A Note on the Stability of Chen's Lindahl Mechanism

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Abstract

There are no general theoretical results on the stability of the Lindahl mechanism introduced by Chen (2002). We show that despite not fitting the requirements of the Milgrom and Roberts 1990 stability results for supermodular games, if the Chen mechanism induces a supermodular game, then the best reply map will be a contraction. This gives us a easy to identify sufficient condition for dynamic stability of equilibrium.

1 Introduction

Mechanisms that induce supermodular games with a unique equilibrium have tended to be very successful in the lab.[1] Milgrom and Roberts (1990), hereafter MR, have shown if a supermodular game has: first, strategy spaces which are complete lattices; and second, a unique equilibrium; then its equilibrium is stable under a variety of learning behavior such as myopic best reply and fictitious play. Motivated by the aforementioned experimental studies as well as the MR result, Chen (2002) proposed a parametric family of Lindahl mechanisms (i.e., mechanisms that Nash implement Lindahl allocations) which, under some conditions of the mechanism parameters, induced a supermodular game with a unique equilibrium. Citing MR, she concluded

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[1]See Chen and Tang(1998), Healy (2004), or Chen and Gazzale (2004).

that the equilibrium of the induced game would be robustly stable. This conclusion is slightly premature.

The economic environment in which Chen works does not satisfy the conditions of MR – in fact, requires that MR *not* be satisfied. While the Chen mechanism does induce a supermodular game, the strategy space is not compact (a requirement of the MR result) thus, no inferences on stability of equilibrium can be made. As a cautionary note both Van Essen (2009) and Healy and Mathevet (2009), in the same environment as Chen's, have provided examples of other Lindahl mechanisms that induce supermodular games with a unique, *unstable* equilibrium.[2] However, neither paper shows that the Chen mechanism is unstable. It therefore remains to be shown whether the Chen mechanism does induce a game with a unique, stable equilibrium. In this paper, we show sufficient conditions for the Chen mechanism to induce a game whose best reply map is a contraction. These conditions are difficult to work with in practice. However, if the game induced by the Chen mechanism is supermodular, then the sufficient conditions for a contraction mapping are satisfied. This gives us a relatively easy to check sufficient condition for stability.

2 Stability and the Chen Mechanism

Our setting applies to $N \geq 2$ consumers. We restrict attention to economies with one private good, one public good, and a constant returns to scale production technology. The quantity of the public good will be denoted by x, and the private good for consumer i by y_i, where consumers are indexed by subscript i. Each consumer is characterized by the convex consumption set $C_i = \mathbb{R}^2_+$, an initial endowment of the private good $\omega_i > 0$, and no initial endowment of the public good. The public good is produced, using the private good as an input (quantity denoted z), with a constant returns to scale production technology $f(z) = \frac{z}{\beta}$ — i.e., each unit of the public good x requires β units ($\beta > 0$) of the private good. Thus β is the constant (real) marginal cost of production. An allocation in this simple economy is an $(N+1)$- tuple $(x, y_1, ..., y_N) \in \mathbb{R}^{N+1}_+$.

[2] Van Essen's example is with 2 players and induces a supermodular game no matter what value of the mechanism parameters are chosen. Healy and Mathevet's example holds for $N \geq 4$ "even economies."

The Chen mechanism is an institution in which consumers report messages to a "planner" who uses this information to determine an amount of the public good to produce and a tax for each consumer. The message space of consumer i is $M_i = \mathbb{R}^2$ with generic element $m_i = (r_i, s_i)$. Consumer i's action r_i should be interpreted as a *request* from the consumer to the planner for units of the public good. Negative requests are allowed. Consumer i's other action, s_i, is interpreted as his *statement* about the amount of the public good that will be produced. We write $(r_1, r_2, ..., r_N, s_1, ..., s_N) = (r, s)$ for a strategy profile. These messages are collected by the planner authority and used to determine an amount of the public good and a tax for each player i according to outcome functions $\chi(r, s)$ and $\tau^i(r, s)$ respectively. The mechanism has exogenous parameters ξ, $\gamma > 0$ and $\delta \geq 0$ which can be manipulated to affect stability.

Let $\varphi^{\xi,\gamma,\delta}(r, s) = \left(\chi(r, s), (\omega^i - \tau^i(r, s))_{i=1}^N\right)$ be the Chen mechanism with outcome functions defined as follows:

$$\begin{aligned} \chi(r, s) &= \sum_{i=1}^{N} r_i \\ \tau^i(r, s) &= P^i(r, s) \cdot \chi(r, s) + \frac{1}{2}\left(s_i - \chi(r, s)\right)^2 + \frac{\delta}{2}\sum_{j \neq i}\left(s_j - \chi(r, s)\right)^2 \end{aligned}$$

where

$$P^i(r, s) = \frac{\beta}{N} - \xi\left(\sum_{j \neq i} r_j - \frac{1}{N}\sum_{j \neq i} s_j\right)$$

can be thought of as i's personalized price for the public good.

Finally, in order to get a unique interior Nash and Lindahl equilibrium, we restrict attention to the following class of economic, quasi-linear environments.

Quasi-linear Economic Environment: E^Q denotes the set of standard C^2 *quasi-linear environments* – i.e., those in which,

1. For each i, there is a real-valued function v^i such that $u^i(x, y_i) = y_i + v^i(x)$.

2. v^i is C^2, where its second derivative is bounded from above and below– i.e., for each i, there exists $\underline{K}_i, \bar{K}_i \in \mathbb{R}_-$ such that $-\infty < \underline{K}_i \leq v^i_{11} \leq \bar{K}_i < 0$.

3. $\sum_i v^i_1(0) > \beta$ and $\sum_i v^i_1(\frac{\Omega}{\beta}) < \beta$ – i.e., that there is unique, interior Pareto optimal level the public good that does not exhaust the economy's private good supply, where $\Omega = \sum \omega_i$.

4. For each i, $\omega_i - v^i_1(x^{PO})x^{PO} \geq 0$ – i.e., each consumer has enough wealth to cover his or her Lindahl taxes.

In the E^Q environment, a mechanism induces a supermodular game if for each i the following inequalities hold: first, for all $j \neq i$, we have $\frac{\partial u^i}{\partial r_i \partial r_j} \geq 0$, $\frac{\partial u^i}{\partial r_i \partial s_j} \geq 0$, $\frac{\partial u^i}{\partial s_i \partial r_j} \geq 0$, and $\frac{\partial u^i}{\partial s_i \partial s_j} \geq 0$; second, $\frac{\partial u^i}{\partial r_i \partial s_i} \geq 0$.[3] Chen proves that her mechanism Nash implements the Lindahl allocations for general environments (including E^Q) and gives conditions for which her mechanism induces a supermodular game.

Theorem (Chen (2002)): For each $e \in E^Q$, the mechanism $\varphi^{\xi,\gamma,\delta}(r,s)$ Nash implements the unique Lindahl allocation of the economy. Furthermore, if ξ, γ, and δ are set such that

$$\delta \in [1-K, \infty) \text{ and } \xi \in [1-K+(N-1)\delta, N\delta]$$

then the game induced by $\varphi^{\xi,\gamma,\delta}(r,s)$ will be supermodular, where $K = \min_i \underline{K}_i$.

Since the message space of the Chen mechanism is $\mathbb{R}^2$, we cannot appeal to the MR stability result for supermodular games. However, MR is not needed, as shown in the following corollary to her theorem. If the Chen mechanism induces a supermodular game, then the best reply mapping is a contraction. Furthermore, by applying the successive approximations result from the Contraction Mapping Theorem, we know the equilibrium is stable (at least under myopic best reply).

Corollary: If the Chen mechanism induces a supermodular game, then the best reply mapping will be a contraction and the unique equilibrium

[3] See Amir (2005) or Vives (2001) for a good introduction on the theory of supermodular games.

of the game induced by $\varphi^{\xi,\gamma,\delta}(r,s)$ will be stable under myopic best reply learning behavior.

As a proof, we directly calculate the slopes of the best response system and show that when the mechanism induces a supermodular game, the best response system satisfies the following sufficient condition for the best response mapping to be a contraction– i.e., for each i, we have that

$$\begin{aligned}\sum_{j\neq i}\left|\frac{\partial r_i^*}{\partial r_j}\right| + \sum_{j\neq i}\left|\frac{\partial r_i^*}{\partial s_j}\right| &< 1 \\ \sum_{j\neq i}\left|\frac{\partial s_i^*}{\partial r_j}\right| + \sum_{j\neq i}\left|\frac{\partial s_i^*}{\partial s_j}\right| &< 1,\end{aligned}$$

where r_i^* and s_i^* are i's optimal actions given the actions of his rivals.[4] In words, this condition reads that if all players other than i change each of their actions r_j and s_j by 1, player i's total change, for each his actions, is less than 1.

When preferences are added to a mechanism they induce a game. The Chen mechanism induces a game where each player i has following preferences

$$v^i(\sum_k r_k) - P^i(r,s)r_k - \frac{\gamma}{2}(s_i - \sum_k r_k)^2 - \frac{\delta}{2}\sum_{j\neq i}(s_j - \sum_k r_k)^2.$$

The first order conditions that define i's best response system are:

$$\begin{aligned}[r_i] &: \; v_1^i(\cdot) - P^i(r,s) + \gamma(s_i - \sum_k r_k) + \delta\sum_{j\neq i}(s_j - \sum_k r_k) = 0 \\ [s_i] &: \; -\gamma(s_i - \sum_k r_k) = 0.\end{aligned}$$

Let $r_i^* = r_i^*(r_{-i}, s_{-i})$ and $s_i^* = s_i^*(r_{-i}, s_{-i})$ be the solution to the two first order conditions. Plug these solutions into each other to get the following augmented FOC:

[4] That this is a sufficient condition follows from the mean value theorem and that $v_{11}^i \in [\underline{k}_i, \bar{k}_i]$ – by the Weirstrass theorem, for any fixed set of parameters, the total derivative of the best response system is bounded.

$$
\begin{aligned}
v_1^i(r_i^* + \sum_{j \neq i} r_j) - P^i(r,s) + \delta \sum_{j \neq i}(s_j - r_i^* - \sum_k r_k) &= 0 \quad (1) \\
s_i^* - r_i^* - \sum_{j \neq i} r_j &= 0 \quad (2)
\end{aligned}
$$

Differentiating equation (1) with respect to r_j we get

$$v_{11}^i(\frac{\partial r_i^*}{\partial r_j} + 1) + \xi - \delta(N-1)\frac{\partial r_i^*}{\partial r_j} - \delta(N-1) = 0.$$

We can then solve for

$$\frac{\partial r_i^*}{\partial r_j} = \frac{v_{11}^i + \xi - \delta(N-1)}{\delta(N-1) - v_{11}^i}.$$

Differentiating equation (1) with respect to s_j we get

$$v_{11}^i \frac{\partial r_i^*}{\partial s_j} - \frac{\xi}{N} + \delta - \delta(N-1)\frac{\partial r_i^*}{\partial s_j} = 0.$$

We can then solve directly for

$$\frac{\partial r_i^*}{\partial s_j} = \frac{-\frac{\xi}{N} + \delta}{\delta(N-1) - v_{11}^i}.$$

Now checking the first sufficient condition $\sum_{j \neq i} \left| \frac{\partial r_i^*}{\partial r_j} \right| + \sum_{j \neq i} \left| \frac{\partial r_i^*}{\partial s_j} \right| < 1$, we need

$$\sum_{j \neq i} \left| \frac{v_{11}^i + \xi - \delta(N-1)}{\delta(N-1) - v_{11}^i} \right| + \sum_{j \neq i} \left| \frac{-\frac{\xi}{N} + \delta}{\delta(N-1) - v_{11}^i} \right| < 1.$$

Since the mechanism parameters are such that the induced game is supermodular, then $\frac{\partial r_i^*}{\partial r_j}$ and $\frac{\partial r_i^*}{\partial s_j}$ are positive for each j. We dispense with the absolute values in our first sufficient condition yielding

$$\sum_{j \neq i} \frac{v_{11}^i + \xi - \delta(N-1)}{\delta(N-1) - v_{11}^i} + \sum_{j \neq i} \frac{-\frac{\xi}{N} + \delta}{\delta(N-1) - v_{11}^i} < 1.$$

Simplifying this expression reads

$$(N-1)\left(\frac{\xi}{N} - \delta\right) < -\frac{N}{(N-1)} v_{11}^i.$$

Since $v_{11}^i < 0$ (by assumption of being in E^Q) and from supermodularity $\delta \geq \frac{\xi}{N}$ therefore this condition is satisfied.

Next, we check the second condition $\sum_{j\neq i}\left|\frac{\partial s_i^*}{\partial r_j}\right| + \sum_{j\neq i}\left|\frac{\partial s_i^*}{\partial s_j}\right| < 1$. Differentiating equation (2) we find $\frac{\partial s_i^*}{\partial r_j} = \frac{\partial r_i^*}{\partial r_j} + 1$ and $\frac{\partial s_i^*}{\partial s_j} = \frac{\partial r_i^*}{\partial s_j}$, therefore we need that

$$\sum_{j\neq i}\left|\frac{v_{11}^i + \xi - \delta(N-1)}{\delta(N-1) - v_{11}^i} + 1\right| + \sum_{j\neq i}\left|\frac{-\frac{\xi}{N} + \delta}{\delta(N-1) - v_{11}^i}\right| < 1.$$

Our game is supermodular, therefore $\frac{\partial s_i^*}{\partial r_j}$ and $\frac{\partial s_i^*}{\partial s_j}$ for all i and j are positive – i.e., we have

$$(N-1)\left(\frac{v_{11}^i + \frac{(N-1)}{N}\xi - N\delta}{\delta(N-1) - v_{11}^i} + 1\right) < 1.$$

Simplifying, this expression reads

$$(N-1)\left(\frac{\xi}{N} - \delta\right) < \delta - \frac{1}{(N-1)}v_{11}^i.$$

Since $v_{11}^i < 0$ (by assumption of being in E^Q) and $\delta \geq \frac{\xi}{N}$ (from the supermodularity conditions) this condition is satisfied.

Thus, if the parameters of the Chen mechanism induces a supermodular game, the best response mapping will be a contraction and the equilibrium will be unique and stable under the myopic best reply learning behavior.

3 Conclusion

We have shown that having Chen's Lindahl mechanism induce a supermodular game, in the quasi-linear economic environment, is sufficient to guarantee stability of the unique Nash equilibrium of the game. This property is not a universal phenomenon among incentive compatible Lindahl mechanisms as demonstrated by both Healy and Mathevet (2009) and Van Essen (2009), but it is not unique either. Van Essen (2009) introduces another Lindahl mechanism for which this is the case. Why this condition holds for some mechanisms and not others is an open question.

4 Resources

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