

A Method to Solve a Class of Two-Dimensional Screening Problems.

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Abstract

We develop a method allowing to solve a class of two-dimensional screening problems in which an agent's private information is two-dimensional, but the allocation is one-dimensional. Our method is based on identifying the 'isoquants' - the sets of types who obtain the same allocation. We provide a general characterization of the optimal solution, and apply it to explicitly compute the solution when the utility function is linear-quadratic and the types are distributed uniformly. The optimal solution exhibits discontinuity on the boundary for a large set of parameter values.

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1 Introduction

The problem of multidimensional screening is commonly recognized as challenging. This problem involves a monopolistic firm (or a duopoly) selling their product(s) to customers with privately known multidimensional types. Many questions in this area remain unanswered. In particular, optimal screening mechanisms have been derived only in a limited number of cases. Armstrong (1996) characterizes the optimal screening mechanism in the case when the utility and cost function make a so-called 'cost-based' tariff optimal. Rochet and Choné (2000) derive necessary and sufficient conditions for a multidimensional quasilinear setting. Yet it is difficult to use these conditions to derive an optimal mechanism.

In this paper we contribute to the analysis of the multidimensional screening problem by developing a method for solving one class of such problems. In our setup an agent's type is two-dimensional, while the allocation is one-dimensional. That is, the firm sells a single line of goods which can be consumed in different quantities. Alternatively, the firm may produce goods of different quality.

On the demand side, a consumer may have several uses for the same product, and her value of the product in each use may be her private information. Similarly, an individual may

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purchase high quality wine both for consumption and also as a status goods which she may show to his friends before consuming it. An education customer may value the education both for its An industrial customer's valuation for the input may depend on the technology that this customer uses to process the input, as well as on the demand for the final product.

In this paper, we develop a general method for solving this class of screening problems. Let us describe the main idea of our method in detail. At the core of it is the characterization of the isoquants, the sets of types who consume the same quantity (purchase the goods of the same quality).

Specifically, consider a consumer with utility function $u(q, \theta, \alpha)$ where q is the quantity of the good that she consumes, and the pair $(\theta, \alpha) \in \mathbf{R}^2$ is the consumer's type which constitutes her private information. By the Taxation Principle, the firm can implement any mechanism by offering a tariff $T(q)$ to the customer. That is, $T(q)$ is the amount of money that the customer has to pay for the quantity q of the good. Let us suppose that $T(q)$ is differentiable.

If a consumer of type (θ, α) chooses to consume quantity q , then the following first-order condition has to hold:

$$u_q(q, \theta, \alpha) = T^{*'}(q)$$

Consequently, if two different types (θ_1, α_1) and (θ_2, α_2) consume the same quantity q , we have:

$$u_q(q, \theta_1, \alpha_1) = u_q(q, \theta_2, \alpha_2)$$

The above relation between the types who consume the same quantity allows us to characterize the isoquants, the sets of types who consume the same quantities. Indeed, we demonstrate that if type (θ_1, α_1) consumes the quantity q_1 , then the same quantity is consumed by all types $I(q_1, \theta_1, \alpha_1) \equiv \{(\theta, \alpha) | u_q(q_1, \theta, \alpha) = u_q(q_1, \theta_1, \alpha_1)\}$.

We then show that all isoquants reach the lower boundary of the set Ω of types assigned positive quantities. Therefore, if we choose the quantity allocations on the lower boundary $\partial_L \Omega$ of the region of positive quantities Ω , then the quantity allocations at all points of the type space will be uniquely determined. Thus our overall problem boils down to choosing the quantity allocations on the lower boundary $\partial_L \Omega$ and the 'level' at which this lower boundary lies in type space.

This method allows us to obtain necessary and sufficient conditions for the solution of this problem. Furthermore, under linear quadratic utility function with uniform distribution of types over a square, we obtain a closed form solution for the optimal mechanism under all admissible parameter configurations.

The linear quadratic case has been previously analyzed by Laffont, Maskin, Rochet (1987) for a limited consternation of parameter values and under an endogenous assumption. They have conjectured the properties a solution (e.g. horizontal linear lower boundary) and have derived an allocation profile satisfying these properties. In our analysis, we consider the full family of linear quadratic utility functions. We confirm that the mechanism derived by Laffont, Maskin, Rochet (1987) is optimal under their parameter values. For different parameter values the solution exhibits qualitatively different features. In particular, the lower boundary of the region of types who get a positive quantity is non-linear. Importantly, the quantity allocation is discontinuous near the boundary for a class of parameter values. These properties are economically significant. In particular, they arise because the optimal tariff is convex, i.e.

exhibits quantity premia. Furthermore, the discontinuity of the quantity assignment is also economically significant. It implies that in economic situations we might encounter

Wilson (1993) solves several specific examples of multidimensional non-linear pricing and identifies the necessary condition on the optimal tariff -its so-called ‘integrability.’ Laffont, Maskin, Rochet (1987) make the first-steps towards solving the two-dimensional screening problem that we are concerned with. Under and endogenous assumption, they solve this problem for a particular subset of parameters. We confirm that their solution is correct. But we also show that under different parameter values the solution is qualitatively different. We will discuss the details below. One of the first studies of multidimensional pricing problem is (McAfee and McMillan 1988) They derive the generalized single-crossing condition for this problem. Necessary and sufficient conditions for the optimal mechanism to sell to consumers whose utility function is additively separable in each good has been derived by Rochet and Choné (2000). Yet, these condition remain hard to verify. Armstrong (1996) solves a class of multidimensional problems, also with additive separability, in environments where the so-called cost-based tariffs are optimal. Armstrong and Rochét (1999) describe a method for characterizing optimal mechanisms in the case with two-dimensional types, with two possible values for each element of the type. An insightful survey of the literature on multidimensional screening is presented by Rochet and Stole (2001). Basov (2003) presents mathematical techniques for solving multidimensional screening problems and provides necessary conditions in the form of differential equations that a solution has to satisfy.

The rest of the paper is organized as follows. In the next section we present the model. In section 3 we introduce our method and show how it can be used to simplify the problem to that of optimization along the boundary. In section 4 we apply our method to explicitly characterize the optimal mechanism when a consumer’s utility function is linear-quadratic and the distribution of types is uniform.

2 Model

A monopoly supplier of a single good faces a population of consumers with privately known preferences for the good. Consumers have private information about their two-dimensional types (θ, α) . A consumer of type (θ, α) gets utility $u(q, \theta, \alpha) - t$ from consuming quantity $q \in \mathbf{R}_+$ of the good, acquired at cost t . The distribution function $F(\theta, \alpha)$ of consumer types in the population is common knowledge. We assume that $F(\cdot)$ is twice continuously differentiable function with density function $f(\theta, \alpha)$ with a rectangular support $[a, b] \times [c, d]$. Renormalizing, we can without loss of generality take the support to be $[0, 1] \times [0, 1]$. The consumer’s reservation utility level is 0.

We assume that the firm’s cost of production is zero. This assumption is without loss of generality when firm’s cost is additively separable across consumers. In this case, we can simply subtract the firm’s cost $c(q)$ of producing quantity q from consumers’ utility and operate with the modified consumer utility function $u(q, \theta, \alpha) - c(q)$ throughout.¹ To handle the case in which the monopolist’s aggregate cost $C(Q)$ is an increasing function of aggregate output

¹This case applies, for example, a monopolist supplies a single consumer randomly drawn from a population whose preferences are distributed according to $F(\cdot)$.

$Q = \int_0^1 q(\theta)f(\theta)d\theta$ we would need one extra step. Precisely, for any given constant marginal cost level c , our model would predict the corresponding aggregate output level Q selected by the firm. Equilibrium then obtains whenever $C'(Q) = c$.

We maintain the following assumptions on preferences and technology throughout the paper:

- Assumption 1** *The function $u(q, \theta, \alpha) : \mathbf{R}_+ \times [0, 1]^2$ is of class C^3 . Furthermore,*
- (i) $u(0, \theta, \alpha) = 0$ for all $(\theta, \alpha) \in [0, 1]^2$;
 - (ii) $u(q, 0, \alpha) \leq 0$ for all α and q ;
 - (iii) $u_q(q, \theta, \alpha) > 0$, $u_\theta(q, \theta, \alpha) > 0$ and $u_\alpha(q, \theta, \alpha) > 0$, for all $q > 0$ and $(\theta, \alpha) \in [0, 1]^2$;
 - (iv) $u_{\theta q}(q, \theta, \alpha) > 0$, $u_{\alpha q}(q, \theta, \alpha) > 0$, for all $(\theta, \alpha) \in (0, 1]^2$ and $q > 0$;
 - (v) $u_{qq}(q, \theta, \alpha) < 0$ for all θ, α and q .

Assumption 1 is fairly standard. Part (ii) guarantees that it will be optimal for the monopolist not to serve all consumers. Part (iv) requires consumer's utility functions to be supermodular. Part (v) ensures that consumers' demand functions are downward sloping.

We will also use the following assumption specific to the higher-dimensional type space, which we term "single-crossing":

- Assumption 2** (*Single Crossing*) $\frac{d}{dq} \frac{u_{q\alpha}}{u_{q\theta}} > 0$ for all $q > 0$.

Assumption 2 guarantees that the inverse demand functions for types (θ', α') and (θ, α) can intersect at most once, as the next lemma demonstrates.

Lemma 1 *Suppose Assumption 2 holds and $\alpha' > \alpha$. Then $u_q(q, \theta', \alpha') = u_q(q, \theta, \alpha)$ implies $u_{qq}(q, \theta', \alpha') > u_{qq}(q, \theta, \alpha)$.*

Assumption 2 should not be confused with the single-crossing condition in one-dimensional screening problems, which guarantees that consumers' indifference curves in (q, t) space intersect at most once. In fact, the latter condition is extremely restrictive, as it implies that consumers' demand curves do not intersect at all, i.e. can be ranked. In the next section, we will show that Assumption 2 has many important consequences: it implies that isoquants in (θ, α) space cannot intersect, and must "fan out".

3 Analysis

3.1 Preliminaries

The monopolist's problem is to select a nonlinear pricing schedule, i.e. a lower semi-continuous function $T : \mathbf{R}_+ \rightarrow \mathbf{R}_+$, so as to maximize her profits subject to consumer maximization.² Thus the monopolist solves

²The Taxation principle implies that the monopolist cannot gain by offering more complicated screening mechanisms. Below, we show that the schedule $T(\cdot)$ may be taken to be continuous, so the assumption of l.s.c. is without loss of generality.

$$\max_{T(\cdot)} \int_0^1 \int_0^1 T(q(\theta, \alpha)) f(\theta, \alpha) d\theta d\alpha \quad (1)$$

where $q(\theta, \alpha)$ satisfies:

$$u(q(\theta, \alpha), \theta, \alpha) - T(q(\theta, \alpha)) \geq u(q, \theta, \alpha) - T(q), \text{ for all } q, \theta, \alpha \quad (2)$$

$$u(q(\theta, \alpha), \theta, \alpha) - T(q(\theta, \alpha)) \geq 0, \text{ for all } \theta, \alpha. \quad (3)$$

We will call an allocation $q : [0, 1]^2 \rightarrow \mathbf{R}_+$ implementable if there exists a pricing schedule T for which (2) and (3) hold.

The firm's profit maximization problem can be written as follows:

$$\max_{q(\theta, \alpha) \geq 0, t(\theta, \alpha)} \int_a^b \int_c^d t(\theta, \alpha) f(\theta, \alpha) d\theta d\alpha \quad (4)$$

$$\text{subject to: } u(q(\theta, \alpha), \theta, \alpha) - t(\theta, \alpha) \geq u(q(\theta', \alpha'), \theta, \alpha) - t(\theta', \alpha') \quad \forall \theta, \theta', \alpha, \alpha' \quad (5)$$

$$u(q(\theta, \alpha), \theta, \alpha) - t(\theta, \alpha) \geq 0 \quad \forall \theta, \alpha \quad (6)$$

Let $U(\theta, \alpha) \equiv u(q(\theta, \alpha), \theta, \alpha) - t(\theta, \alpha)$.

It is easy to show that an implementable quantity schedule $q(\theta, \alpha)$ must be nondecreasing in both θ and α . This has several implications. Since $q(\theta, \alpha)$ is non-decreasing in both arguments and bounded, it is Riemann integrable (Rudin (1976), Theorem 6.9, p.126) and a.e. differentiable (Royden (1987), Theorem 3, p.100). Hence, $u(q(\theta), \theta)$ is also bounded, Riemann integrable and a.e. differentiable. Let $U(\theta) \equiv u(q(\theta), \theta) - t^s(\theta)$. Incentive constraints (5) imply that $U(\theta)$ is increasing in both arguments and satisfies the following inequalities $\forall \theta, \theta' \in [0, 1]$:

$$u(q(\theta', \alpha), \theta, \alpha) - u(q(\theta', \alpha), \theta', \alpha) \leq U(\theta, \alpha) - U(\theta', \alpha) \leq u(q(\theta, \alpha), \theta, \alpha) - u(q(\theta, \alpha), \theta', \alpha)$$

By the intermediate value theorem, $\exists \lambda_1, \lambda_2 \in [0, 1]$ s.t. $u_\theta(q(\theta'), \lambda_1 \theta + (1 - \lambda_1) \theta') (\theta - \theta') \leq U(\theta) - U(\theta') \leq u_\theta(q(\theta), \lambda_2 \theta + (1 - \lambda_2) \theta') (\theta - \theta')$. Since, $u_\theta(q(\theta'), \theta) \leq \max_{\theta \in [0, 1]} u_\theta(q^*(1), \theta) < \infty$, $U(\theta)$ is absolutely continuous.

Therefore by Theorem 14, p.110 in Royden (1987), we have $\frac{dU(\theta, \alpha)}{d\theta} = u_\theta(q(\theta), \theta)$ and $U(\cdot)$ is equal to the integral of its derivative, i.e. $U(\theta, \alpha) - U(\theta', \alpha) = \int_{\theta'}^\theta u_\theta(q(s, \alpha), s, \alpha) ds$. Making a standard substitution and integrating by parts, we can rewrite the firm's maximization problem as follows:

$$\max_{q(\theta, \alpha) \geq 0} \int_0^1 \int_0^1 (u(q(\theta, \alpha), \theta, \alpha) f(\theta|\alpha) - u_\theta(q(\theta, \alpha), \theta, \alpha) (1 - F(\theta|\alpha))) d\theta f(\alpha) d\alpha - \int_0^1 U(0, \alpha) f(\alpha) d\alpha \quad (7)$$

$$\text{subject to: } \int_0^\theta u_\theta(q(s, \alpha), s, \alpha) ds + U(0, \alpha) \geq$$

$$u(q(\theta', \alpha'), \theta, \alpha) - u(q(\theta', \alpha'), \theta', \alpha') + \int_0^{\theta'} u_\theta(q(s, \alpha'), s, \alpha') ds + U(0, \alpha') \quad \forall \theta, \theta', \alpha \neq \alpha' \quad (8)$$

$$U(0, \alpha) \geq 0 \text{ for all } \alpha \quad (9)$$

$$q(\theta, \alpha) \text{ is nondecreasing in } \theta \quad (10)$$

Given that $u(q, 0, \alpha) \equiv 0$ by Assumption (1) (i), $U(0, \alpha)$ must be constant in α since otherwise the incentive constraints of some type $(0, \alpha)$ will fail. Inspecting Problem (7) and constraints (8) and (9), it is easy to see that it is optimal to set $U(0, \alpha) = 0$ for all α . So, we can rewrite our problem as:

$$\max_{q(\theta, \alpha) \geq 0} \int_0^1 \int_0^1 (u(q(\theta, \alpha), \theta, \alpha) f(\theta|\alpha) - u_\theta(q(\theta, \alpha), \theta, \alpha)(1 - F(\theta|\alpha))) f(\alpha) d\alpha d\theta \quad (11)$$

$$\text{s.t.: } \int_0^\theta u_\theta(q(s, \alpha), s, \alpha) ds \geq u(q(\theta', \alpha'), \theta, \alpha) - u(q(\theta', \alpha'), \theta', \alpha') + \int_0^{\theta'} u_\theta(q(s, \alpha'), s, \alpha') ds \quad \forall \theta, \theta', \alpha \neq \alpha' \quad (12)$$

$$q(\theta, \alpha) \text{ is nondecreasing in } \theta \quad (13)$$

3.2 Solution Method.

We start the investigation of this problem by establishing how one can reduce the set of incentive constraints (12). But in order to do this, we need a set of preliminary results. Consider some mechanism which can be represented by a tariff $T(q)$.

Define

$$\begin{aligned} Q^*(\theta, \alpha) &= \arg \max_q \{u(q, \theta, \alpha) - T(q)\} \\ s(\theta, \alpha) &= \max_q \{u(q, \theta, \alpha) - T(q)\}. \end{aligned}$$

Then we have:

Lemma 2 *Let q be an implementable allocation. Then s is a continuous function, $Q^*(\theta, \alpha)$ is a non-empty valued u.h.c. correspondence, and every selection of $Q^*(\theta, \alpha)$ is an increasing function.*³

Since $q \equiv 0$ is always implementable,⁴ we will from now on assume that there exists $\theta < 1$ and $\alpha < 1$ such that $q(\theta, \alpha) > 0$. Define $\underline{\theta}(\alpha) = \inf\{\theta | q(\theta, \alpha) > 0\}$, and adopt the convention that the infimum of the empty set is 1; then:

Lemma 3 *The function $\underline{\theta}(\alpha)$ is continuous, non-increasing in α , and strictly decreasing in α whenever $q(\underline{\theta}(\alpha), \alpha) > 0$.*

Let us temporarily assume that $\underline{\theta}(0) < 1$; the arguments for the case where $\underline{\theta}(0) = 1$ are entirely parallel.⁵ Define the four boundaries of the set $\Omega = cl\{(\theta, \alpha) | q(\theta, \alpha) > 0\}$ as follows:

$$\begin{aligned} L_1 &= \{(\theta, \alpha) : \theta = \underline{\theta}(\alpha)\} \\ L_2 &= \{(\theta, \alpha) : \alpha = 1 \text{ and } \theta \geq \underline{\theta}(1)\} \\ U_1 &= \{(\theta, \alpha) : \alpha = 0 \text{ and } \theta \geq \underline{\theta}(0)\} \\ U_2 &= \{(\theta, \alpha) : \theta = 1\}. \end{aligned}$$

³More precisely, if q' is a selection from Q^* then $q'(\theta', \alpha') \geq q(\theta, \alpha)$ whenever $\theta' \geq \theta$ and $\alpha' \geq \alpha$.

⁴The tariff $T(0) = 0$ and $T(q) = u(q, 1, 1) + \varepsilon$, for some $\varepsilon > 0$ implements $q \equiv 0$.

⁵In that case define $\underline{\alpha} = \inf\{\alpha : \underline{\theta}(\alpha) < 1\}$. Then $U_1 = \emptyset$, $U_2 = \{(1, \alpha) : \alpha \geq \underline{\alpha}\}$, $L_1 = \{(\theta, \alpha) : \theta = \underline{\theta}(\alpha) \text{ and } \alpha \geq \underline{\alpha}\}$.

and let $U = U_1 \cup U_2$ be the “upper boundary” and $L = L_1 \cup L_2$ be the “lower boundary” of Ω . See Figure 1.

Our goal now is to characterize the behavior of q on the set Ω . Since the set of feasible choices for consumers is one-dimensional (it is described by the graph of the function $T(\cdot)$), and since Ω is two-dimensional, there will necessarily be a lot of bunching in the allocation q . The role of our single-crossing assumption is to lend structure to this pattern of bunching.

Lemma 4 *Suppose Assumption 2 holds, and $q_1 \in Q^*(\theta_1, \alpha_1)$. Then for all (θ_2, α_2) s.t. $\alpha_1 > \alpha_2$ and s.t. $u_q(q_1, \theta_2, \alpha_2) = u_q(q_1, \theta_1, \alpha_1)$, one has $q_1 \in Q^*(\theta_2, \alpha_2)$.*

Lemma 4 has the following immediate implication:

Lemma 5 *Problem (11)- (12) is solved by a solution to the following problem:*

$$\max_{q(\theta, \alpha) \geq 0} \int_0^1 \int_0^1 (u(q(\theta, \alpha), \theta, \alpha) f(\theta|\alpha) - u_\theta(q(\theta, \alpha), \theta, \alpha)(1 - F(\theta|\alpha))) f(\alpha) d\alpha d\theta \quad (14)$$

$$q(\theta_1, \alpha_1) = q(\theta_2, \alpha_2) \text{ if } u_q(q(\theta_2, \alpha_2), \theta_2, \alpha_2) = u_q(q(\theta_2, \alpha_2), \theta_1, \alpha_1), \alpha_2 \geq \alpha_1 \text{ and } q(\theta_2, \alpha_2) > 0 \quad (15)$$

$$q(\theta, \alpha) \text{ is nondecreasing in } \theta \text{ and } \alpha \quad (16)$$

Precisely, Lemma 5 holds because we are able to characterize the isoquants, the sets of types who get the same quantity allocation q . Let $\hat{I}(q) \equiv \{(\theta, \alpha) | q(\theta, \alpha) = q\}$ be a set of types who consume quantity q . Incentive compatibility requires all types in $\hat{I}(q)$ to pay the same transfer.

Further, for any $(\theta, \alpha) \in \Omega$ let $I(\theta, \alpha) = \{(\theta', \alpha') \in \Omega, \alpha' > \alpha : u_q(q(\theta, \alpha), \theta', \alpha') = u_q(q(\theta, \alpha), \theta, \alpha)\}$. Then Lemma 4 establishes that $q(\theta, \alpha)$ is an optimal choice for any $(\theta', \alpha') \in I(\theta, \alpha)$. That is, $I(\theta, \alpha)$ is an isoquant of types who are assigned the same allocation as (θ, α) .

We next establish that except when $q(\theta, \alpha) = 0$, the isoquant curve $I(\theta, \alpha)$ cannot intersect the lower boundary to the northwest of the point (θ, α) ,

Lemma 6 *Suppose that Assumption 2 holds. Let (θ, α) be any point in Ω for which $q(\theta, \alpha) > 0$. Then $I(\theta, \alpha)$ does not intersect L at any point to the northwest of (θ, α) .*

Lemma 7 below shows the conclusion of Lemma 4 can be strengthened, provided q is continuous on $\Omega \setminus L$: $I(\theta, \alpha)$ then defines a locus of types that must choose the same quantity as type (θ, α) . Henceforth, we will therefore refer to $I(\theta, \alpha)$ as an “isoquant”.

Lemma 7 *Suppose that Assumption 2 holds, and that q is continuous on $\Omega \setminus L$. Let $(\theta, \alpha) \in \Omega$, and let $(\theta', \alpha') \in I(\theta, \alpha) \cap \Omega$. Then $q(\theta', \alpha') = q(\theta, \alpha)$.*

Finally, we obtain the following characterization of implementable allocations that are continuous on $\Omega \setminus L$:

Theorem 1 *Suppose assumptions 1 and 2 hold. Consider any implementable allocation $q(\theta, \alpha)$ that is continuous on $\Omega \setminus L$.*

Then along the lower boundary L the allocation q is nondecreasing, i.e. $q(\underline{\theta}(\alpha), \alpha)$ is a nondecreasing function of α , $q(\theta, 1)$ is a nondecreasing function of θ , and $\lim_{\alpha \uparrow 1} q(\underline{\theta}(\alpha), \alpha) \leq \lim_{\theta \downarrow \underline{\theta}(1)} q(\theta, 1)$. Finally, for all $(\theta, \alpha) \in L$ the set $Q^(\theta, \alpha)$ is an interval.*

Thus the allocation q is completely determined by its value along the lower boundary L , i.e. for any $(\theta, \alpha) \in \Omega$ there exists $(\theta', \alpha') \in L$ and $q \in Q^(\theta, \alpha)$ such that $(\theta, \alpha) \in I(q, \theta', \alpha')$, where $I(q, \theta, \alpha) = \{(t, a) \in \Omega : u_q(q, t, a) = u_q(q, \theta, \alpha)\}$.*

The expected value of the firm's profits is completely determined by the choice of the point $\underline{\theta}(1)$ and the quantity schedules $\underline{q}(\theta)$ and $q(\theta, 1)$.

Theorem 1 has several interesting implications. First, it shows that all isoquants connect the lower boundary L to the upper boundary U . These isoquants do not intersect in the interior of Ω , and ‘fan out’ as we move from $(\underline{\theta}(0), 0)$ to $(1, 1)$. Secondly, it reduces the problem of specifying an allocation on all of Ω to just specifying its values on the lower boundary L . This essentially reduces the multi-dimensional screening problem to a one-dimensional screening problem.

Our next Lemma makes this even more explicit. Define

$$\begin{aligned}\underline{q}(\alpha) &= q(\underline{\theta}(\alpha), \alpha), \alpha \in [0, 1] \\ \bar{q}(\theta) &= q(\theta, 1), \theta \in [\underline{\theta}(1), 1]\end{aligned}$$

Then the functions $\underline{q}(\cdot)$ and $\bar{q}(\cdot)$ completely determine the lower boundary L_1 and the implementing tariff $T(\cdot)$:

Lemma 8 *Suppose that Assumptions 1 and 2 hold. Consider any implementable allocation $q(\theta, \alpha)$ that is continuous on $\Omega \setminus L$. Then*

- (i) *The functions $\underline{q}(\cdot)$ and $\bar{q}(\cdot)$ are nondecreasing, with $\bar{q}(\underline{\theta}(1)) \geq \underline{q}(1)$;*
- (ii) *The function $\underline{\theta}(\alpha)$ is absolutely continuous, and $\underline{\theta}'(\alpha) = -\frac{u_\alpha}{u_\theta}(\underline{q}(\alpha), \underline{\theta}(\alpha), \alpha)$ for almost every α*
- (iii) *The function $T(q)$ is given by*

$$T(q) = \begin{cases} u(\underline{q}(1), \underline{\theta}(1), 1) - \int_{\underline{q}}^{\bar{q}} u_q(z, \underline{\theta}(\alpha(z)), \alpha(z)) dz & q \in [\underline{q}(0), \underline{q}(1)] \\ u(\underline{q}(1), \underline{\theta}(1), 1) + \int_{\underline{q}(1)}^{\bar{q}} u_q(z, \theta(z), 1) dz & q \in [\underline{q}(1), \bar{q}(1)] \end{cases}, \quad (17)$$

where $\alpha : [\underline{q}(0), \underline{q}(1)] \rightarrow [0, 1]$ is given by $\alpha(q) = \min\{\alpha : q \in Q^*(\underline{\theta}(\alpha), \alpha)\}$, and $\theta : [\underline{q}(1), \bar{q}(1)] \rightarrow [0, 1]$ is given by $\min\{\theta : q \in Q^*(\theta, 1)\}$.

Let us briefly summarize our results. The ‘lower’ boundary of the set of types who are assigned positive quantities is described by a function $\underline{\theta}(\alpha)$ or, equivalently, $\underline{q}(\theta)$ defined in the following way: $\underline{\theta}(\alpha) = \inf\{\alpha | q(\theta, \alpha) > 0\}$, or symmetrically, $\underline{q}(\theta) = \inf\{\alpha | q(\theta, \alpha) > 0\}$. By continuity of the utility function, $\underline{\theta}(\alpha)$ and $\underline{q}(\theta)$ are continuous functions.

By Lemma 8, the choice of the mechanism boils down to choosing a boundary profile which consists of three elements: (i) the ‘corner-point’ $\underline{\theta}(1)$, (ii) the quantity schedule $\underline{q}(\alpha)$ for all $\alpha \in [0, 1]$ on the lower boundary L_1 , (iii) the quantity schedule $q(\theta, 1)$ on the lower boundary

L_2 for $\theta \in [\underline{\theta}(1), 1]$. By Lemma 5 the ‘boundary profile’ $(\underline{\theta}(1), \underline{q}(\alpha), q(\theta, 1))$ induces quantity allocation profile $q(\theta, \alpha)$ according to (15) for a set of types (θ, α) away from the boundary.

We will say that the ‘boundary profile’ $(\underline{\theta}(1), \underline{q}(\alpha), q(\theta, 1))$ is *admissible* if it has the following properties:

- Property 1** (i) $q(\theta, 1)$ is increasing in θ and $q(1, \alpha) \geq \underline{q}(\alpha)$ for all $\alpha \in [0, 1]$.
(ii) If $u_q(q(\theta, 1), \theta, 1) = u_q(q(\theta, 1), \underline{\theta}(\alpha), \alpha)$ for some θ and α , then $q(\theta, 1) = q(\alpha)$.
(iii) $\underline{q}(\alpha)$ is monotone on $[0, 1]$

Conditions (i) and (iii) is necessary for a ‘boundary profile’ $(\underline{\theta}(1), q(\alpha), q(\theta, 1))$ to be incentive compatible, while condition (ii) is imposed without loss of generality by Lemma 5.

3.3 Reformulation of the Problem.

To derive the optimal mechanism using our isoquant approach we will make use of the following function $s(\theta, q, \alpha, \hat{\alpha})$. For $\theta \in [\underline{\theta}, \bar{\theta}]$, and some $\alpha, \hat{\alpha} \in [\underline{\alpha}, \bar{\alpha}]$, $\alpha \leq \hat{\alpha}$, let $s(\theta, q, \alpha, \hat{\alpha})$ be given by the solution to the following equation:

$$u_q(q, \theta, \hat{\alpha}) = u_q(q, s, \alpha) \quad (18)$$

if such solution exists. Also, let $s(\theta, q, \alpha, \hat{\alpha}) = \infty$ if (18) has no solution.

The function $s(\theta, q, \alpha, \hat{\alpha})$ is fundamental in our analysis, because according to Lemma 5, type $(s(\theta, q(\theta, \hat{\alpha}), \alpha, \hat{\alpha}), \alpha, \hat{\alpha})$ is assigned the quantity $q(\theta, \hat{\alpha})$. Differentiating (18), we get:

$$\frac{ds(\theta, q(\theta, \hat{\alpha}), \alpha, \hat{\alpha})}{d\theta} = \frac{dq(\theta, \hat{\alpha})}{d\theta} \frac{u_{qq}(q(\theta, \hat{\alpha}), \theta, \hat{\alpha}) - u_{qq}(q(\theta, \hat{\alpha}), s(\cdot), \alpha)}{u_{q\theta}(q(\theta, \hat{\alpha}), s(\cdot), \alpha)} + \frac{u_{q\theta}(q(\theta, \hat{\alpha}), \theta, \hat{\alpha})}{u_{q\theta}(q(\theta, \hat{\alpha}), s(\cdot), \alpha)} \quad (19)$$

Using the definition of $s(\theta, q(\theta, \hat{\alpha}), \alpha, \hat{\alpha})$ and (19) we will simplify our problem (14) further. Let $\Lambda(q, \theta, \alpha) \equiv u(q, \theta, \alpha)f(\theta, \alpha) - u_\theta(q, \theta, \alpha)(1 - F(\theta|\alpha))f(\alpha)$. We have:

Lemma 9 Change of variables. Fix some $\underline{q}(\alpha)$ and $q(\theta, a)$ satisfying Property 1. Consider some $\bar{\alpha} \in [0, 1]$.

If $(\theta_1, \bar{\alpha}) \in I(q(\theta'_1, 1), \theta'_1, 1)$ and $(\theta_2, \bar{\alpha}) \in I(q(\theta'_2, 1), \theta'_2, 1)$, then making the change of variables $\tilde{\theta} = s(\bar{\alpha}, \theta, 1, q(\theta, 1))$, we get:

$$\int_{\theta_1}^{\theta_2} \Lambda(q(\tilde{\theta}, \bar{\alpha}), \tilde{\theta}, \bar{\alpha})d\tilde{\theta} = \int_{\theta'_1}^{\theta'_2} \Lambda(q(\theta, 1), s(\bar{\alpha}, \theta, 1, q(\theta, 1)), \bar{\alpha}) \frac{ds(\bar{\alpha}, \theta, 1, q(\theta, 1))}{d\theta} d\theta \quad (20)$$

If $(\theta_1, \bar{\alpha}) \in I(q(\underline{\theta}(\alpha_1), \alpha_1), \underline{\theta}(\alpha_1), \alpha_1)$ and $(\theta_2, \bar{\alpha}) \in I(q(\underline{\theta}(\alpha_2), \alpha_2), \underline{\theta}(\alpha_2), \alpha_2)$, then making the change of variables $\theta = s(\bar{\alpha}, \underline{\theta}(\alpha), \alpha, q(\underline{\theta}(\alpha), \alpha))$, we get:

$$\int_{\theta_1}^{\theta_2} \Lambda(q(\tilde{\theta}, \bar{\alpha}), \tilde{\theta}, \bar{\alpha})d\tilde{\theta} = \int_{\alpha_1}^{\alpha_2} \Lambda(q(\underline{\theta}(\alpha), \alpha), s(\bar{\alpha}, \underline{\theta}(\alpha), \alpha, q(\underline{\theta}(\alpha), \alpha)), \bar{\alpha}) \frac{ds(\bar{\alpha}, \underline{\theta}(\alpha), \alpha, q(\underline{\theta}(\alpha), \alpha))}{d\alpha} d\alpha = \int_{\alpha_1}^{\alpha_2} \Lambda(\underline{q}(\alpha), s(\bar{\alpha}, \underline{\theta}(\alpha), \alpha, \underline{q}(\alpha)), \bar{\alpha}) \frac{ds(\bar{\alpha}, \underline{\theta}(\alpha), \alpha, \underline{q}(\alpha))}{d\alpha} d\alpha \quad (21)$$

Alternatively, denoting $\underline{\theta}_1 = \underline{\theta}(\alpha_1)$ and $\underline{\theta}_2 = \underline{\theta}(\alpha_2)$, and also using the function $\underline{\alpha}(\theta)$ -the inverse of $\underline{\theta}(\alpha)$ and the function $\underline{q}(\theta) \equiv \underline{q}(\underline{\alpha}(\theta))$, we can rewrite (21) as follows:

$$\begin{aligned} \int_{\underline{\theta}_1}^{\underline{\theta}_2} \Lambda(q(\tilde{\theta}, \bar{\alpha}), \tilde{\theta}, \bar{\alpha}) d\tilde{\theta} &= \int_{\underline{\theta}_1}^{\underline{\theta}_2} \Lambda(q(\theta, \underline{\alpha}(\theta)), s(\bar{\alpha}, \underline{\theta}(\alpha), \alpha, q(\underline{\theta}(\alpha), \alpha)), \bar{\alpha}) \frac{ds(\bar{\alpha}, \underline{\theta}(\alpha), \alpha, q(\underline{\theta}(\alpha), \alpha))}{d\theta} d\theta = \\ \int_{\underline{\theta}_1}^{\underline{\theta}_2} \Lambda(\underline{q}(\theta), s(\bar{\alpha}, \underline{\theta}(\alpha), \alpha, \underline{q}(\theta)), \bar{\alpha}) \frac{ds(\bar{\alpha}, \underline{\theta}(\alpha), \alpha, \underline{q}(\theta))}{d\theta} d\theta \end{aligned} \quad (22)$$

As the next step in the solution to the problem, we will consider two cases, Case A and Case C, corresponding to two different ways of subdividing the region $\Omega \equiv \{(\theta, \alpha) | \theta \in [\underline{\theta}(\alpha), 1], \alpha \in [0, 1]\}$ into three subregions based on the shape of the isoquants.

Case A is characterized by the following property: $q(\underline{\theta}(1), 1) \leq q(1, 0)$. In Case C, we have: $q(\underline{\theta}(1), 1) > q(1, 0)$.

Let us start from case A. Let θ^* be such that $q(\theta^*, 1) = q(1, 0)$. By Lemma 4 and (15), θ^* can be found as a solution to the following equation:

$$u_q(q(\theta^*, 1), \theta^*, 1) = u_q(q(\theta^*, 1), 1, 0) \quad (23)$$

Then $\theta^u(\alpha)$ is defined via $(\theta^u(\alpha), \alpha) \in I(q(\theta^*, 1), \theta^*, 1)$ for all α . Also, define $\theta^l(\alpha)$ via $(\theta^l(\alpha), \alpha) \in I(q(\underline{\theta}(1), 1), \underline{\theta}(1), 1)$ for all α ,

Case A. We have the following three regions:

‘Upper A’ - consists of all points above the isoquant $I(q(\theta^*, 1), \theta^*, 1)$, i.e. $(\theta, \alpha) \in$ ‘Upper A’ iff $\theta \geq \theta^u(\alpha)$.

The region ‘Mid A’: consists of all points between the isoquants $I(q(\theta^*, 1), \theta^*, 1)$ and $I(q(\underline{\theta}(1), 1), \underline{\theta}(1), 1)$. That is, $(\theta, \alpha) \in$ ‘Mid A’ iff $\theta^l(\alpha) \leq \theta \leq \theta^u(\alpha)$.

The region ‘Lower A’: consists of all points between the lower boundary $\underline{\theta}(\alpha)$ and the isoquant $I(q(\underline{\theta}(1), 1), \underline{\theta}(1), 1)$, i.e. $(\theta, \alpha) \in$ ‘Lower A’ iff $\underline{\theta}(\alpha) \leq \theta \leq \theta^u(\alpha)$.

In case C, let $\alpha^* \in [0, 1]$ be such that $q(\underline{\theta}(\alpha^*), \alpha^*) = q(0, 1)$. Also, define $\theta^r(\alpha)$ via $(\theta^r(\alpha), \alpha) \in I(q(\underline{\theta}(\alpha^*), \alpha^*), \underline{\theta}(\alpha^*), \alpha^*)$ for all α .

The region ‘Upper C’ consists of all points above the isoquant $I(q(\underline{\theta}(1), 1), \underline{\theta}(1), 1)$, i.e. $(\theta, \alpha) \in$ ‘Upper C’ iff $\theta \geq \theta^l(\alpha)$.

The region ‘Mid C’: consists of all points between the isoquants $I(q(\underline{\theta}(\alpha^*), \alpha^*), \underline{\theta}(\alpha^*), \alpha^*)$ and $I(q(\underline{\theta}(1), 1), \underline{\theta}(1), 1)$. That is, $(\theta, \alpha) \in$ ‘Mid C’ iff $\theta^r(\alpha) \leq \theta \leq \theta^l(\alpha)$.

Finally, the region ‘Lower C’ consists of all points between the lower boundary $\underline{\theta}(\alpha)$ and the isoquant $I(q(\underline{\theta}(\alpha^*), \alpha^*), \underline{\theta}(\alpha^*), \alpha^*)$, i.e. $(\theta, \alpha) \in$ ‘Lower C’ iff $\underline{\theta}(\alpha) \leq \theta \leq \theta^r(\alpha)$.

Next, let us define $\hat{s}(\alpha_1, \theta_2, \alpha_2)$ as the solution \hat{s} to the following equation:

$$u_q(q(\hat{s}, \alpha_1), \hat{s}, \alpha_1) = u_q(q(\hat{s}, \alpha_1), \theta_2, \alpha_2) \quad (24)$$

Also, let us define $z(\theta_1, \theta_2, \alpha_2, q)$ as the solution z to the following equation:

$$u_q(q, \theta_1, z) = u_q(q, \theta_2, \alpha_2) \quad (25)$$

Define $\underline{s}(\theta, \alpha)$ as the solution \underline{s} to the following equation:

$$u_q(q(\underline{s}, \underline{\alpha}(\underline{s})), \underline{s}, \underline{\alpha}(\underline{s})) = u_q(q(\underline{s}, \underline{\alpha}(\underline{s})), \theta, \alpha) \quad (26)$$

Finally, define $\underline{z}(\theta, \alpha)$ as the solution \underline{z} to the following equation:

$$u_q(q(\underline{\theta}(\underline{z}), \underline{z}), \underline{\theta}(\underline{z}), \underline{z}) = u_q(q(\underline{\theta}(\underline{z}), \underline{z}), \theta, \alpha) \quad (27)$$

Now, we can compute the expected value of the firm's profit function in different regions in cases A and C.

Case A.

The expected value of the firm's profits in region 'Upper A' is equal to:

$$\begin{aligned} & \int_0^1 \int_{s(\bar{\alpha}, \theta^*, 1, q(\theta^*, 1))}^1 \Lambda(q(\tilde{\theta}, \bar{\alpha}), \tilde{\theta}, \bar{\alpha}) d\tilde{\theta} d\bar{\alpha} = \int_0^1 \int_{\theta^*}^{\hat{s}(1, 1, \bar{\alpha})} \Lambda(q(\theta, 1), s(\bar{\alpha}, \theta, 1, q(\theta, 1)), \bar{\alpha}) \frac{ds(\bar{\alpha}, \theta, 1, q(\theta, 1))}{d\theta} d\theta d\bar{\alpha} \\ & = \int_{\theta^*}^1 \int_{z(1, \theta, 1, q(\theta, 1))}^1 \Lambda(q(\theta, 1), s(\bar{\alpha}, \theta, 1, q(\theta, 1)), \bar{\alpha}) \frac{ds(\bar{\alpha}, \theta, 1, q(\theta, 1))}{d\theta} d\bar{\alpha} d\theta \end{aligned} \quad (28)$$

The expected value of the firm's profit function in the region 'Mid A' is equal to:

$$\begin{aligned} & \int_0^1 \int_{s(\bar{\alpha}, \underline{\theta}(1), 1, q(\underline{\theta}(1), 1))}^{s(\bar{\alpha}, \theta^*, 1, q(\theta^*, 1))} \Lambda(q(\tilde{\theta}, \bar{\alpha}), \tilde{\theta}, \bar{\alpha}) d\tilde{\theta} d\bar{\alpha} = \int_0^1 \int_{\underline{\theta}(1)}^{\theta^*} \Lambda(q(\theta, 1), s(\bar{\alpha}, \theta, 1, q(\theta, 1)), \bar{\alpha}) \frac{ds(\bar{\alpha}, \theta, 1, q(\theta, 1))}{d\theta} d\theta d\bar{\alpha} \\ & = \int_{\underline{\theta}(1)}^{\theta^*} \int_0^1 \Lambda(q(\theta, 1), s(\bar{\alpha}, \theta, 1, q(\theta, 1)), \bar{\alpha}) \frac{ds(\bar{\alpha}, \theta, 1, q(\theta, 1))}{d\theta} d\bar{\alpha} d\theta \end{aligned} \quad (29)$$

The expected value of the firm's profit function in the region 'Lower A' is equal to:

$$\begin{aligned} & \int_0^1 \int_{\underline{\theta}(\bar{\alpha})}^{s(\bar{\alpha}, \underline{\theta}(1), 1, q(\underline{\theta}(1), 1))} \Lambda(q(\tilde{\theta}, \bar{\alpha}), \tilde{\theta}, \bar{\alpha}) d\tilde{\theta} d\bar{\alpha} = \\ & \int_0^1 \int_{\theta(\bar{\alpha})}^{\underline{\theta}(1)} \Lambda(q(\theta, \underline{\alpha}(\theta)), s(\bar{\alpha}, \theta, \underline{\alpha}(\theta), q(\theta, \underline{\alpha}(\theta)))) \frac{ds(\bar{\alpha}, \theta, \underline{\alpha}(\theta), q(\theta, \underline{\alpha}(\theta)))}{d\theta} d\theta d\bar{\alpha} \end{aligned} \quad (30)$$

Case C. The expected value of the firm's profit function in the region 'Upper C' is equal to:

$$\begin{aligned} & \int_{z(1, \underline{\theta}(1), 1, q(\underline{\theta}(1), 1))}^1 \int_{s(\alpha, \underline{\theta}(1), 1, q(\underline{\theta}(1), 1))}^1 \Lambda(q(\tilde{\theta}, \alpha), \tilde{\theta}, \alpha) d\tilde{\theta} d\alpha = \\ & \int_{z(1, \underline{\theta}(1), 1, q(\underline{\theta}(1), 1))}^1 \int_{\underline{\theta}(1)}^{\hat{s}(1, 1, \alpha)} \Lambda(q(\theta, 1), s(\alpha, \theta, 1, q(\theta, 1)), \alpha) \frac{ds(\alpha, \theta, 1, q(\theta, 1))}{d\theta} d\theta d\alpha = \\ & \int_{\underline{\theta}(1)}^1 \int_{z(1, \theta, 1, q(\theta, 1))}^1 \Lambda(q(\theta, 1), s(\alpha, \theta, 1, q(\theta, 1)), \alpha) \frac{ds(\alpha, \theta, 1, q(\theta, 1))}{d\theta} d\alpha d\theta \end{aligned} \quad (31)$$

The expected value of the firm's profit function in the region 'Mid C' is equal to:

$$\begin{aligned}
& \int_0^1 \int_{\max\{\underline{\alpha}(\tilde{\theta}), z(\theta, 1, 0, q(1, 0))\}}^{z(\tilde{\theta}, \underline{\theta}(1), 1, q(\underline{\theta}(1), 1))} \Lambda(q(\tilde{\theta}, \bar{\alpha}), \tilde{\theta}, \bar{\alpha}) d\bar{\alpha} d\tilde{\theta} = \\
& \int_0^1 \int_{\max\{\underline{\theta}(\alpha), s(\bar{\alpha}, 1, 0, q(1, 0))\}}^{\min\{1, s(\bar{\alpha}, \underline{\theta}(1), 1, q(\underline{\theta}(1), 1))\}} \Lambda(q(\tilde{\theta}, \bar{\alpha}), \tilde{\theta}, \bar{\alpha}) d\tilde{\theta} d\bar{\alpha} = \\
& \int_0^1 \int_{\max\{\underline{\theta}(\alpha), \underline{s}(1, 0)\}}^{\min\{1, s(\alpha, \underline{\theta}(1), 1, q(\underline{\theta}(1), 1))\}} \Lambda(\underline{q}(\theta), s(\alpha, \theta, \underline{\alpha}(\theta), \underline{q}(\theta)), \alpha) \frac{ds(\alpha, \theta, \underline{\alpha}(\theta), \underline{q}(\theta))}{d\theta} d\theta d\alpha \\
& = \int_{\underline{\theta}(1)}^{\underline{s}(1, 0)} \int_{z(1, \theta, \underline{\alpha}(\theta), \underline{q}(\theta))}^{\underline{\alpha}(\theta)} \Lambda(\underline{q}(\theta), s(\alpha, \theta, \underline{\alpha}(\theta), \underline{q}(\theta)), \alpha) \frac{ds(\alpha, \theta, \underline{\alpha}(\theta), \underline{q}(\theta))}{d\theta} d\alpha d\theta \quad (32)
\end{aligned}$$

The expected value of the firm's profit function in the region 'Lower C' is equal to:

$$\begin{aligned}
& \int_0^{\underline{z}(1, 0)} \int_{\underline{\theta}(\alpha)}^{s(\alpha, 1, 0, q(1, 0))} \Lambda(q(\tilde{\theta}, \bar{\alpha}), \tilde{\theta}, \bar{\alpha}) d\tilde{\theta} d\bar{\alpha} = \\
& \int_{\underline{s}(1, 0)}^1 \int_{\underline{\alpha}(\theta)}^{z(\theta, 1, 0, q(1, 0))} \Lambda(q(\tilde{\theta}, \bar{\alpha}), \tilde{\theta}, \bar{\alpha}) d\bar{\alpha} d\tilde{\theta} = \\
& \int_{\underline{s}(1, 0)}^1 \int_{\underline{z}(\theta, 0)}^{\underline{z}(1, 0)} \Lambda(\underline{q}(\alpha), \theta, z(\theta, \underline{\theta}(\alpha), \alpha, \underline{q}(\alpha))) \frac{dz(\theta, \underline{\theta}(\alpha), \alpha, \underline{q}(\alpha))}{d\alpha} d\alpha d\theta \\
& = \int_0^{\underline{z}(1, 0)} \int_{\underline{\theta}(\alpha)}^{\underline{s}(1, 0)} \Lambda(\underline{q}(\theta), s(\alpha, \theta, \underline{\alpha}(\theta), \underline{q}(\theta)), \alpha) \frac{ds(\alpha, \theta, \underline{\alpha}(\theta), \underline{q}(\theta))}{d\theta} d\theta d\alpha \quad (33)
\end{aligned}$$

Inspecting the expressions (28)-(33), we conclude that the firm's expected profits in cases A and C depend on the schedules $\underline{q}(\theta)$ (equivalently, $\underline{q}(\alpha)$) and $q(\theta, 1)$ ($\theta \in [\underline{\theta}(1), 1]$), as well as on the form of the 'lower boundary' $\underline{\theta}(\alpha)$. But since the shape of the lower boundary $\underline{\theta}(\alpha)$ is determined by $\underline{q}(\theta)$ and the level of the boundary $\underline{\theta}(1)$, we have:

Thus, we have used our 'isoquants' method to convert the problem from a two-dimensional one into a one-dimensional problem. Although the resulting one-dimensional problem is rather complex, it is significantly easier to solve.

In the next section we explicitly derive a solution in the case of linear quadratic utility with uniform distribution of types:

4 Linear-Quadratic-Uniform Case.

In this section, we derive an explicit solution for the linear quadratic case where a consumer's utility function is given by $u(q, \theta, \alpha) = \theta q - \frac{(b-\alpha)}{2} q^2$, and the probability distribution $F(\alpha, \theta)$ of the type (θ, α) is uniform on $[0, 1]^2$.

We can choose a more general specification by allowing α to be uniformly distributed on any interval $[0, a]$ or, equivalently, by using the following specification of the utility: $u(q, \theta, \alpha) = \theta q - \frac{(b-\alpha)}{c} q^2$ where $c > 0$. However the solution in any such case can be obtained by a simple renormalization of the solution for our basic case with $a = 1$ and $c = 2$. With the linear

quadratic functional form, we have: $\Lambda(q, \theta, \alpha) = (2\theta - 1)q - \frac{(b-\alpha)}{2}q^2$, and the appropriate density functions are $f(\alpha, \theta) = 1$, $f(\theta) = 1$ and $f(\alpha) = 1$ for all θ and α . It is easy to compute that $s(\alpha_1, \theta_2, \alpha_2, q) = \theta_2 + (\alpha_2 - \alpha_1)q$, $z(\theta_1, \theta_2, \alpha_2, q) = \alpha_2 + \frac{\theta_2 - \theta_1}{q}$, and $\underline{\theta}'(\alpha) = -\frac{q(\alpha)}{2}$.

Proposition 1 *The optimal quantity schedule is characterized as follows:*

If $b \geq 3/2$, then:

$q(\theta, \alpha) = 0$ for all (θ, α) such that $\theta < 1/2$, i.e. the lower boundary $\underline{\alpha}(\cdot)$ is such that $\underline{\alpha}(\theta) = 1$ if $\theta \leq 1/2$ and $\underline{\alpha}(\theta) = 0$ if $\theta > 1/2$.

$q(\theta, \alpha) = \frac{3\theta-1}{2b+1-3\alpha}$ when (θ, α) satisfies $\theta + \alpha \frac{2}{2b+1} \geq 1$.

$q(\theta, \alpha) = \frac{2(2\theta-1)}{2b+1-4\alpha}$ when (θ, α) satisfies $\theta + \alpha \frac{2}{2b+1} \leq 1$, $\theta \geq 1/2$.

If $1 < b < 3/2$, then let \hat{q} , $\hat{\theta}$, $\underline{\theta}(1)$ and k be given by respectively:

$$\hat{q} \text{ is the unique solution on } (0, 3/4] \text{ to the following equation:} \quad (1 + bq - bq^2(3/2 + 2b))^2 = (1 - bq)(1 + bq - bq^2(5/2 + b))^3; \quad (34)$$

$$\underline{\theta}(1) = \frac{1 + \sqrt{(1 - b\hat{q})(4(1 + b\hat{q}) - 2b\hat{q}^2(5 + 2b))}}{3} \quad (35)$$

$$k = \frac{4b\hat{q}^2}{3}(1 - b\hat{q}) \quad (36)$$

$$\hat{\theta} = 1 - \frac{2b\hat{q}}{3} \quad (37)$$

Then, the lower boundary $\underline{\alpha}(\theta)$ is given by:

$$\underline{\alpha}(\theta) = \begin{cases} 0 & \text{if } \theta > \hat{\theta}, \\ \frac{2b}{3} + \frac{2}{k} \left(\theta(1 - \theta)^2 - \hat{\theta}(1 - \hat{\theta})^2 \right) & \text{if } \theta \leq \hat{\theta}, \end{cases}$$

The optimal quantity allocation is given by:

$$q(\theta, \alpha) = \begin{cases} 0 & \text{if } \alpha < \underline{\alpha}(\theta) \\ \frac{\theta - \hat{\theta}}{\frac{2b}{3} - \alpha} & \text{if } \theta \geq \hat{\theta}, \alpha(1 - \hat{\theta}) + \frac{2}{3}b\theta \leq \frac{2}{3}b; \\ \text{solves: } \theta + (\alpha - \frac{2b}{3})q = 2 \left(1 - \sqrt{4 - 3(1 + \frac{k}{q})} \right) \left(\frac{1}{3} + \frac{q}{k} \left(\frac{1}{3} + \frac{2}{3}\sqrt{4 - 3(1 + \frac{k}{q})} \right)^2 \right) & \\ \text{if } \alpha(1 - \hat{\theta}) + \frac{2}{3}b\theta > \frac{2}{3}b, \alpha > \underline{\alpha}(\theta), \text{ and } \theta + \alpha \frac{3\theta(1)-1}{2(b-1)} < \frac{\theta(1)(2b+1)-1}{2(b-1)} & \\ \frac{3\theta-1}{2b+1-3\alpha} & \text{if } \theta + \alpha \frac{3\theta(1)-1}{2(b-1)} \geq \frac{\theta(1)(2b+1)-1}{2(b-1)} \end{cases}$$

or if $\alpha < \underline{\alpha}(\theta) \equiv \frac{2b}{3}$ $\theta < \theta^$, $\alpha < \frac{2}{3}b$;*

In the remainder of this section we will prove this proposition. We start by considering the boundary $\underline{\theta}(\alpha)$ in the Linear-Quadratic case. As we know, the quantity schedule $\underline{q}(\alpha)$ is increasing in α along this boundary.

The proof will now proceed via the following steps:

Step (i): Obtain closed-form expressions for the firm's profits in all regions in cases A and C.

Step (ii): Determine which case applies under different parameter values;

Step (iii): Combine the expected payoffs in different regions to obtain an expression for the total expected payoffs and maximize over the boundary quantity schedules.

Let us start from region ‘Upper A’ and ‘Upper C.’ The expression for the firm’s expected profits in these cases is given by (28) and (31) respectively which differ only in the lower limit of integration with respect to θ : this limit is equal to θ^* in Case A defined by (23). In case C this limit of integration is equal to $\underline{\theta}(1)$. Next, note that $z(1, \theta, 1, q(\theta, 1)) = \frac{\theta-1}{q(\theta, 1)} + 1$ and $s(\alpha, \theta, 1, q(\theta, 1), \alpha) = \theta + (1 - \alpha)q(\theta, 1)$.

Let $q(\theta) = q(\theta, 1)$ and consider the following expression which is equal to (28) when $\theta^l = \theta^*$ and is equal to (31) if $\theta^l = \underline{\theta}(1)$:

$$\begin{aligned} & \int_{\theta^l}^1 \int_{\frac{\theta-1}{q(\theta)}+1}^1 \left((2\theta + 2(1 - \alpha)q(\theta) - 1)q(\theta) - \frac{(b - \alpha)q^2(\theta)}{2} \right) (1 + (1 - \alpha)q'(\theta)) d\alpha d\theta = \\ & \int_{\theta^l}^1 \int_{\frac{\theta-1}{q(\theta)}+1}^1 \left((2\theta - 1)q(\theta) + \left(2 - \frac{b + 3\alpha}{2} \right) q(\theta)^2 \right) (1 + (1 - \alpha)q'(\theta)) d\alpha d\theta \end{aligned} \quad (38)$$

Integrating the integrand of (38) with respect to α first, we obtain:

$$\begin{aligned} & \int_{\frac{\theta-1}{q(\theta)}+1}^1 \left((2\theta - 1)q(\theta) + \left(2 - \frac{b + 3\alpha}{2} \right) q(\theta)^2 \right) (1 + (1 - \alpha)q'(\theta)) d\alpha \\ & = (2\theta - 1)(1 - \theta) + q(\theta)^2 \frac{(1 - b)(1 - \theta)}{2q(\theta)} + \frac{3(1 - \theta)^2}{4} + q'(\theta)q(\theta) \frac{(2\theta - 1)(1 - \theta)^2}{2q(\theta)^2} + \\ & q'(\theta)q(\theta)^2 \frac{(1 - b)(1 - \theta)^2}{4q(\theta)^2} + q'(\theta)q(\theta)^2 \frac{(1 - \theta)^3}{2q(\theta)^3} \\ & = \frac{(5\theta - 1)(1 - \theta)}{4} + q(\theta) \frac{(1 - b)(1 - \theta)}{2} + q'(\theta) \frac{(1 - b)(1 - \theta)^2}{4} + \frac{q'(\theta)}{2q(\theta)} \theta(1 - \theta)^2 \\ & = \frac{(5\theta - 1)(1 - \theta)}{4} + q(\theta)(1 - b)(1 - \theta) + \left(\frac{q(\theta)(1 - b)(1 - \theta)^2}{4} \right)' + \frac{q'(\theta)}{2q(\theta)} \theta(1 - \theta)^2 \end{aligned} \quad (39)$$

Performing the integration of (39) with respect to θ on the interval $[\theta_l, 1]$, we obtain that (38) is equal to:

$$\int_{\theta_l}^1 \frac{(5\theta - 1)(1 - \theta)}{4} d\theta - \frac{q(\theta^l)(1 - b)(1 - \theta^l)^2}{4} + \int_{\theta_l}^1 \frac{q'(\theta)}{2q(\theta)} \theta(1 - \theta)^2 + q(\theta)(1 - b)(1 - \theta) d\theta \quad (40)$$

Neither the first, nor the second term of (40) depend on the values of $q(\theta)$ on $(\theta^l, 1]$. Then consider only the sum of the third and the fourth terms in (40). Integrating by parts, we get:

$$\begin{aligned} & \int_{\theta_l}^1 \frac{q'(\theta)}{2q(\theta)} \theta(1 - \theta)^2 d\theta + q(\theta)(1 - b)(1 - \theta) d\theta = -\frac{\theta^l(1 - \theta^l)^2 \log(q(\theta^l))}{2} \\ & + \int_{\theta_l}^1 \frac{\log(q(\theta))(1 - \theta)(3\theta - 1)}{2} + q(\theta)(1 - b)(1 - \theta) d\theta \end{aligned} \quad (41)$$

The maximization of (41) with respect to $q(\cdot)$ gives us the first quantity allocation function

$$q(\theta) \equiv q(\theta, 1) = \frac{3\theta - 1}{2(b - 1)} \quad (42)$$

If $\theta^l = \theta^*$, i.e. Case A applies, then we have $1 = \theta^* + q(\theta^*, 1)$, i.e. $1 = \theta^* + \frac{3\theta^* - 1}{2(b - 1)}$, and hence $\theta^* = \frac{2b - 1}{2b + 1}$.

Let us now compute the firm's expected profit in the region 'Mid A.' Substituting the linear quadratic functional form into (29) we get:

$$\begin{aligned} & \int_{\underline{\theta}(1)}^{\theta^*} \int_0^1 \left((2\theta - 1)q(\theta) + \left(2 - \frac{b + 3\alpha}{2} \right) q(\theta)^2 \right) (1 + (1 - \alpha)q'(\theta)) d\alpha d\theta = \\ & \int_{\underline{\theta}(1)}^{\theta^*} (2\theta - 1)q(\theta) + \frac{(5 - 2b)q(\theta)^2}{4} + q'(\theta)q(\theta)\frac{2\theta - 1}{2} + q'(\theta)q(\theta)^2\frac{3 - b}{4} d\theta \end{aligned} \quad (43)$$

To derive the optimal quantity in this region, we can use the calculus of variations techniques. In particular, recall the Euler equation for maximization of the functional $\int_a^b F(q(\theta), q'(\theta), \theta) d\theta$ is: $F_q = \frac{dF_{q'}}{d\theta}$. We have: $F(q(\theta), q'(\theta), \theta) = (2\theta - 1)q(\theta) + \frac{(5 - 2b)q(\theta)^2}{4} + q'(\theta)q(\theta)\frac{2\theta - 1}{2} + q'(\theta)q(\theta)^2\frac{3 - b}{4}$. Thus, the Euler equation is:

$$(2\theta - 1) + \frac{(5 - 2b)q(\theta)}{2} + q'(\theta)\frac{2\theta - 1}{2} + q'(\theta)q(\theta)\frac{3 - b}{2} = q'(\theta)\frac{2\theta - 1}{2} + q(\theta) + q'(\theta)q(\theta)\frac{3 - b}{2}$$

Simplifying, we obtain:

$$q(\theta) \equiv q(\theta, 1) = \frac{2\theta - 1}{b - 1.5} \quad (44)$$

All together, (42) and (44) provide a solution for the case $b > 3/2$ described in the statement of the proposition. This is when Case A applies. Note that the lower boundary in this case is flat and given by $\theta = 1/2$, i.e. $\underline{\theta}(0) = \underline{\theta}(1) = 1/2$. Therefore, the lower region (i.e. 'Lower A') is degenerate.

Now let us consider $b \in (1, 1.5]$. In this case, the solution on the upper part is given by (42) with $\theta^l = \underline{\theta}$. Hence, we need to study the regions 'Mid C' and 'Lower C.' s

Substituting the expressions for the functional forms into (32) we get the following expression for the firm's expected profits in the region 'Mid C':

$$\begin{aligned} & \int_{\underline{\theta}(1)}^{\underline{s}(1,0)} \int_{z(1,\theta,\underline{\alpha}(\theta),\underline{q}(\theta))}^{\underline{\alpha}(\theta)} \Lambda(\underline{q}(\theta), s(\alpha, \theta, \underline{\alpha}(\theta), \underline{q}(\theta)), \alpha) \frac{\partial s(\alpha, \theta, \underline{\alpha}(\theta), \underline{q}(\theta))}{\partial \theta} d\alpha d\theta = \\ & \int_{\underline{\theta}(1)}^{\hat{\theta}} \int_{\underline{\alpha}(\theta) - \frac{1 - \theta}{\underline{q}(\theta)}}^{\underline{\alpha}(\theta)} \left((2\theta - 1)\underline{q}(\theta) + \left(2\underline{\alpha}(\theta) - \frac{b + 3\alpha}{2} \right) \underline{q}(\theta)^2 \right) (1 + (\alpha - \underline{\alpha}(\theta))\underline{q}'(\theta)) d\alpha d\theta \end{aligned} \quad (45)$$

Let us integrate with respect to α first. We obtain:

$$\begin{aligned}
& \int_{\underline{\alpha}(\theta) - \frac{1-\theta}{\underline{q}(\theta)}}^{\underline{\alpha}(\theta)} \left((2\theta - 1)\underline{q}(\theta) + \left(2\underline{\alpha}(\theta) - \frac{b+3\alpha}{2} \right) \underline{q}(\theta)^2 \right) (1 + (\alpha - \underline{\alpha}(\theta))\underline{q}'(\theta)) d\alpha = (2\theta - 1)(1 - \theta) \\
& + \frac{\underline{q}(\theta)(\underline{\alpha}(\theta) - b)(1 - \theta)}{2} + \frac{3(1 - \theta)^2}{4} - \frac{\underline{q}'(\theta)(2\theta - 1)(1 - \theta)^2}{\underline{q}(\theta)} - \frac{\underline{q}'(\theta)(1 - \theta)^3}{\underline{q}(\theta)} - \underline{q}'(\theta) \frac{(1 - \theta)^2(\underline{\alpha}(\theta) - b)}{4} = \\
& = \frac{(5\theta - 1)(1 - \theta)}{4} - \frac{\underline{q}'(\theta)\theta(1 - \theta)^2}{\underline{q}(\theta)} - \left(\frac{\underline{q}(\theta)(\underline{\alpha}(\theta) - b)(1 - \theta)^2}{4} \right)' - \frac{(1 - \theta)^2}{2} \quad (46)
\end{aligned}$$

Finally, let us consider the region ‘Lower C’. Let V_{LC} denote the firm’s expected payoff in this region. First, substitute the expressions for the linear quadratic functional forms into (33). Also, define $\hat{\theta}$ as follows:

$$\hat{\theta} + \underline{\alpha}(\hat{\theta})\underline{q}(\hat{\theta}) = 1 \quad (47)$$

Next, let $\hat{\alpha} = \underline{\alpha}(\hat{\theta})$ and $\hat{q} = \underline{q}(\hat{\theta})$ in the region *LowerC*. Then we have

$$\begin{aligned}
V_{LC} &= \int_0^{\underline{z}(1,0)} \int_{\underline{\theta}(\alpha)}^{s(0,\underline{\theta}(\alpha),\alpha,\underline{q}(\alpha))} \Lambda(\underline{q}(\alpha), z(\theta, \underline{\theta}(\alpha), \alpha, \underline{q}(\alpha)), \alpha) \frac{\partial z(\theta, \underline{\theta}(\alpha), \alpha, \underline{q}(\alpha))}{\partial \alpha} d\theta d\alpha = \\
& \int_0^{\hat{\alpha}} \int_{\underline{\theta}(\alpha)}^{\underline{\theta}(\alpha) + \alpha \underline{q}(\alpha)} \left(\left(\frac{3}{2}\theta + \frac{1}{2}\underline{\theta}(\alpha) - 1 \right) \underline{q}(\alpha) - \frac{\underline{q}^2(\alpha)}{2}(b - \alpha) \right) \left(\frac{1}{2} + \frac{\underline{q}'(\alpha)(\theta - \underline{\theta}(\alpha))}{\underline{q}^2(\alpha)} \right) d\theta d\alpha \quad (48)
\end{aligned}$$

Integrating (48) with respect to θ , we obtain:

$$\begin{aligned}
& \int_{\underline{\theta}(\alpha)}^{\underline{\theta}(\alpha) + \alpha \underline{q}(\alpha)} \left(\left(\frac{3}{2}\theta + \frac{1}{2}\underline{\theta}(\alpha) - 1 \right) \underline{q}(\alpha) - \frac{\underline{q}^2(\alpha)}{2}(b - \alpha) \right) \left(\frac{1}{2} + \frac{\underline{q}'(\alpha)(\theta - \underline{\theta}(\alpha))}{\underline{q}^2(\alpha)} \right) d\theta d\alpha = \\
& \underline{q}^2(\alpha)\alpha \left(\underline{\theta}(\alpha) - \frac{1}{2} + \frac{3\alpha \underline{q}(\alpha)}{8} \right) - \frac{\underline{q}^3(\alpha)}{4}(b - \alpha)\alpha + \frac{\underline{q}'(\alpha)\underline{q}(\alpha)\alpha^2}{2}(2\underline{\theta}(\alpha) - 1 + \alpha \underline{q}(\alpha)) - \frac{\underline{q}'(\alpha)\underline{q}^2(\alpha)}{4}(b - \alpha)\alpha^2 \\
& = \underline{q}^2(\alpha)\alpha \frac{2\underline{\theta}(\alpha) - 1}{2} + \underline{q}^3(\alpha)\alpha \left(\frac{5\alpha}{8} - \frac{b}{4} \right) + \frac{\underline{q}'(\alpha)\underline{q}(\alpha)\alpha^2}{2} \left(2\underline{\theta}(\alpha) - 1 + \frac{3\alpha \underline{q}(\alpha)}{2} - \frac{b\underline{q}(\alpha)}{2} \right) = \\
& \left(\underline{q}^2(\alpha)\alpha \frac{2\underline{\theta}(\alpha) - 1}{2} + \frac{\underline{q}'(\alpha)\underline{q}(\alpha)\alpha^2}{2}(2\underline{\theta}(\alpha) - 1) - \frac{\underline{q}^3(\alpha)\alpha^2}{4} \right) + \left(\frac{\underline{q}^3(\alpha)\alpha^2}{2} + \frac{\underline{q}'(\alpha)\underline{q}^2(\alpha)\alpha^3}{2} \right) + \\
& - \left(\frac{\underline{q}^3(\alpha)\alpha(b - \alpha)}{6} - \frac{\underline{q}^3(\alpha)\alpha^2}{12} + \frac{\underline{q}'(\alpha)\underline{q}(\alpha)^2\alpha^2(b - \alpha)}{4} \right) + \frac{\underline{q}(\alpha)^3\alpha^2}{8} - \frac{\underline{q}(\alpha)^3\alpha b}{12} \quad (49)
\end{aligned}$$

Integrating this expression with respect to α on the interval $[0, \hat{\alpha}]$ yields the value of the firm’s objective in the region lower C:

$$V_{LC} = \hat{q}^2 \hat{\alpha}^2 \frac{2\underline{\theta}(\hat{\alpha}) - 1}{4} + \frac{\hat{q}^3 \hat{\alpha}^3}{6} - \frac{\hat{q}^3 \hat{\alpha}^2 (b - \hat{\alpha})}{12} + \int_0^{\hat{\alpha}} \frac{\underline{q}(\alpha)^3 \alpha^2}{8} - \frac{\underline{q}(\alpha)^3 \alpha b}{12} d\alpha \quad (50)$$

Note that the value of $q(\theta, \alpha)$ in regions *MidC*, *LowerC*, or *UpperC* cannot be set independently of its value in the other regions, because incentive compatibility requires $q(\theta, \alpha)$ to be increasing in its arguments. Therefore, we need to consider these regions and the firm's expected profits in them jointly. Rewriting the above, the profits in the regions Upper C, Mid C and Lower C are given by the following expressions:

$$V_{UC} = \int_{\underline{\theta}(1)}^1 \frac{(5\theta - 1)(1 - \theta)}{4} d\theta + \frac{q'(\theta)}{2q(\theta)} \theta(1 - \theta)^2 + q(\theta)(1 - b)(1 - \theta) d\theta - \frac{q(\underline{\theta}(1), 1)(1 - b)(1 - \underline{\theta}(1))^2}{4} \quad (51)$$

$$\begin{aligned} &= \int_{\underline{\theta}(1)}^1 \frac{(5\theta - 1)(1 - \theta)}{4} + \frac{\log(q(\theta))(1 - \theta)(3\theta - 1)}{2} + q(\theta)(1 - b)(1 - \theta) d\theta \\ &\quad - \frac{q(\underline{\theta}(1), 1)(1 - b)(1 - \underline{\theta}(1))^2}{4} - \frac{\underline{\theta}(1)(1 - \underline{\theta}(1))^2 \log(q(\underline{\theta}(1), 1))}{2} \\ V_{MC} &= \int_{\underline{\theta}(1)}^{\hat{\theta}} \frac{(5\theta - 1)(1 - \theta)}{4} - \frac{q'(\theta)}{q(\theta)} \frac{\theta(1 - \theta)^2}{2} - \frac{(1 - \theta)^2}{2} - \frac{d\left(\frac{q(\theta)(\underline{\alpha}(\theta) - b)(1 - \theta)^2}{4}\right)}{d\theta} d\theta \quad (52) \\ &= \int_{\underline{\theta}(1)}^{\hat{\theta}} \frac{(5\theta - 1)(1 - \theta)}{4} - \frac{\log(q(\theta))(1 - \theta)(3\theta - 1)}{2} - \frac{(1 - \theta)^2}{2} d\theta \\ &\quad + \frac{q(\underline{\theta}(1))(1 - b)(1 - \underline{\theta}(1))^2}{4} - \frac{q(\hat{\theta})(\hat{\alpha} - b)(1 - \hat{\theta})^2}{4} + \frac{\underline{\theta}(1)(1 - \underline{\theta}(1))^2 \log(q(\underline{\theta}(1)))}{2} - \frac{\hat{\theta}(1 - \hat{\theta})^2 \log(q(\hat{\theta}(1)))}{2} \quad (53) \end{aligned}$$

$$V_{LC} = \hat{q}^2 \hat{\alpha}^2 \frac{2\underline{\theta}(\hat{\alpha}) - 1}{4} + \frac{\hat{q}^3 \hat{\alpha}^3}{6} - \frac{\hat{q}^3 \hat{\alpha}^2 (b - \hat{\alpha})}{12} + \int_0^{\hat{\alpha}} \frac{q(\alpha)^3 \alpha^2}{8} - \frac{q(\alpha)^3 \alpha b}{12} d\alpha \quad (54)$$

Note that the firm's expected profits EV are given by:

$$EV = V_{UC} + V_{MC} + V_{LC} \quad (55)$$

We will now establish the following intermediate lemma

- Lemma 10** (i) In the region *Mid C*, $\underline{q}(\theta) = \frac{k}{(3\theta - 1)(1 - \theta)}$ for some $k > 0$.
(ii) In the region *Lower C*, $\hat{\alpha} = \frac{2b}{3}$ and $\underline{q}(\alpha) = 0$ for all $\alpha \in [0, \hat{\alpha}]$.
(iii) In the region *Mid C*, $\underline{q}(\underline{\theta}(1)) = \frac{3\underline{\theta}(1) - 1}{2(b - 1)}$.
(iv) In the region *Upper C*, $q(\theta, 1) = \frac{3\theta - 1}{2(b - 1)}$.
(v) $\hat{q} = \underline{q}(\hat{\alpha})$.

We will now use Lemma 10 to complete the proof of Proposition 1 First, equation (37), i.e. $\hat{\theta} = 1 - \frac{2b\hat{q}}{3}$ immediately follows from part (ii) of Lemma 10 and the fact that $1 = \hat{\theta} + \hat{\alpha}\hat{q}$.

Second, the value of k in (36) immediately follows from part (i) of Lemma 10. In particular, it obtains when we take $k = \hat{q}(3\hat{\theta} - 1)(1 - \hat{\theta})$ and substitute $\hat{\theta} = 1 - \frac{2b}{3}\hat{q}$ in it.

Next, to derive \hat{q} in (34) and $\underline{\theta}(1)$ in (35), we will obtain a system of two equations. First, note that $1 - \hat{\alpha} = \int_{\underline{\theta}(1)}^{\hat{\theta}} \frac{2}{\underline{q}(\theta)} d\theta$. Substituting $\underline{q}(\theta) = \frac{k}{(1 - \theta)(3\theta - 1)}$ into this integral, integrating

it, and using $\hat{\alpha} = \frac{2}{3}b$ and $\hat{\theta} = 1 - \frac{2b\hat{q}}{3}$, we obtain the following equation:

$$\underline{\theta}(1)(1 - \underline{\theta}(1))^2 = \frac{2}{3}b\hat{q}^2(1 - b\hat{q} + \frac{2}{9}b^2\hat{q}) \quad (56)$$

To obtain the second equation, note that by parts (iii) and (iv) of Lemma 10, $\underline{q}(\underline{\theta}(1)) = \frac{k}{(3\underline{\theta}(1)(1-\underline{\theta}))} = \frac{3\underline{\theta}(1)-1}{2(b-1)}$. So, $k = \frac{(3\underline{\theta}(1)-1)^2(1-\underline{\theta})}{2(b-1)}$. Combining this with (36), we obtain our second equation:

$$(3\underline{\theta}(1) - 1)^2(1 - \underline{\theta}) = \frac{4b\hat{q}^2}{3}(1 - b\hat{q})2(b - 1) \quad (57)$$

To solve the system of equations (56)-(57), first, multiply equation (56) by 9 and add to (57). After simplification, we obtain a quadratic equation in $\underline{\theta}(1)$ which solves to yield equation (35) i.e.:

$$\underline{\theta}(1) = \frac{1 + \sqrt{(1 - b\hat{q})(4(1 + b\hat{q}) - 2b\hat{q}^2(5 + 2b))}}{3}$$

Finally, to obtain the expression for (34), we substitute the above expression for $\underline{\theta}(1)$ into (57) and simplify.

The expression for the lower boundary $\underline{\alpha}(\theta)$ follows because $\hat{\alpha} = \underline{\alpha}(\hat{q}) = \frac{2b}{3}$ and $\underline{\alpha}(\theta) - \hat{\alpha} = \int_{\theta}^{\hat{\theta}} \frac{2}{\underline{q}(\theta)} d\theta = \int_{\theta}^{\hat{\theta}} \frac{2(3\theta-1)(1-\theta)}{k} d\theta$.

Finally, the expressions for $q(\theta, \alpha)$ are easily computed given the expressions for the isoquants, and the values of $\underline{q}(\cdot)$ on the lower boundary and $q(\theta, 1)$ on the right boundary. This completes the proof of Proposition 1. The following figures illustrate the isoquants in the cases A and C.

Figure 1: Typical shape of the isoquants

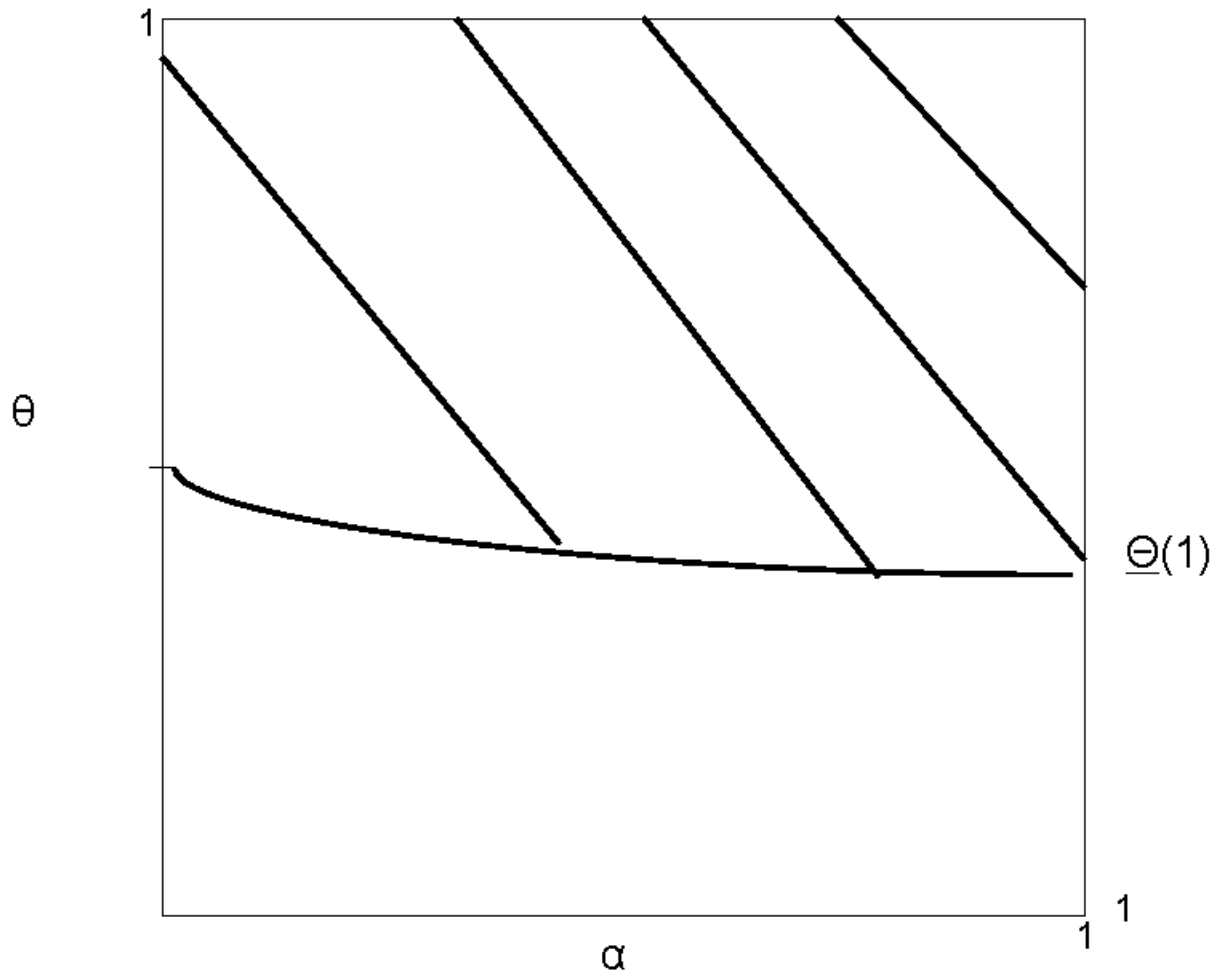


Figure 2: Linear- quadratic case with uniform distribution. Isoquants in Case A: $b > 3/2$.

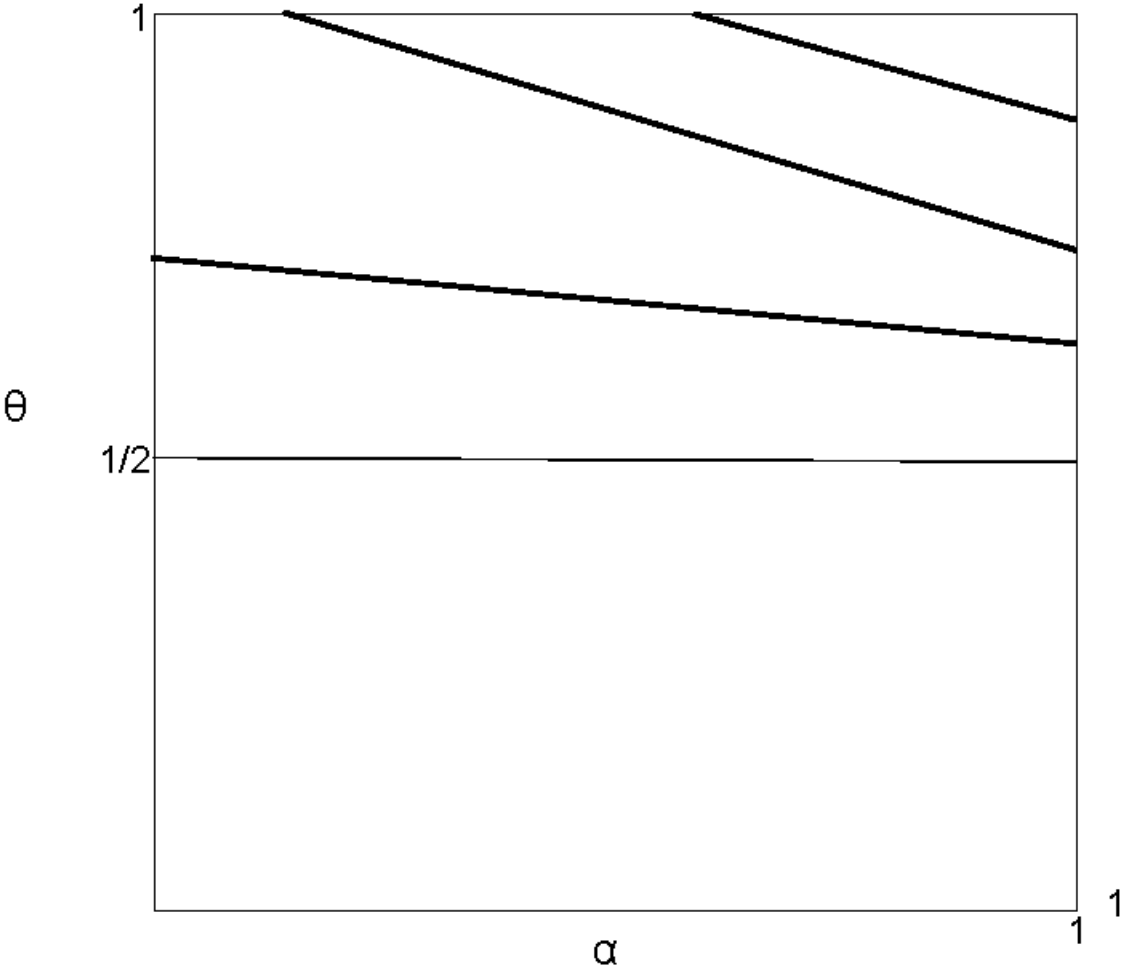
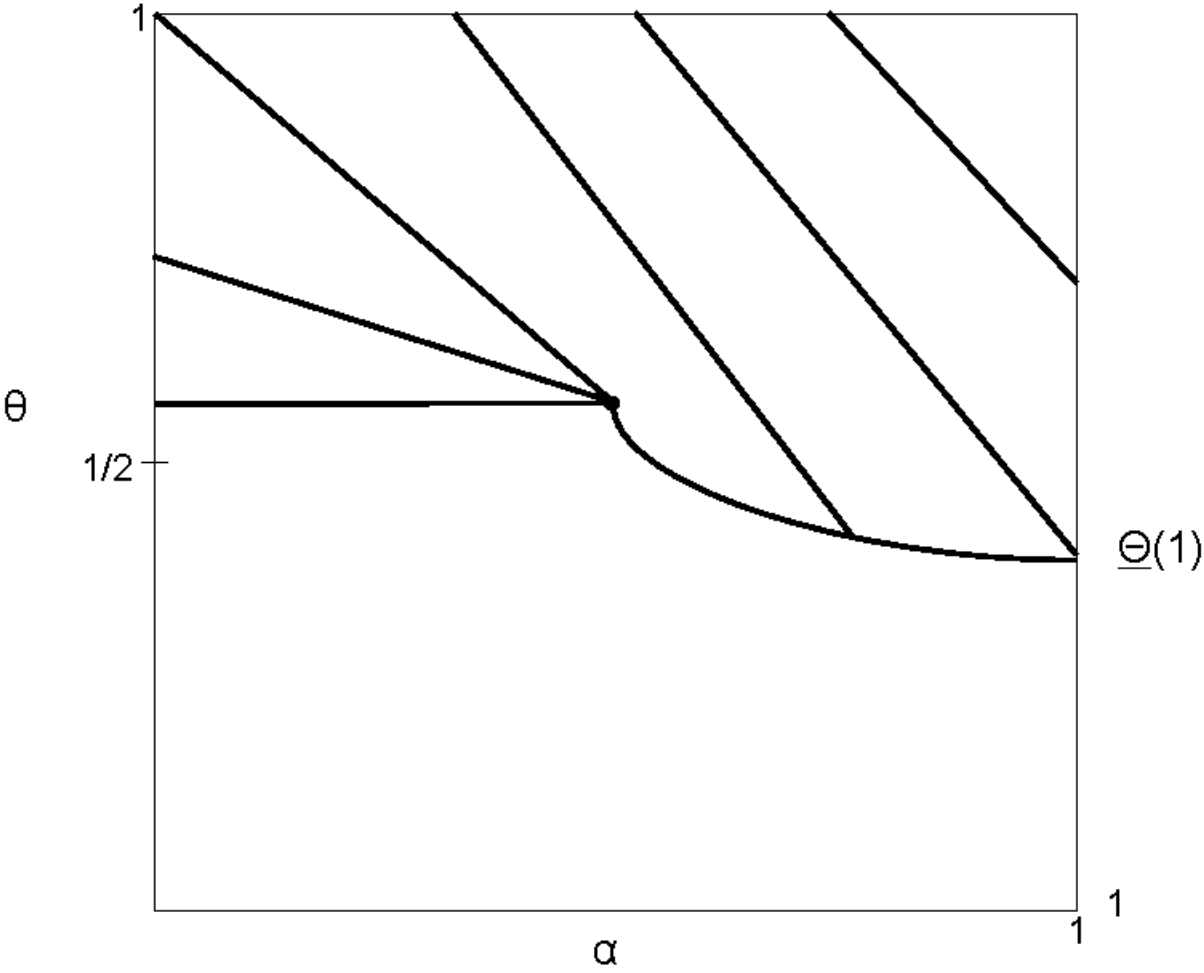


Figure 3: Linear- quadratic case with uniform distribution. Isoquants in Case C: $b < 3/2$.



5 Appendix

Proof of Lemma 1: Observe that the relation $u_q(q, \theta', \alpha') = u_q(q, \theta, \alpha)$ implicitly describes an “isoquant” $\theta' = \tilde{\theta}(\alpha')$, at the level q . Along this isoquant we have $u_{q\theta} \frac{d\tilde{\theta}}{d\alpha} + u_{q\alpha} = 0$. Hence $u_{qq}(q, \theta', \alpha') - u_{qq}(q, \theta, \alpha) = \int_{\alpha}^{\alpha'} [u_{qq\theta}(q, \tilde{\theta}(a), a) \frac{d\tilde{\theta}}{da} + u_{qq\alpha}(q, \tilde{\theta}(a), a)] da = \int_{\alpha}^{\alpha'} [-u_{qq\theta} \frac{u_{q\alpha}}{u_{q\theta}} + u_{qq\alpha}] da > 0$, proving the desired result. *Q.E.D.*

Proof of Lemma 2: By assumption 2, $q(\theta, \alpha) \in Q^*(\theta, \alpha)$, so Q^* is not empty valued. Upper hemi-continuity of Q^* follows from the generalized Theorem of the Maximum (Ausubel and Deneckere). Finally, monotonicity of Q^* follows because u is strictly supermodular. *Q.E.D.*

Proof of Lemma 3: By definition of $\underline{\theta}(\alpha)$, we have $q(\theta, \alpha) = 0$ for all $\theta < \underline{\theta}(\alpha)$, so we must have $s(\theta, \alpha) = 0$ for all $\theta \leq \underline{\theta}(\alpha)$. The definition of $\underline{\theta}(\alpha)$ and the monotonicity of Q^* also imply that $q(\theta, \alpha) > 0$ for all $\theta > \underline{\theta}(\alpha)$. It then follows from the envelope theorem that $s(\theta, \alpha) = \int_{\underline{\theta}(\alpha)}^{\theta} u_{\theta}(q(t, \alpha), t) dt > 0$ for all $\theta > \underline{\theta}(\alpha)$. Thus, the lower boundary can equivalently be defined as $\underline{\theta}(\alpha) = \inf\{\theta | s(\theta, \alpha) > 0\}$.

Now by the generalized Theorem of the maximum (Ausubel and Deneckere), $s(\cdot)$ is a C^0 function. The characterization $\underline{\theta}(\alpha) = \inf\{\theta | s(\theta, \alpha) > 0\}$ then implies that $\underline{\theta}(\alpha)$ is a C^0 function.

Note that $\underline{\theta}(\alpha)$ satisfies the identity $s(\underline{\theta}(\alpha), \alpha) = 0$. Since $q(\theta, \alpha) = 0$ for all $\theta < \underline{\theta}(\alpha)$, monotonicity of $q(\cdot, \cdot)$ in α implies that $q(\theta, \alpha') = 0$ for all $\alpha' < \alpha$ and all $\theta < \underline{\theta}(\alpha)$, so we have $\underline{\theta}(\alpha') \geq \underline{\theta}(\alpha)$ for all $\alpha' < \alpha$ and all α .

Finally, suppose that $q(\underline{\theta}(\alpha), \alpha) > 0$. Monotonicity of q in α then implies that for all $a > \alpha$ we have $q(\underline{\theta}(\alpha), a) > 0$, so $u_{\alpha}(q(\underline{\theta}(\alpha), a), \underline{\theta}(\alpha), a) > 0$. By the envelope theorem, we then have for all $\alpha' > \alpha$ $s(\underline{\theta}(\alpha), \alpha') = s(\underline{\theta}(\alpha), \alpha) + \int_{\alpha}^{\alpha'} u_{\alpha}(q(\underline{\theta}(\alpha), a), \underline{\theta}(\alpha), a) da > 0$. Continuity of s then implies that $\underline{\theta}(\alpha') = \inf\{\theta | s(\theta, \alpha') > 0\} < \underline{\theta}(\alpha)$. *Q.E.D.*

Proof of Lemma 4: If $q_1 \in Q^*(\theta_1, \alpha_1)$ then $u(q_1, \theta_1, \alpha_1) - T(q_1) \geq u(q, \theta_1, \alpha_1) - T(q)$, for all q . Rearranging, we have $T(q) \geq u(q, \theta_1, \alpha_1) - u(q_1, \theta_1, \alpha_1) + T(q_1)$, for all q . We now claim that $u(q, \theta_1, \alpha_1) - u(q_1, \theta_1, \alpha_1) \geq u(q, \theta_2, \alpha_2) - u(q_1, \theta_2, \alpha_2)$, for all q and (θ_2, α_2) satisfying the hypothesis of the Lemma. It then follows from the previous inequality that $T(q) \geq u(q, \theta_2, \alpha_2) - u(q_1, \theta_2, \alpha_2) + T(q_1)$, for all q so that $q_1 \in Q^*(\theta_2, \alpha_2)$.

To prove the claim, we need to show that $u(q, \theta_1, \alpha_1) - u(q, \theta_2, \alpha_2) \geq u(q_1, \theta_1, \alpha_1) - u(q_1, \theta_2, \alpha_2)$, for all $q \in \mathbf{R}_+$, i.e. that q_1 minimizes $\psi(q) \equiv u(q, \theta_1, \alpha_1) - u(q, \theta_2, \alpha_2)$. Now by assumption q_1 is a stationary point of ψ , and single-crossing implies that q_1 is a strict local minimum (see Lemma 1). In fact, single crossing implies that there is no other point $q \in \mathbf{R}_+$ such that $\psi'(q) = 0$, i.e. that q_1 is a global minimum of ψ , as was to be demonstrated. *Q.E.D.*

Proof of Lemma 6: We first show that Assumption 2 implies that $\frac{u_{q\alpha}}{u_{q\theta}} - \frac{u_{\alpha}}{u_{\theta}} > 0$ whenever $q > 0$. For fixed (θ, α) define $\varphi(q) = \frac{u_{q\alpha}}{u_{q\theta}}(q, \theta, \alpha) - \frac{u_{\alpha}}{u_{\theta}}(q, \theta, \alpha)$, so that $\varphi'(q) = \frac{d}{dq} \frac{u_{q\alpha}}{u_{q\theta}} - \varphi(q) \left(\frac{u_{q\theta}}{u_{\theta}} \right)$.

It then follows from Assumption 2 that for any $q > 0$ s.t. $\varphi(q) \leq 0$ we have $\varphi'(q) > 0$. Thus $\varphi(q) \leq 0$ for some $q > 0$ would imply $\phi(q') < \phi(q)$ for all $q' < q$, and in particular also that $\lim_{q' \rightarrow 0} \varphi(q') < 0$. But since u_α and u_θ both converge to zero as $q \rightarrow 0$, it follows from l'Hospital's rule that $\lim_{q \rightarrow 0} \varphi(q) = 0$, a contradiction. We conclude that we must have $\varphi(q) > 0$ whenever $q > 0$.

Next, let $(\theta(\alpha), \alpha) \in \Omega$ be s.t. $q(\theta, \alpha) > 0$. Let $(\tilde{\theta}(a), a)$ parameterize the isoquant $I(\theta, \alpha)$, i.e. $\tilde{\theta}(a)$ solves $u_q(q(\theta, \alpha), \tilde{\theta}(a), a) = u_q(q(\theta, \alpha), \theta, \alpha)$. Then we have

$$\tilde{\theta}'(a) = -\frac{u_{q\alpha}(q(\theta, \alpha), \tilde{\theta}(a), a)}{u_{q\theta}(q(\theta, \alpha), \tilde{\theta}(a), a)} < 0. \quad (58)$$

By the envelope theorem $s((\tilde{\theta}(\alpha'), \alpha') = s(\theta, \alpha) + \int_\alpha^{\alpha'} \{u_\theta(q(\theta, \alpha), \tilde{\theta}(a), a)\tilde{\theta}'(a) + u_\alpha(q(\theta, \alpha), \tilde{\theta}(a), a)\} da$. Since $s(\theta, \alpha) \geq 0$, and since $q(\theta, \alpha) > 0$ and assumption 2 imply that $\frac{u_\alpha}{u_\theta} - \frac{u_{q\alpha}}{u_{q\theta}} = -\varphi < 0$, we have $s((\tilde{\theta}(\alpha'), \alpha') > 0$ for all $\alpha' < \alpha$. It follows that $\tilde{\theta}(\alpha') > \underline{\theta}(a)$ for all $\alpha' < \alpha$. *Q.E.D.*

Proof of Lemma 7: We first claim that every point of Ω strictly to the right of the curve $I(\theta, \alpha)$ must be assigned a quantity no smaller than $q(\theta, \alpha)$.

Suppose to the contrary that there existed such a point (θ'', α'') that had $q(\theta'', \alpha'') < q(\theta, \alpha)$. By Lemma 4 and monotonicity of Q^* , we necessarily have $\theta'' < \theta$. Furthermore since $I(\theta'', \alpha'')$ is flatter than $I(\theta, \alpha)$ and since Lemma 6 implies $I(\theta'', \alpha'')$ cannot intersect L to the northwest of (θ'', α'') , $I(\theta'', \alpha'')$ must intersect $I(\theta, \alpha)$ at a point to the southeast of (θ, α) . Otherwise, the point on (θ, α''') on $I(\theta'', \alpha'')$ would have $\alpha''' > \alpha$, and $q(\theta, \alpha''') = q(\theta'', \alpha'') < q(\theta, \alpha)$ would violate monotonicity of Q^* in the second coordinate. Now consider the point (θ, α'') . By monotonicity of Q^* we have $q(\theta, \alpha'') \geq q(\theta, \alpha) > q(\theta'', \alpha'')$. Let $\theta''' = \min\{t \in [\theta'', \theta] : q(t, \alpha'') = q(\theta, \alpha)\}$; by continuity of q on $\Omega \setminus L$ such a θ''' exists. It follows by Lemma 4 that there exists α''' with $\alpha < \alpha''' \leq \alpha''$ such that $(\theta, \alpha''') \in I(\theta''', \alpha'')$. Monotonicity of Q^* then implies that $q(\theta, a) = q(\theta, \alpha)$ for all $a \in [\alpha, \alpha''']$. But since $q(t, \alpha'')$ is continuous in t and since $q(t, \alpha'') < q(\theta, \alpha)$ for all $t \in [\theta'', \theta''')$, there will exist a $t \in [\theta'', \theta''')$ such that $I(t, \alpha'')$ intersects the segment $\{(\theta, a) : a \in (\alpha, \alpha''')\}$ at a point (θ, a) with $a > \alpha$. But then $q(\theta, a) = q(t, \alpha'') < q(\theta, \alpha)$, violating monotonicity of Q^* . This proves the claim.

An entirely parallel argument also shows that every point in Ω to the left of the curve $I(\theta, \alpha)$ must be assigned a quantity no greater than $q(\theta, \alpha)$. Continuity of q then implies that every point on $I(\theta, \alpha)$ must be assigned a quantity equal to $q(\theta, \alpha)$. *Q.E.D.*

Proof of Theorem 1. Lemma 7 implies that from every point (θ, α) along L there emanates an isoquant $I(\theta, \alpha)$. Since isoquants are downward sloping in (θ, α) space, and since Lemma 6 isoquants are steeper than the lower boundary $\underline{\theta}(\alpha)$, all isoquants emanating from L must end on U . Since isoquants cannot cross in the interior of Ω without violating monotonicity of Q^* , and since the allocation must be monotone along U , it follows that q must be nondecreasing along L .

Suppose now that there exist a point $(\theta, \alpha) \in L$ such that $q_1 \in Q^*(\theta, \alpha)$ and $q_2 \in Q^*(\theta, \alpha)$, where $q_1 < q_2$. Let $(\theta_1, \alpha_1) = I(q_1, \theta, \alpha) \cap U$ and $(\theta_2, \alpha_2) = I(q_2, \theta, \alpha) \cap U$. By continuity of q on U , for each $q_3 \in [q_1, q_2]$ there exists a point (θ_3, α_3) on the segment of U connecting (θ_1, α_1)

to (θ_2, α_2) such that $q(\theta_3, \alpha_3) = q_3$. Furthermore, since $I(\theta_3, \alpha_3)$ cannot cross $I(q_1, \theta, \alpha)$ or $I(q_2, \theta, \alpha)$ without violating monotonicity of Q^* , it must be that $I(\theta_3, \alpha_3)$ intersects L at (θ, α) . Upper-hemicontinuity of Q^* at (θ, α) then implies that $q_3 \in Q^*(\theta, \alpha)$.

The convexity of Q^* along L then implies that the collection isoquants span all of Ω , i.e. that $\{I(q, \theta, \alpha) : q \in Q^*(\theta, \alpha) \text{ and } (\theta, \alpha) \in L\} = \Omega$. *Q.E.D.*

Proof of Lemma 8. Part (i) immediately follows from Theorem 1.

To prove (ii), observe that incentive compatibility implies

$$\begin{aligned} s(\underline{\theta}(\alpha), \alpha) &= u(\underline{q}(\alpha), \underline{\theta}(\alpha), \alpha) - T(\underline{q}(\alpha)) \leq u(\underline{q}(\alpha'), \underline{\theta}(\alpha), \alpha) - T(\underline{q}(\alpha')) \\ s(\underline{\theta}(\alpha'), \alpha') &= u(\underline{q}(\alpha'), \underline{\theta}(\alpha'), \alpha') - T(\underline{q}(\alpha')) \leq u(\underline{q}(\alpha), \underline{\theta}(\alpha'), \alpha') - T(\underline{q}(\alpha)) \end{aligned}$$

Consequently, we have

$$u(\underline{q}(\alpha'), \underline{\theta}(\alpha), \alpha) - u(\underline{q}(\alpha'), \underline{\theta}(\alpha'), \alpha') \leq s(\underline{\theta}(\alpha), \alpha) - s(\underline{\theta}(\alpha'), \alpha') \leq u(\underline{q}(\alpha), \underline{\theta}(\alpha), \alpha) - u(\underline{q}(\alpha), \underline{\theta}(\alpha'), \alpha')$$

Using the mean value theorem, and the fact that $s(\underline{\theta}(\alpha), \alpha) = s(\underline{\theta}(\alpha'), \alpha') = 0$, we obtain

$$u_\theta(\underline{q}(\alpha'), \underline{\theta}(\alpha_0), \alpha_0)(\underline{\theta}(\alpha) - \underline{\theta}(\alpha')) + u_\alpha(\underline{q}(\alpha'), \underline{\theta}(\alpha_0), \alpha_0)(\alpha - \alpha') \leq 0 \quad (59)$$

$$u_\theta(\underline{q}(\alpha), \underline{\theta}(\alpha_1), \alpha_1)(\underline{\theta}(\alpha) - \underline{\theta}(\alpha')) + u_\alpha(\underline{q}(\alpha), \underline{\theta}(\alpha_1), \alpha_1)(\alpha - \alpha') \geq 0 \quad (60)$$

for some α_0 and α_1 between α and α' . If $\alpha < \alpha'$ then inequalities (59) and (60) imply

$$-\frac{u_\alpha}{u_\theta}(\underline{q}(\alpha'), \underline{\theta}(\alpha_0), \alpha_0) \leq \frac{\underline{\theta}(\alpha) - \underline{\theta}(\alpha')}{\alpha - \alpha'} \leq -\frac{u_\alpha}{u_\theta}(\underline{q}(\alpha), \underline{\theta}(\alpha_1), \alpha_1)$$

If $\alpha' < \alpha$ then inequalities (60) and (60) imply

$$-\frac{u_\alpha}{u_\theta}(\underline{q}(\alpha), \underline{\theta}(\alpha_1), \alpha_1) \leq \frac{\underline{\theta}(\alpha) - \underline{\theta}(\alpha')}{\alpha - \alpha'} \leq -\frac{u_\alpha}{u_\theta}(\underline{q}(\alpha'), \underline{\theta}(\alpha_0), \alpha_0)$$

Let $L = \max_{(\theta, \alpha) \in [0, 1]^2} \frac{u_\alpha}{u_\theta}(\underline{q}(1), \theta, \alpha)$. The previous two inequalities show that $\underline{\theta}(\alpha)$ is Lipschitz continuous with Lipschitz constant L . It follows that $\underline{\theta}(\alpha)$ is absolutely continuous. Furthermore, at all continuity points of $\underline{q}(\cdot)$, which excludes all but at most a countable set of α , taking limits in the above two inequalities yields

$$\underline{\theta}'(\alpha) = -\frac{u_\alpha}{u_\theta}(\underline{q}(\alpha), \underline{\theta}(\alpha), \alpha).$$

Finally, let us prove (iii). It follows from incentive compatibility that if $q \in Q^*(\underline{\theta}(\alpha), \alpha)$ and $q' \in Q^*(\underline{\theta}(\alpha'), \alpha')$ then:

$$u(q, \underline{\theta}(\alpha), \alpha) - u(q', \underline{\theta}(\alpha), \alpha) \leq T(q) - T(q') \leq u(q, \underline{\theta}(\alpha'), \alpha') - u(q', \underline{\theta}(\alpha'), \alpha')$$

Using the mean value theorem, we obtain:

$$u_q(z_0, \underline{\theta}(\alpha), \alpha)(q - q') \leq T(q) - T(q') \leq u_q(z_1, \underline{\theta}(\alpha'), \alpha')(q - q') \quad (61)$$

for some z_0 and z_1 between q and q' . Let $M = \max_{(\theta, \alpha) \in [0, 1]^2} u_q(0, \theta, \alpha)$. Then it follows from (61) that T is Lipschitz continuous with Lipschitz constant M , and hence absolutely continuous. If $Q^*(\underline{\theta}(\alpha), \alpha)$ is multi-valued then let $q_1 = \min Q^*(\underline{\theta}(\alpha), \alpha)$ and $q_2 = \max Q^*(\underline{\theta}(\alpha), \alpha)$. It follows from (61) that for any $q \in (q_1, q_2)$ we have $T'(q) = u_q(q, \underline{\theta}(\alpha), \alpha)$. Next consider any point $q = \underline{q}(\alpha)$ at which both $\underline{q}(\alpha)$ and $\alpha(q)$ are continuous. Since $\alpha(q)$ and $\underline{q}(\alpha)$ are both monotone, this excludes at most countably many q . Rewriting (61) as

$$u_q(z_0, \underline{\theta}(\alpha), \alpha)(q - q') \leq T(q) - T(q') \leq u_q(z_1, \underline{\theta}(\alpha(q')), \alpha(q'))(q - q'),$$

Dividing through by $(q - q')$ and taking limits as $q' \rightarrow q$, it follows that $T'(q) = u_q(q, \underline{\theta}(\alpha), \alpha)$. To complete the proof of (3), note that $s(\underline{\theta}(1), 1) = 0$ implies that $T(\underline{q}(1)) = u(\underline{q}(1), \underline{\theta}(1), 1)$. Since T is absolutely continuous, and since $T'(q) = u_q(q, \underline{\theta}(\alpha(q)), \alpha(q))$ holds for almost all q , the formula for $T(q)$ follows for all $q \in [\underline{q}(0), \underline{q}(1)]$. For $q \in (\underline{q}(1), \bar{q}(1)]$ the argument is analogous. Q.E.D.

The proof of Lemma 10: To prove statement (i), let us integrate the second term under the integral in (52) by parts. We have:

$$\int_{\underline{\theta}(1)}^{\hat{\theta}} -\frac{\underline{q}'(\theta)}{\underline{q}(\theta)} \frac{\theta(1-\theta)^2}{2} d\theta = -\log(\underline{q}(\hat{\theta})) \frac{\hat{\theta}(1-\hat{\theta})^2}{2} + \log(\underline{q}(\underline{\theta}(1))) \frac{\underline{\theta}(1)(1-\underline{\theta}(1))^2}{2} + \int_{\underline{\theta}(1)}^{\hat{\theta}} \log \underline{q}(\theta) \frac{(1-3\theta)(1-\theta)}{2} d\theta \quad (62)$$

Consider the optimal schedule $\underline{q}(\theta)$ on the interval $(\underline{\theta}(1), \hat{\theta})$. It has to maximize (62) subject to the constraint that the boundary points of the region *Mid C* remain unchanged, i.e. $1 - \hat{\alpha} = \int_{\underline{\theta}(1)}^{\hat{\theta}} \frac{2}{\underline{q}(\theta)} d\theta$. For otherwise, the expected profit in the region *Mid C*, V_{MC} , can be increased without affecting the values of the profits in the other regions, *Upper C* and *Lower C*. Hence, the optimal $\underline{q}(\theta)$, $\theta \in (\underline{\theta}(1), \hat{\theta})$, maximizes the following Lagrangian:

$$\mathcal{L} = \int_{\underline{\theta}(1)}^{\hat{\theta}} \log(\underline{q}(\theta)) \frac{(1-3\theta)(1-\theta)}{2} d\theta + \lambda \left((1 - \hat{\alpha}) - \int_{\underline{\theta}(1)}^{\hat{\theta}} \frac{2}{\underline{q}(\theta)} d\theta \right) \quad (63)$$

The first-order condition with respect to $\underline{q}(\theta)$ is:

$$\frac{1}{2\underline{q}(\theta)} (1-3\theta)(1-\theta) = \frac{2\lambda}{(\underline{q}(\theta))^2}.$$

Then statement (i) obtains if we set $k = 4\lambda$.

To prove part (ii), first, suppose that $\hat{\alpha} < \frac{2b}{3}$. Let us show that the value of firm's expected profits EV given by (55) can be increased by raising $\hat{\alpha}$ and decreasing \hat{q} so that $\hat{\alpha}\hat{q}$ remains constant. Note that $\hat{\theta} = 1 - \hat{\alpha}\hat{q}$. So $\hat{\theta}$ does not change after such perturbation. Consequently, the value of V_{UC} in (51) does not change. Note that the value of $\underline{q}(\hat{\alpha})$ in the region *Mid C* can be left unchanged since it is only required that for any α in the region *Mid C*, $\underline{q}\alpha \geq \hat{q}$, and by lowering \hat{q} we are not violating this condition. Further, an inspection of (46) shows that the value of V_{MC} increases as we increase α .

Further, since $\hat{\alpha} < \frac{2b}{3}$, an inspection of (54) shows that it is optimal to set $\underline{q}(\alpha) = 0$ for all $\alpha < \hat{\alpha}$ and so the last integral term in (54) is equal to zero. Then it is easy to see that the

value of (54) increases as we lower \hat{q} keeping $\hat{\alpha}\hat{q}$ constant, because all terms in (54) remain constant after such perturbation except for the third one, the value of which increases.

Next, suppose that $\hat{\alpha} > \frac{2b}{3}$. Then a similar argument shows that the firm's expected profits EV in (55) increases if we decrease $\hat{\alpha}$ while keeping $\hat{\alpha}\hat{q}$ constant.

To see the argument, note that if $\hat{\alpha} > \frac{2b}{3}$, then the maximization of $\int_0^{\hat{\alpha}} \underline{q}(\alpha)^3 \alpha \left(\frac{\alpha}{8} - \frac{b}{12} \right) d\alpha$ requires that $\underline{q}(\alpha) = 0$ for $\alpha \in [0, \frac{2b}{3})$, and $\underline{q}(\alpha) = \underline{q}(\hat{\alpha})$ for $\alpha \in [\frac{2b}{3}, \hat{\alpha}]$. Then we can integrate the last term in (54), use the fact that $1 = \hat{\theta} + \hat{\alpha}\underline{q}(\hat{\alpha})$ in the first term and simplify to get:

$$\begin{aligned}
V_{LC} &= \underline{q}(\hat{\alpha})^2 \hat{\alpha}^2 \frac{2\hat{\theta}(\hat{\alpha}) - 1}{4} + \frac{\underline{q}(\hat{\alpha})^3 \hat{\alpha}^3}{6} - \frac{\underline{q}(\hat{\alpha})^3 \hat{\alpha}^2 (b - \hat{\alpha})}{12} + \int_{\frac{2b}{3}}^{\hat{\alpha}} \underline{q}(\hat{\alpha})^3 \alpha \left(\frac{\alpha}{8} - \frac{b}{12} \right) d\alpha \\
&= \underline{q}(\hat{\alpha})^2 \hat{\alpha}^2 \frac{1 - 2\underline{q}(\hat{\alpha})\hat{\alpha}}{4} + \frac{\underline{q}(\hat{\alpha})^3 \hat{\alpha}^3}{4} - \frac{\underline{q}(\hat{\alpha})^3 \hat{\alpha}^2 b}{12} + \frac{\underline{q}(\hat{\alpha})^3 \hat{\alpha}^3}{24} - \frac{\underline{q}(\hat{\alpha})^3 \hat{\alpha}^2 b}{24} + \frac{\underline{q}(\hat{\alpha})^3 b^3}{6 \times 27} \\
&= \frac{\underline{q}(\hat{\alpha})^2 \hat{\alpha}^2}{4} - \frac{5\underline{q}(\hat{\alpha})^3 \hat{\alpha}^3}{24} - \frac{\underline{q}(\hat{\alpha})^3 \hat{\alpha}^2 b}{8} + \frac{\underline{q}(\hat{\alpha})^3 b^3}{6 \times 27}
\end{aligned} \tag{64}$$

When we add to the value of (64) the value of the last term in (53), we obtain an expression that decreases in $\hat{\alpha}$ whereas the other parts of V_{MC} and the value of V_{UC} are not affected by an increase in $\hat{\alpha}$.

To prove parts (iii) and (iv), note that, as we have shown above, separate maximization of the value of V_{UC} in (51) requires setting $q(\theta, 1) = \frac{3\theta-1}{2(b-1)}$ for all $\theta \in [\underline{\theta}(1), 1]$. So, (iv) will follow once we establish (iii).

To establish this, substitute $\underline{q}(\theta) = \frac{k}{(3\theta-1)(1-\theta)}$ into the expression V_{MC} in (52) and note that this expression is decreasing in k . So, k should be set as low as possible, which implies (v). Also, to minimize k , \hat{q} should be chosen so that (iv) holds, for otherwise k can be further decreased. Q.E.D.

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