidding $b_i^{\bar{x}} \neq \theta_i^{\bar{x}}$ and for some bids by the other bidders is strictly better. We first use the FSA to show that there are truthful bids for the other bidders that put i in a sufficiently rich set of situations where he would regret using $\theta_i^{\bar{x}}$ –i.e., that there are truthful bids by the other bidders that could always make truthful bidding pivotal.

By FSA, if $b_i^{\bar{x}} > \theta_i^{\bar{x}}$, then there are type realizations, and therefore truthful bids, by the other bidders such that any of the following situations could occur: first, $b_i^{\bar{x}} > \theta_i^{\bar{x}} \geq \max\{0, c^{\bar{x}} - \sum_{j\neq i} \theta_j^{\bar{x}}\}$; second, $\max\{0, c^{\bar{x}} - \sum_{j\neq i} \theta_j^{\bar{x}}\} > b_i^{\bar{x}} > \theta_i^{\bar{x}}$; and last, $b_i^{\bar{x}} \geq \max\{0, c^{\bar{x}} - \sum_{j\neq i} \theta_j^{\bar{x}}\} > \theta_i^{\bar{x}}$.

Suppose $b_i^{\bar{x}} > \theta_i^{\bar{x}}$. If $b_i^{\bar{x}} > \theta_i^{\bar{x}} \geq \max\{0, c^{\bar{x}} - \sum_{j\neq i} \theta_j^{\bar{x}}\}$, then the last possible unit is produced, under both bids, and the payoffs of both strategies are equivalent; second, if $\max\{0, c^{\bar{x}} - \sum_{j\neq i} \theta_j^{\bar{x}}\} \geq b_i^{\bar{x}} > \theta_i^{\bar{x}}$, then the last unit is not produced, under both bids, and the payoffs of both strategies are equivalent; and finally, if $b_i^{\bar{x}} > \max\{0, c^{\bar{x}} - \sum_{j\neq i} \theta_j^{\bar{x}}\} > \theta_i^{\bar{x}}$, then the auction stops at $\bar{x}+1$ under bid $b_i^{\bar{x}}$ and $\bar{x}$ under $\theta_i^{\bar{x}}$ yielding a higher payoff for truthful revelation of bidding type in round $\bar{x}$. A similar argument applies to $b_i^{\bar{x}} < \theta_i^{\bar{x}}$. We therefore eliminate all strategies that specify non truthful bidding at time $\bar{x}$.

(2). (Inductive Hypothesis): Suppose after k iterations of weak dominance all bids other than truthful revelation of bidding type have been eliminated for periods: $t = \bar{x} - k + 1$ to $\bar{x}$.

(3). If (2) is true, then i's choice at $t = \bar{x} - k$ can have no influence on subsequent moves by anybody. If i's strategy specifies a bid that is above his true marginal valuation in round $t = \bar{x} - k$, then by the FSA there is positive probability that bidders other than i have valuations leading them to truthful bids such that 1 of 3 cases could occur: first, i will profitably keep the auction continuing (which he could have done by bidding truthfully); second, he will profitably stop the auction (which he could have done by bidding truthfully); or i will unprofitably keep the auction continuing when he would have preferred it to stop (which he could have done by bidding truthfully). Hence, all strategies specifying bids greater than $\theta_i^{(\bar{x}-k)}$ in round $t = \bar{x} - k$ never do any better than truthful revelation of bidding type and sometimes do worse and therefore can be eliminated. By a similar reasoning, all strategies specifying bids below than $\theta_i^{(\bar{x}-k)}$ in round $t = \bar{x} - k$ never do any better than truthful revelation of bidding type and sometimes do worse and therefore can be eliminated.

Thus, by induction, we can eliminate all non-truthful bids after $\bar{x} + 1$ rounds of IEWDS.

Furthermore, from lemma 1, we know truthful revelation of bidding type cannot be eliminated by IEWDS. Thus, no matter what strategies have already been eliminated, we can perform the operation above and leave truthful revelation of bidding type as the unique outcome after $\bar{x} + 1$ more rounds of iterated dominance. We conclude that truth telling is the unique outcome of IEWDS independent of the order of elimination. ■

Proof of Theorem 4:. Let $\hat{\beta}_i$ be i's truth telling strategy. The proof will proceed as follows: first, we show $\hat{\beta}_i$ always does at least as well as β_i against arbitrary β_{-i}; second, we show that if the appropriate assumptions on types and cost are met, if i does not use $\hat{\beta}_i$ there is some profile of strategies by bidders other than i where i does strictly worse than if he had followed his truth telling strategy.

Suppose β_{-i} is the strategy profile followed by bidders other than i. It specifies a bid for each bidder other than i at each round where the auction continues. Since this is all of the information each bidder knows at each information set where the auction continues, no bidder can distinguish between 'auction continuing' strategies of their rivals. It also means that at any information set where the auction is continued, no bidder knows whether or not they were pivotal any specific rounds.

Consider the truthful bidding strategy $\hat{\beta}_i$. Suppose that given $(\hat{\beta}_i, \beta_{-i})$ the auction ends at round L yielding i a payoff of $u_i(\hat{\beta}_i, \beta_{-i})$. Now consider the consequences of deviating to an alternative strategy $\beta_i \neq \hat{\beta}_i$.

First, if β_i also ends the auction in round L, then it gives i the same payoff – i.e., $u_i(\hat{\beta}_i, \beta_{-i}) = u_i(\beta_i, \beta_{-i})$. This is since whether or not i is pivotal in any given round is independent of i's action in a round; and by the "No Bid Information condition," bidders other than i cannot distinguish between β_i and $\hat{\beta}_i$ for rounds 1 to L and therefore cannot respond to the change.

Second, if β_i ends the auction earlier than truthful bidding in round $E < L$. Bidder i's surplus for the first $E - 1$ rounds is exactly the same as when he was bidding truthfully so there are no gains. Furthermore, since truthful revelation of bidding type guarantees non-negative surplus at each round, bidder i is potentially foregoing positive payoffs from round E to $L-1$.

Last, if β_i ends the auction later than truthful bidding in round $M > L$, then i's surplus for the first $L-1$ rounds is exactly the same as under $\hat{\beta}_i$. This is since, by the "No Bid Information" constraint, bidders other than i cannot respond to the change and the fact that i's tax is independent of own action. Since the auction ended in round L when i was bidding truthfully, i must be pivotal in round L under β_i. Furthermore since the auction ended in round L when i was bidding truthfully, then it must have been that $\sum_{k=1}^{N} b_k^L < c^L$ or $\theta_i^L < c^L - \sum_{j \neq i} b_j^L$. Therefore the tax for changing i's bid in round L is bigger than the gains from having the auction continuing for that round. However, this observation is not enough. It may be the case that gains in later rounds offset the losses from this round. However this cannot happen due to the "Monotonic Bidding Constraint" which guarantees $c^t - \sum_{j \neq i} b_j^t$ is a weakly increasing function in t; additionally, since marginal valuations are assumed to be weakly decreasing, i's payoff from continuing past round L is strictly decreasing– i.e., $u_i(\beta_i, \beta_{-i}) < u_i(\hat{\beta}_i, \beta_{-i})$. Thus, for all β_{-i}, $u_i(\hat{\beta}_i, \beta_{-i}) \geq u_i(\beta_i, \beta_{-i})$.

The second part of the proof proceeds in the same manner as Claim 2 above. ■

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