Nonparametric Identification of Risk Aversion in First-Price Auctions Under Exclusion Restrictions*

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Abstract

This paper studies the nonparametric identification of the first-price auction model with risk averse bidders within the private value paradigm. First, we show that the benchmark model is nonindentified from observed bids. We also derive the restrictions imposed by the model on observables and show that these restrictions are weak. Second, we establish the nonparametric identification of the bidders' utility function under exclusion restrictions. Our primary exclusion restriction takes the form of an exogenous bidders' participation leading to a latent distribution of private values independent of the number of bidders. The key idea is to exploit the property that the bid distribution varies with the number of bidders while the private value distribution does not. We then extend these results to endogenous bidders' participation when the exclusion restriction takes the form of instruments that do not affect the bidders' private value distribution. Though derived for a benchmark model, our results extend to more general cases such as a binding reserve price, affiliated private values and asymmetric bidders. Lastly, possible estimation methods are proposed.

Key words: Risk Aversion, Private Value, Nonparametric Identification, Exclusion Restrictions

JEL classification: C14, D44

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1 Introduction

The empirical and experimental literature indicates that risk aversion is an important component of bidders' behavior in auctions. For instance, Baldwin (1995) and Athey and Levin (2001) find that bidders diversify risk when bidding on species in US Forest Service auctions in which payments are based on ex post harvested quantities. Adopting a structural approach, Campo, Guerre, Perrigne and Vuong (2007) and Lu and Perrigne (2006) also find significant risk aversion in US Forest Service auctions. Following Li and Tan's (2000) theoretical results, Perrigne (2003) shows that a random reserve price leads empirically to larger revenues when bidders are risk averse thereby justifying this observed auction design. In eBay auctions, Ackerberg, Hirano and Shahriar (2006) rationalize the popularity of the Buy-it-now option by bidders' risk aversion. In experiments, bidders' risk aversion is used to explain the frequently observed overbidding over the neutral Bayesian Nash equilibrium strategy as in Cox, Smith and Walker (1988) among others. Relying on a different equilibrium concept, Goeree, Holt and Palfrey (2002) also find significant bidders' risk aversion. Lastly, using recent structural econometric methods, Bajari and Hortacsu (2005) show that risk aversion provides the best fit to some experimental data among a set of competing models including learning ones.

With the exception of Lu and Perrigne (2006) which exploit additional bidding data, the previous papers all rely on a parameterization of the bidders' utility function, yet various concepts of risk aversion have different implications on agents' behavior. There is, however, no general agreement on which concept of risk aversion is the most appropriate to explain observed phenomena such as in finance through the diversification of portfolios, insurance when low risk car drivers tend to buy more insurance than needed, or in auctions through overbidding.[1] A typical controversy is whether risk aversion is absolute or relative to economic agent's wealth. As a matter of fact, little is known on the shape of agents' utility functions. Several families of utility functions have been developed to embody some economic properties related to risk aversion. See Gollier (2001) for an extensive survey on risk aversion. Which family is relevant is an empirical question.

Given the importance of risk aversion in auctions and our ignorance about bidders' utility functions, we address the nonparametric identification of the latter in this paper. First, we show that the general model is not identified from observed bids. We then derive the theoretical restrictions imposed by the model on observables and show that these restrictions are weak. In particular, we show that any smooth bid distribution can be rationalized by a model with general risk aversion. Such a striking result implies that risk aversion does not impose testable restrictions on bids. This can explain why risk aversion is so difficult to detect in observed bids.

Second, we show that the bidders' utility function is nonparametrically identified under some exclusion restrictions. To clarify ideas, our primary exclusion restriction takes the form of an exogenous bidders' participation leading to a latent distribution of private values that is independent of the number of bidders. Exclusion restrictions are widely used in econometrics. A typical example is the use of instrumental variables in labor economics to address the endogeneity of education in the estimation of the wage equation. Other examples can be found in Matzkin (1994) in a nonparametric context. Exclusion restrictions have also been used in the structural auction literature. Athey and Haile (2002) and Haile, Hong and Shum (2003) exploit some exclusion restrictions to test for common values in first-price sealed-bid auctions. Both papers assume exogenous participation to detect the winner's curse. In a different framework, Bajari and Hortacsu (2005) use exogenous participation to estimate an auction model with constant relative risk aversion from experimental data.[2] We then extend our results to a model with endogenous bidders' participation where the exclusion restriction takes the form of instruments that do not

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[1] For an empirical analysis of risk aversion in car insurance, see Cohen and Einav (2007).

[2] Exogenous participation is not necessary to estimate the model in their paper. Such a restriction avoids the use of a conditional quantile restriction as in Campo, Guerre, Perrigne and Vuong (2007).

affect the bidders' private value distribution but affect bidders' participation.[3] Though we consider first a benchmark model with symmetric bidders, independent private values and no reserve price, our results extend to a binding reserve price, affiliated private values and asymmetric bidders, whether asymmetry arises from private values and/or heterogenous preferences. As such, our results apply to a large class of auction models.

The paper is organized as follows. Section 2 presents the benchmark model and establishes its nonparametric identification from observed bids. In view of this, Section 3 exploits exclusion restrictions to achieve nonparametric identification of the bidders' utility function and private value distribution. We then characterize the theoretical restrictions that observed bids need to satisfy. Section 4 extends our nonidentification and identification results to a binding reserve price, affiliated private values and asymmetric bidders. Section 5 concludes with a discusion of possible estimation methods. An appendix collects the proofs.

2 Model and Nonidentification

A first subsection presents the independent private value (IPV) first-price sealed-bid auction model with risk averse bidders and properties of its equilibrium strategy. A second subsection establishes the nonparametric nonidentification of the model.

2.1 The Benchmark Model

A single and indivisible object is sold through a first-price sealed-bid auction. Within the IPV paradigm, each bidder knows his own private value v_i but not other bidders' private values. There are I potential bidders with $I \in \mathcal{I}$ a finite subset of $\{2, 3, 4, \ldots\}$. Private values are drawn independently from a distribution $F(\cdot|I)$, which is absolutely continuous with density $f(\cdot|I)$ on a support $[\underline{v}(I), \overline{v}(I)] \subset I\!\!R_+$. The distribution $F(\cdot|I)$ and the number of potential bidders I are common knowledge. Let $U(\cdot)$ be the bidders' von Neuman Morgenstern (vNM) utility function with $U(0) = 0$, $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$. All bidders are thus identical *ex ante* and the game is symmetric. Bidder i maximizes his

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[3] Haile, Hong and Shum (2003) also introduce such instruments to deal with an endogenous number of bidders.

expected utility

$$\mathrm{E}\Pi_i = U(v_i - b_i)\Pr(b_i \geq b_j, j \neq i) \tag{1}$$

with respect to his bid b_i, where b_j is the jth player's bid. See Case 1 in Maskin and Riley (1984) and Krishna (2002). Because the scale is irrelevant, we impose the normalization $U(1) = 1$. The risk neutral case is obtained when $U(\cdot)$ is the identity function.[4]

From Maskin and Riley (1984), if a symmetric Bayesian Nash equilibrium strategy $s(\cdot) = s(\cdot, U, F, I)$ exists, then it is strictly increasing and continuous on $[\underline{v}(I), \overline{v}(I)]$ and differentiable on $(\underline{v}(I), \overline{v}(I)]$.[5] Thus (1) becomes $\mathrm{E}\Pi_i = U(v_i - b_i)F^{I-1}(s^{-1}(b_i)|I)$, where $s^{-1}(\cdot)$ denotes the inverse of $s(\cdot)$. Hence, imposing bidder i's optimal bid b_i to be $s(v_i)$ gives the following differential equation

$$s'(v_i) = (I-1)\frac{f(v_i|I)}{F(v_i|I)}\lambda(v_i - b_i) \text{ for all } v_i \in (\underline{v}(I), \overline{v}(I)], \tag{2}$$

where $\lambda(\cdot) = U(\cdot)/U'(\cdot)$. From Maskin and Riley (1984), the boundary condition is $s(\underline{v}(I)) = \underline{v}(I)$ because $U(0) = 0$. Moreover, the second-order conditions are satisfied.

When the reserve price is nonbinding, existence of a pure equilibrium strategy follows from Maskin and Riley (2000) and Athey (2001), while its uniqueness is established by Maskin and Riley (2003) using an argument similar to Lebrun (1999). The main contribution of Theorem 1 below is to derive the smoothness of the equilibrium strategy, which is used in the next subsection. Determining the smoothness of the equilibrium strategy is difficult when the differential equation (2) does not have an explicit solution, which is the case for general utility functions $U(\cdot)$. We assume that $U(\cdot)$ and $F(\cdot|I)$ belong to $\mathcal{U}_R$ and $\mathcal{F}_R$ defined as follows, respectively.

Definition 1: *For $R \geq 1$, let $\mathcal{U}_R$ be the set of utility functions $U(\cdot)$ satisfying*
(i) $U : [0, +\infty) \to [0, +\infty)$, $U(0) = 0$ and $U(1) = 1$,
(ii) $U(\cdot)$ is continuous on $[0, +\infty)$, and admits $R + 2$ continuous derivatives on $(0, +\infty)$ with $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$ on $(0, +\infty)$,

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[4] With bidders' wealth w, the expected profit becomes $[U(w+v_i-b_i)-U(w)]\Pr(b_i \geq b_j, j \neq i)+U(w)$. Different wealths w_i lead to an asymmetric game if the w_is are common knowledge and to a multisignal game if the w_is are private information. See Che and Gale (1998) for a multisignal auction model.

[5] From Maskin and Riley (1984, Remark 2.3), the only equilibria are symmetric when $F(\cdot)$ has bounded support.

(iii) $\lim_{x \downarrow 0} \lambda^{(r)}(x)$ *is finite for* $1 \leq r \leq R+1$*, where* $\lambda^{(r)}(\cdot)$ *is the* r*th derivative of* $\lambda(\cdot)$*.*

Conditions (i) and (ii) have been discussed previously. Note that $\lim_{x \downarrow 0} \lambda(x) = 0$ since $U(0) = 0$ and $U'(\cdot)$ is nonincreasing. Thus, from (ii) and (iii) it follows that $\lambda(\cdot)$ admits $R+1$ continuous derivatives on $[0, +\infty)$. These regularity assumptions are weak as they are satisfied by many vNM utility functions.

Definition 2: *For* $R \geq 1$*, let* $\mathcal{F}_R$ *be the set of distributions* $F(\cdot|I)$*,* $I \in \mathcal{I}$*, satisfying*
(i) $F(\cdot|I)$ *is a c.d.f. with support of the form* $[\underline{v}(I), \overline{v}(I)]$*, where* $0 \leq \underline{v}(I) < \overline{v}(I) < +\infty$*,*
(ii) $F(\cdot|I)$ *admits* $R+1$ *continuous derivatives on* $[\underline{v}(I), \overline{v}(I)]$*,*
(iii) $f(\cdot|I) > 0$ *on* $[\underline{v}(I), \overline{v}(I)]$*.*

Altogether (i)–(iii) imply that $f(\cdot|I)$ is bounded away from zero on $[\underline{v}(I), \overline{v}(I)]$.

Theorem 1: *Let* $R \geq 1$*. Suppose that* $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$*, then for each* $I \in \mathcal{I}$*, there exists a unique (symmetric) equilibrium strategy* $s(\cdot)$*. Moreover, this strategy satisfies:*
(i) $\forall v \in (\underline{v}(I), \overline{v}(I)]$*,* $s(v) < v$ *with* $s(\underline{v}(I)) = \underline{v}(I)$*,*
(ii) $\forall v \in [\underline{v}(I), \overline{v}(I)]$*,* $s'(v) > 0$ *with* $s'(\underline{v}) = (I-1)\lambda'(0)/[(I-1)\lambda'(0)+1] < 1$*,*
(iii) $s(\cdot)$ *admits* $R+1$ *continuous derivatives on* $[\underline{v}(I), \overline{v}(I)]$*.*

The proof of Theorem 1 can be found in the supplementary material.

2.2 Nonidentification Results

We study identification of the structure $[U, F]$ from observables. We assume that the number I of bidders is observed, as in a first-price sealed-bid auction with a nonbinding reserve price. We also assume that the distribution $G(\cdot|I)$ of an equilibrium bid is known. Thus the identification problem reduces to whether the structure $[U, F]$ can be recovered uniquely from the knowledge of (I, G). A related issue is whether any bid distribution $G(\cdot|I)$ can be rationalized by a structure $[U, F]$. Such a question relates to the possibility of testing the validity of the auction model.

Following Guerre, Perrigne and Vuong (2000), we express (2) using the distribution $G(\cdot|I)$ of an equilibrium bid. For every $b \in [\underline{b}(I), \overline{b}(I)] = [\underline{v}(I), s(\overline{v}(I))]$, we have $G(b|I) = F(s^{-1}(b)|I) = F(v|I)$ with density $g(b|I) = f(v|I)/s'(v)$, where $v = s^{-1}(b)$. Thus (2) can be written as

$$1 = (I-1)\frac{g(b_i|I)}{G(b_i|I)}\lambda(v_i - b_i) \text{ for all } b_i \in (\underline{b}(I), \overline{b}(I)]. \tag{3}$$

Since $U(\cdot) \geq 0$ and $U''(\cdot) \leq 0$, we have $\lambda'(\cdot) = 1 - U(\cdot)U''(\cdot)/U'(\cdot)^2 \geq 1$. Thus $\lambda(\cdot)$ is strictly increasing. Solving (3) for v_i and using $\underline{b}(I) = \underline{v}(I)$ with $\lambda^{-1}(0) = 0$ give

$$v_i = b_i + \lambda^{-1}\left(\frac{1}{I-1}\frac{G(b_i|I)}{g(b_i|I)}\right) \equiv \xi(b_i, U, G, I) \text{ for all } b_i \in [\underline{b}(I), \overline{b}(I)], \tag{4}$$

where $\lambda^{-1}(\cdot)$ denotes the inverse of $\lambda(\cdot)$. This equation expresses each bidder's private value as a function of his corresponding bid, the bid distribution, the number of bidders and the utility function. Note that $\xi(\cdot)$ is the inverse of the bidding strategy $s(\cdot)$.

The equilibrium bid distribution $G(\cdot|I)$ satisfies some regularity properties implied by the smoothness of $s(\cdot)$ given in Theorem 1 and the regularity assumptions on $[U, F]$.

Definition 3: *For $R \geq 1$, let $\mathcal{G}_R$ be the set of distributions $G(\cdot|I)$, $I \in \mathcal{I}$, satisfying*
(i) $G(\cdot|I)$ is a c.d.f. with support of the form $[\underline{b}(I), \overline{b}(I)]$, where $0 \leq \underline{b}(I) < \overline{b}(I) < +\infty$,
(ii) $G(\cdot|I)$ admits $R+1$ continuous derivatives on $[\underline{b}(I), \overline{b}(I)]$,
(iii) $g(\cdot|I) > 0$ on $[\underline{b}(I), \overline{b}(I)]$,
(iv) $g(\cdot|I)$ admits $R+1$ continuous derivatives on $(\underline{b}(I), \overline{b}(I)]$,
(v) $\lim_{b \downarrow \underline{b}(I)} d^r[G(b|I)/g(b|I)]/db^r$ exists and is finite for $r = 1, \ldots, R+1$.

The regularity properties (i)–(iii) are similar to those of Definition 2 for $F(\cdot|I)$. They imply that $g(\cdot|I)$ is bounded away from zero on $[\underline{b}(I), \overline{b}(I)]$ and $\lim_{b\downarrow\underline{b}} G(b|I)/g(b|I) = 0$ so that $\lim_{b\downarrow\underline{b}(I)} \xi(b, U, G, I) = \underline{b}(I)$. Properties (iv) and (v) are specific to the auction model. In particular, (iv) says that $g(\cdot|I)$ is smoother than $f(\cdot|I)$, extending a similar property noted by Guerre, Perrigne and Vuong (2000) for the risk neutral model. Combined with (iii) and (iv), (v) implies that $G(\cdot|I)/g(\cdot|I)$ is $R+1$ continuously differentiable on $[\underline{b}(I), \overline{b}(I)]$.

The following lemma provides necessary and sufficient conditions for rationalizing a distribution of observed bids by an IPV auction model with risk aversion. Hereafter, we say that a distribution is *rationalized* by an auction model with risk aversion if there exists a structure $[U, F]$ whose equilibrium bid distribution is identical to the given distribution.

Lemma 1: *Let $R \geq 1$, and $\mathbf{G}(\cdot|I)$ be the joint distribution of $(b_1, \ldots, b_I)$ conditional on $I \in \mathcal{I}$. There exists an IPV auction model with risk aversion $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ that rationalizes $\mathbf{G}(\cdot|\cdot)$ if and only if*

(i) $\mathbf{G}(b_1,\ldots,b_I|I)=\prod_{i=1}^{I}G(b_i|I)$*, with* $G(\cdot|\cdot)\in\mathcal{G}_R$*,*
(ii) $\exists\lambda:I\!\!R_+\to I\!\!R_+$ *with* $R+1$ *continuous derivatives on* $[0,+\infty)$*,* $\lambda(0)=0$ *and* $\lambda'(\cdot)\geq 1$ *such that* $\xi'(\cdot)>0$ *on* $[\underline{b}(I),\overline{b}(I)]$*, where* $\xi(b,U,G,I)=b+\lambda^{-1}\left[G(b|I)/((I-1)g(b|I))\right]$.

Condition (i) is related to the IPV paradigm and requires that bids be i.i.d., where $G(\cdot|\cdot)$ satisfies the regularity properties of Definition 3. Condition (ii) arises from $\xi(\cdot,U,G,I)$ being the inverse of the equilibrium strategy, which is strictly increasing for each $I\in\mathcal{I}$.[6]

The next proposition shows that any distribution $G(\cdot|\cdot)\in\mathcal{G}_R$ can be rationalized by an IPV auction model with a utility function displaying risk aversion.

Proposition 1: *Let* $R\geq 1$*. A bid distribution* $G(\cdot|\cdot)$ *can be rationalized by a risk aversion structure* $[U,F]\in\mathcal{U}_R\times\mathcal{F}_R$ *if and only if* $G(\cdot|\cdot)\in\mathcal{G}_R$*.*

Proposition 1 is striking as it says that any bid distribution in $\mathcal{G}_R$ can be rationalized by risk aversion. It implies that the restriction (ii) in Lemma 1 is redundant. Specifically, our proof indicates that we can always find a function $\lambda(\cdot)$ corresponding to a utility function $U(\cdot)\in\mathcal{U}_R$ so that (ii) is satisfied whenever $G(\cdot|\cdot)\in\mathcal{G}_R$. Alternatively, the IPV auction model with general risk aversion does not impose any restriction on observed bids beyond their independence and the smoothness conditions embodied in $\mathcal{G}_R$. Thus, by allowing for risk aversion, one does enlarge the set of rationalizable bid distributions relative to the risk neutral case studied in Guerre, Perrigne and Vuong (2000).[7]

A model is a set of structures $[U,F]$. Hereafter, a structure $[U,F]$ is *nonidentified* if there exists another structure $[\tilde{U},\tilde{F}]$ within the model that leads to the same equilibrium bid distribution. If no such structure $[\tilde{U},\tilde{F}]$ exists for any $[U,F]$, the model is (globally) identified. Given the weakness of the restrictions imposed by the model, it is not surprising that the model with risk aversion is not identified.

Proposition 2: *Let* $R\geq 1$*. Any structure* $[U,F]\in\mathcal{U}_R\times\mathcal{F}_R$ *is not identified.*

Proposition 2 says that the risk aversion model is nonparametrically nonidentified. This contrasts with Guerre, Perrigne and Vuong (2000) who show that the risk neutral model

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[6] As shown in the proof of Lemma 1, if (ii) is satisfied, then $G(\cdot|I)$ is rationalized by the structure $[U,F]$, where $U(x)=\exp\int_1^x(1/\lambda(t))dt$ and $F(\cdot|I)$ is the distribution of $\xi(b,U,G,I)$ with $b\sim G(\cdot|I)$. Because $\lambda(x)\approx\lambda'(0)x$ in the neighborhood of 0, $\int_1^0(1/\lambda(t))dt=-\infty$ so that $U(0)=0$, as required.

[7] Campo, Guerre, Perrigne and Vuong (2007) show another interesting result: Any distribution $G(\cdot|\cdot)\in\mathcal{G}_R$ can be rationalized by some constant relative or absolute risk aversion model.

is nonparametrically identified. Thus nonidentification arises from the unknown utility function $U(\cdot)$, which is restricted to the identity function under risk neutrality.

In view of this result, one can entertain several strategies to identify the model. A first natural strategy is to require more data. For instance, the availability of ascending auction data allows one to identifiy nonparametrically $[U, F]$ as shown in Lu and Perrigne (2006). In particular, since risk aversion does not affect bidding in ascending auctions, ascending auction data allow one to identify $F(\cdot|I)$ as in Athey and Haile (2002), from which $U(\cdot)$ is identified using sealed-bid auction data. A second strategy is to impose minimal parametric restrictions on the structure $[U, F]$ through a parameterization of $U(\cdot)$ and/or $F(\cdot|\cdot)$. This is pursued in Campo, Guerre, Perrigne and Vuong (2007) where $U(\cdot)$ and one quantile of $F(\cdot|\cdot)$ are parameterized to identify semiparametrically the model. In the next section, we explore another identifying strategy based on exclusion restrictions to identify nonparametrically the risk aversion model.

3 Identification Under Exclusion Restrictions

To clarify ideas, we first consider the case of an exogenous participation. In a second subsection, we consider the case of an endogenous participation with instruments that do not affect the underlying private value distribution.

3.1 Exogenous Participation

We assume that $F(\cdot|\cdot)$ does not depend on the number I of bidders, which corresponds to the exclusion restriction $F(\cdot|I) = F(\cdot)$. Because bidders' private values are independent of participation, the latter is exogenous. This exclusion restriction has been made implicitly in a majority of structural empirical studies involving various data sets such as Paarsch (1992) for timber sales, Bajari and Hortacsu (2003) for eBay data, etc. A notable exception is Athey, Levin and Seira (2004) for US Forest Service auctions. The functions $U(\cdot)$ and $F(\cdot)$ satisfy Definitions 1 and 2 with $R \geq 1$. The previous definitions and equations defining the model need to be revised accordingly with $F(\cdot)$ and $f(\cdot)$ defined on the support $[\underline{v}, \overline{v}]$, while the bid distribution $G(\cdot|I)$ still depends on I through $s(\cdot, U, F, I)$ but its support is now $[\underline{v}, s(\overline{v})]$, where the upper bound depends implicitly on I. The key idea of our nonparametric identification result is to exploit variations in the quantiles of the bid distribution with the number of bidders, while the corresponding quantiles of the private value distribution remain the same. Our result relies on a property of the equilibrium strategy, namely that $s(\cdot)$ is increasing in bidders' participation. In simple terms, increased competition renders bidding more aggressive.[8]

Let $I_2 > I_1 \geq 2$ be two different numbers of bidders. We use the index $j = 1, 2$ to refer to the level of competition. Because the equilibrium strategy varies with the number of bidders, the bid distribution also varies with I giving $s_j(\cdot)$ and $G_j(\cdot) \equiv G(\cdot|I_j)$. Though the lower bound of the bid distribution remains the same because of the boundary condition, the upper bound $\overline{b}_j$ varies with I. The next lemma gives some lower and upper bounds for each equilibrium strategy in terms of the other equilibrium strategy. In particular, it establishes that the equilibrium strategy strictly increases in the number of bidders. As far as we know, the latter property was obtained for the risk neutral case and the CRRA case, but not with a general risk aversion $U(\cdot)$.

Lemma 2: *Under the previous assumptions, we have*

$$\frac{I_1 - 1}{I_2 - 1} s_2(v) + \frac{I_2 - I_1}{I_2 - 1} \underline{v} < s_1(v) < s_2(v) < \frac{I_2 - 1}{I_1 - 1} s_1(v) + \frac{I_1 - I_2}{I_1 - 1} \underline{v}$$

for any $v \in (\underline{v}, \overline{v}]$.

The preceding lemma provides some testable implications in terms of stochastic dominance between the observed equilibrium bid distributions as well as their quantiles.[9] Let $G_1(\cdot) \prec_{\underline{b}} G_2(\cdot)$ denote that the distribution $G_1(\cdot)$ is strictly (first-order) stochastically dominated by $G_2(\cdot)$ except at the common lower bound $\underline{b}$. That is, $G_1(b) > G_2(b)$ for any $b \in (\underline{b}, \overline{b}_1]$, where the support of $G_j(\cdot)$ is $[\underline{b}, \overline{b}_j]$, which is a compact subset with nonempty interior of $[0, \infty)$. For $j = 1, 2$, let $b_j(\alpha)$ denote the α-quantile of $G_j(\cdot)$, i.e. $G_j[b_j(\alpha)] = \alpha$ for $\alpha \in [0, 1]$. Because $b_j = s_j(v)$ where $s_j(\cdot)$ is strictly increasing on $[\underline{v}, \overline{v}]$, $b_j(\alpha) = s_j[v(\alpha)]$, where $v(\alpha)$ is the α-quantile of $F(\cdot)$. Hence, from Lemma 2 the quantiles of $G_1(\cdot)$ and $G_2(\cdot)$ satisfy

$$\frac{I_1 - 1}{I_2 - 1} b_2(\alpha) + \frac{I_2 - I_1}{I_2 - 1} \underline{b} < b_1(\alpha) < b_2(\alpha) < \frac{I_2 - 1}{I_1 - 1} b_1(\alpha) + \frac{I_1 - I_2}{I_1 - 1} \underline{b} \qquad (5)$$

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[8] Identification of the bidders' utility function when the equilibrium strategies are nonincreasing in competition is discussed in Section 4.4.

[9] See Barrett and Donald (2003) for consistent tests of stochastic dominance.

for any $\alpha \in (0,1]$. Equivalently, let $G_{jk}(\cdot)$ denote the distribution of $[(I_k - 1)b_j + (I_j - I_k)\underline{b}]/[I_j - 1]$, where $j, k = 1, 2$, and $b_j = s_j(v)$.[10] Thus, the lower bound of the support of $G_{jk}(\cdot)$ is $\underline{b}$ and we have $G_{21}(\cdot) \prec_{\underline{b}} G_1(\cdot) \prec_{\underline{b}} G_2(\cdot) \prec_{\underline{b}} G_{12}(\cdot)$.

When the number I of bidders can take more than two values, the previous results imply several testable stochastic dominance relations among the observed bid distributions. Several of them are actually redundant. For instance, suppose that $I \in [\underline{I}, \overline{I}]$ with $2 \leq \underline{I} < \overline{I} < \infty$. The above implies that there are $4[1+2+\ldots+(\overline{I}-\underline{I})] = 2(\overline{I}-\underline{I})(\overline{I}-\underline{I}+1)$ stochastic dominance relations. The next corollary shows that there are at most $2(\overline{I}-\underline{I}+1)$ relevant relations.

Corollary 1: *Suppose that $I \in \mathcal{I} \equiv [\underline{I}, \overline{I}]$ with $2 \leq \underline{I} < \overline{I}$. Under the previous assumptions, the quantiles $b(\alpha, I)$ of the equilibrium bid distribution $G(\cdot|I)$ satisfy*

$$\max\left\{b(\alpha, I-1), \frac{I-1}{I} b(\alpha, I+1) + \frac{1}{I}\underline{b}\right\} < b(\alpha, I) < \min\left\{b(\alpha, I+1), \frac{I-1}{I-2} b(\alpha, I-1) - \frac{1}{I-2}\underline{b}\right\}$$

for any $\alpha \in (0,1]$ and any $I \in [\underline{I}, \overline{I}]$.[11] *Equivalently, let $b(I)$ denote the equilibrium bid with I bidders. Let $\underline{G}_I(\cdot)$ denote the distribution of the maximum of $b(I-1)$ and $[(I-1)b(I+1) + \underline{b}]/I$ and $\overline{G}_I(\cdot)$ denote the distribution of the minimum of $b(I+1)$ and $[(I-1)b(I-1) - \underline{b}]/(I-2)$. Hence, $\underline{G}_I(\cdot) \prec_{\underline{b}} G(\cdot|I) \prec_{\underline{b}} \overline{G}_I(\cdot)$, for any $I \in [\underline{I}, \overline{I}]$.*

Given two numbers of bidders $I_2 > I_1$, we now turn to the nonparametric identification of $[U(\cdot), F(\cdot)]$ or equivalently $[\lambda(\cdot), F(\cdot)]$ as $U(x) = \exp \int_1^x 1/\lambda(t)dt$ using the normalization $U(1) = 1$. The key idea is to use the invariance of the quantile $v(\alpha)$ for two different numbers of bidders I_1 and I_2. Specifically, using (4) leads to the *compatibility conditions*

$$b_2(\alpha) + \lambda^{-1}\left(\frac{1}{I_2 - 1}\frac{\alpha}{g_2(b_2(\alpha))}\right) = b_1(\alpha) + \lambda^{-1}\left(\frac{1}{I_1 - 1}\frac{\alpha}{g_1(b_1(\alpha))}\right), \tag{6}$$

for all $\alpha \in [0,1]$. We then show that there exists a unique inverse function $\lambda^{-1}(\cdot)$ defined on the range $\mathcal{R}_1$ of the function $R_1(\alpha)$, where $\alpha \in [0,1]$ and

$$R_j(\alpha) = \frac{1}{I_j - 1}\frac{\alpha}{g_j[b_j(\alpha)]} \tag{7}$$

for $j = 1, 2$. Note that the range $\mathcal{R}_j$ of $R_j(\cdot)$ is of the form $[0, \overline{r}_j]$ with $0 < \overline{r}_j < \infty$ because $g_j(\cdot)$ is bounded away from zero and continuous on $[0, \overline{b}_j]$ by Definition 3 and

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[10] When $j = k$, $G_{jk} = G_j(\cdot)$.

[11] Obviously, $b(\cdot, I-1)$ is dropped when $I = \underline{I}$, while $b(\cdot, I+1)$ is dropped when $I = \overline{I}$.

Lemma 1. Moreover, $R_j(\alpha) = \lambda[v(\alpha) - s_j(v(\alpha))]$ from (4). Thus, identifying $\lambda^{-1}(\cdot)$ on $\mathcal{R}_j$ is equivalent to identifying $\lambda(\cdot)$ on the range of the markdown/rent $v - s_j(v)$, where $v \in [\underline{v}, \overline{v}]$. Because $s_1(\cdot) < s_2(\cdot)$ on $(\underline{v}, \overline{v}]$ by Lemma 2, we have $R_1(\cdot) > R_2(\cdot)$ on $(0, 1]$. Thus, $\overline{r}_2 < \overline{r}_1$ so that $\mathcal{R}_2 \subset \mathcal{R}_1$. Hence, $\lambda(\cdot)$ is identified nonparametrically on the largest set of possible markdowns $[0, \max_{v\in[\underline{v},\overline{v}]} v - s_1(v)]$.[12] The next proposition provides explicit expressions for $\lambda(\cdot)$ and $F(\cdot)$.

Proposition 3: *Under the previous assumptions,* $\lambda^{-1}(\cdot)$ *is identified nonparametrically on* $\mathcal{R}_1$. *Specifically,* $\lambda^{-1}(0) = 0$ *and for any* $u_0 \in \mathcal{R}_1 \setminus \{0\}$, $\lambda^{-1}(\cdot)$ *is given by*

$$\lambda^{-1}(u_0) = \sum_{t=0}^{+\infty} \Delta b(\alpha_t),$$

where $\Delta b(\alpha) = b_2(\alpha) - b_1(\alpha)$, *and the sequence* $\{\alpha_t\}$ *is strictly decreasing with* $0 < \alpha_t \leq 1$ *satisfying the nonlinear recursive relation* $R_1(\alpha_t) = R_2(\alpha_{t-1})$ *with initial condition* $R_1(\alpha_0) = u_0$. *Moreover,* $F(\cdot)$ *is identified nonparametrically on* $[\underline{v}, \overline{v}]$ *with* $F(\cdot) = G_j[\xi_j^{-1}(\cdot)]$ *for* $j = 1, 2$.

Our proof is constructive and shows that the sequence $\{\alpha_t\}$ exists though is not necessarly unique. When $R_1(\cdot)$ is strictly increasing, i.e. when the markdown is strictly increasing in v, such a sequence is unique. When $R_1(\cdot)$ is not strictly increasing, the sequence $\{\alpha_t\}$ may not be unique, but all such sequences must lead to the same value for $\sum_{t=0}^{\infty} \Delta b(\alpha_t)$, which then defines $\lambda^{-1}(u_0)$ uniquely. The construction of $\{\alpha_t\}$ is illustrated in Figure 1. Figure 1 displays the equilibrium strategies $s_1(\cdot) < s_2(\cdot)$. For $\alpha_0 \in (0, 1]$, consider the α_0-quantile $v(\alpha_0)$ of $F(\cdot)$. The markdown $v(\alpha_0) - b_1(\alpha_0)$ is the sum of $\Delta b(\alpha_0)$, which is known and $\lambda^{-1}[R_2(\alpha_0)]$, which is unknown. The latter is equal to the markdown $v(\alpha_1) - b_1(\alpha_1)$, which is also the sum of $\Delta b(\alpha_1)$ and $\lambda^{-1}[R_2(\alpha_1)]$. Continuing this construction gives the sequence $\{\alpha_t\}$ and determines $\lambda^{-1}[R_2(\alpha_0)]$ as the infinite series of known differences in bid quantiles $\Delta(\alpha_t)$.

An important related question is to characterize all the restrictions on the observed equilibrium bid distributions that arise from the auction model with exogenous participation. In particular, it is useful to assess whether the observed bid distributions, which vary with I, can be rationalized by a structure $[U(\cdot), F(\cdot)]$ that is independent of I. In

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[12] In general, $\max_{v\in[\underline{v},\overline{v}]} v - s_j(v) \neq \overline{v} - s_j(\overline{v})$. On the other hand, if the markdown $v - s_j(v)$ is increasing in v, then $\max_{v\in[\underline{v},\overline{v}]} v - s_j(v) = \overline{v} - s_j(\overline{v})$. Moreover, $R_j(\cdot)$ would be increasing in α and $\overline{r}_j = R_j(1)$.

other words, these restrictions allow one to test the model and its assumptions. Violation of one of these restrictions leads to reject the model for explaining the observed bids. In particular, it could mean that the exogeneity of bidders' participation is not justified. Lemma 3 provides such restrictions when I takes two different values $I_2 > I_1$.

Lemma 3: *Let* $\mathcal{I} = \{I_1, I_2\}$ *with* $I_1 < I_2$. *Let* $\mathbf{G}_j(\cdot, \ldots, \cdot)$ *be the joint distribution of* $(b_1, \ldots, b_{I_j})$, $j = 1, 2$. *The equilibrium bid distributions* $\mathbf{G}_j(\cdot)$, $j = 1, 2$, *are rationalized by a structure* $[U(\cdot), F(\cdot)]$ *independent of* I *if and only if*

(i) For each $j = 1, 2$, $\mathbf{G}_j(b_1, \ldots, b_{I_j}) = \prod_{i=1}^{I_j} G_j(b_i)$, *where* $G_j(\cdot) = G(\cdot|I_j)$ *with support of the form* $[\underline{b}, \overline{b}_j]$ *and* $\{G(\cdot|I); I \in \mathcal{I}\} \in \mathcal{G}_R$,

(ii) The α*-quantiles of* $G_1(\cdot)$ *and* $G_2(\cdot)$ *satisfy* $b_1(\alpha) < b_2(\alpha)$ *for* $\alpha \in (0, 1]$, *i.e.* $G_1(\cdot) \prec_{\underline{b}} G_2(\cdot)$,

(iii) $\exists \lambda(\cdot) : I\!R_+ \to I\!R_+$ *with* $R + 1$ *continuous derivatives on* $[0, +\infty)$, $\lambda(0) = 0$ *and* $\lambda'(\cdot) \geq 1$ *such that*

(a) the compatibility condition (6) is satisfied for every $\alpha \in [0, 1]$,

(b) for $I_j \in \mathcal{I}$, $\xi_j'(\cdot) > 0$ *on* $[\underline{b}, \overline{b}_j]$, *where* $\xi_j(b) = b + \lambda^{-1}[G_j(b)/((I_j - 1)g_j(b))]$.

Unlike Lemma 2, which only provides some (testable) implications, Lemma 3 characterizes all the theoretical restrictions imposed by the model with an exogenous bidders' participation. Relative to the general case of Section 2 in which $F(\cdot)$ can vary with I, the set of bid distributions that can be rationalized is much reduced because of the restrictions (ii) and (iii)(a). Together, Proposition 3 and Lemma 3 say that exogenous participation allows one to identify and test the model in sharp contrast to Propositions 1 and 2 for the model without the exclusion restriction. As in Corollary 1, Lemma 3 can be extended straightforwardly to the case where $I \in \mathcal{I} \equiv [\underline{I}, \overline{I}]$. Specifically, (i) and (iii)-(b) hold, while (ii) and (iii)-(a) still hold for all pairs $(I_j, I_k) \in \mathcal{I}, k \neq j$.

3.2 Endogenous Participation

From an economic point of view, we can expect that the private value distribution depends on the number of bidders. A simple argument is that higher valued objects may attract more bidders. In this subsection, we introduce additional variables and possible unobserved ones that can explain auctioned objects heterogeneity and bidders' participation. In particular, bidders' participation may be endogenous and some of these additional variables can play the role of instrumental variables through exclusion restrictions.

We first consider the case of unobserved heterogeneity in bidder's participation. Let W be a vector of observed variables characterizing heterogeneity across auctioned objects. These variables are assumed to affect both bidders' private value distribution and bidders' participation. Bidders' participation is modeled as $I = I(W, \epsilon)$, where ϵ can be interpreted as a term of unobserved heterogeneity or as a traditional error term. We assume $v \perp \epsilon | W$, namely bidders' private values are independent of ϵ given the auction characteristics W. This assumption translates into the exclusion restriction $F(v|W, \epsilon) = F(v|W)$, while the observed bid distribution is $G(b|I, W)$ since $b = s(v, U, F, I)$. This model is similar to the one in the previous subsection since bidders' private values are independent of I given W. Thus the difference between the two models is the introduction of the vector of conditioning variables W. This exclusion restriction allows one to exploit variations in bidding behavior under two competitive environments, i.e. $I_2 > I_1$, at W given. Proposition 3 applies and the pair $[U(\cdot), F(\cdot|\cdot)]$ is nonparametrically identified, while the quantile becomes $b_j(\alpha, W)$. Regarding Lemma 3, the bids are now conditionally independent given W in (i), while the rest extends straightforwardly.

The term of unobserved heterogeneity ϵ may, however, affect the private value distribution as ϵ can capture some unobserved characteristics such as quality affecting both private values and bidders' participation. This leads to a model of endogenous participation. The introduction of additional variables or instruments Z combined with appropriate exclusion restrictions solves this problem. Specifically, bidders' participation is modeled as $I = I(W, Z, \epsilon)$, while we assume $v \perp Z | (W, \epsilon)$, namely the bidders' private values are independent of Z given (W, ϵ). This translates into the exclusion restriction $F(v|W, Z, \epsilon) = F(v|W, \epsilon)$. Hence, the variables Z can be viewed as instruments. This model corresponds to an endogenous number of bidders as the unobserved heterogeneity ϵ affects both bidders' private values and bidders' participation.[13]

This model is reminiscent of entry models used recently in the empirical auction literature. Bajari and Hortacsu (2003) model bidders' entry in eBay coin auctions within a common value framework. Their empirical results show that the variables explaining bidders' entry are the appraisal value of the auctioned object (W), the reserve price for

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[13] Haile, Hong and Shum (2003) adopt a similar framework to test for common value in first-price sealed-bid auctions when I is endogenous.

the auctioned object (Z_1) and the seller's reputation (Z_2), while the bidders' private signal distribution depends on W only. Athey, Levin and Seira (2004) model loggers' entry in US Forest Service sealed auctions and find that entry depends on road costs (Z_1) and scale sale (Z_2) in Norhern sales, and on logging costs (Z_1), road costs (Z_2), density of timber (Z_3) and timber volume (Z_4) in California sales in addition to several variables (W) affecting the bids. Krasnokutskaya and Seim (2005) model firms' participation in road construction procurements and find that variables affecting participation but not bids are the distance of the firm to the project location (Z_1) and the firm's working load or backlog (Z_2). As suggested in these papers, a reduced form analysis of the number of bidders and the observed bids can help in determining the variables W and the instruments Z.

The next result shows how the availability of instruments can help identify the model.

Corollary 2: *The structure $[U, F]$ with endogenous participation and unobserved heterogeneity is identified under the exclusion restriction $F(\cdot|W, Z, \epsilon) = F(\cdot|W, \epsilon)$, additive separability of $I(W, Z, \epsilon)$ in ϵ and $\mathrm{E}[\epsilon|W, Z] = 0$.*

The argument is as follows. The observed bid distribution satisfies $G(b|W, Z, \epsilon) = G(b|I, W, \epsilon)$ as $s(\cdot) = s(\cdot, I, W, \epsilon)$, where $v \sim F(v|W, Z, \epsilon) = F(v|W, \epsilon)$ and $I = I(W, Z, \epsilon)$. The parallel with Proposition 3 appears as we can exploit variations in bidding behavior under two competitive environments while the latent distribution remains the same at (W, ϵ) given. The term of heterogeneity ϵ is, however, unobserved. Under additive separability of ϵ, we have $I(W, Z, \epsilon) = I(W, Z) + \epsilon$, where ϵ takes a finite number of values. Under the assumption $\mathrm{E}(\epsilon|W, Z) = 0$, $I(W, Z) = \mathrm{E}(I|W, Z)$. Because $\mathrm{E}(I|\cdot, \cdot)$ is the regression of I on (W, Z), $\mathrm{E}(I|\cdot, \cdot)$ is nonparametrically identified so that ϵ can be recovered as $\epsilon = I - \mathrm{E}(I|\cdot, \cdot)$. Proposition 3 applies and $[U, F]$ is nonparametrically identified, while the quantile becomes $b_j(\alpha, W, \epsilon)$. Regarding Lemma 3, the bids are now conditionally independent given (W, ϵ) in (i), while the rest extends straightforwardly.[14]

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[14] Endogenous entry with an additive error term can be used for solving the problem of unobserved heterogeneity. For instance, consider the risk neutral model with endogenous entry $I = I(W) + \epsilon$, where the bidders' private value distribution is $F(v|W, \epsilon)$. As above, ϵ can be recovered as $\epsilon = I - \mathrm{E}(I|W, \epsilon)$. Thus, $F(\cdot|W, \epsilon)$ is nonparametrically identified following Guerre, Perrigne and Vuong (2000, Theorem 1). This method differs from Krasnokutskaya (2004) who identifies the term of unobserved heterogeneity in a private value model $F(v|W, \epsilon)$ with exogenous participation $I = I(W)$ using a multiplicative decomposition of private values and applying Li and Vuong's (1998) results.

4 Extensions

This section extends our results to a reserve price, affiliated private values and asymmetric biddders in private values and/or preferences. Except for the first part of Proposition 7, we only sketch the proofs of Propositions 4–7 in the text.

4.1 Reserve Price

A reserve price $p_0 > \underline{v}$ introduces a truncation in the observed bid distribution as only the I^* bidders who have a value above p_0 bid at the auction. Let $G^*(\cdot|I)$ be the truncated bid distribution on $[p_0, \overline{b}(I)]$. We observe I^* the number of active bidders, $I^* \leq I, I \in \mathcal{I}$. Because $G^*(b^*|I) = [F(v|I) - F(p_0|I)]/[1 - F(p_0|I)]$ for $b^* \in [p_0, \overline{b}(I)]$, elementary algebra gives the following inverse equilibrium strategy

$$v = b^* + \lambda^{-1}\left(\frac{1}{I-1}\frac{G^*(b^*|I)}{g_j^*(b|I)} + \frac{1}{I-1}\frac{1}{g^*(b^*|I)}\frac{F(p_0|I)}{1-F(p_0|I)}\right) \equiv \xi(b^*, U, G^*, I, F(p_0|I)). \qquad (8)$$

Definitions 1, 2 and 3 remain the same with the exception that p_0 replaces $\underline{b}(I)$ in Definition 3. Moreover, because $\lim_{b \downarrow p_0} g^*(b|I) = +\infty$ as $s'(p_0) = 0$ from (2), derivatives and limits are infinite at p_0 in Definition 3.[15] Given that I^* is Binomial distributed with parameters $[I, 1 - F(p_0|I)]$, I and $F(p_0|I)$ are identified under the assumption that I is constant across subsets of auctions.

Proposition 4: *Any structure* $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ *with a reserve price is not identified. On the other hand, the structure* $[U, F]$ *with the exclusion restriction* $F(\cdot|I) = F(\cdot)$ *is with* $U(\cdot)$ *identified on* $[0, \max_{v \in [\underline{v}, \overline{v}]} v - s_1(v)]$ *and* $F(\cdot)$ *identified on* $[p_0, \overline{v}]$.

The joint bid distribution $\mathbf{G}^*(\cdot, \ldots, \cdot)$ is rationalized if and only if Lemma 1 is satisfied with $\mathcal{G}_R$ and $\xi(\cdot)$ as defined above. From this rationalization result, any $G^*(\cdot|I) \in \mathcal{G}_R, I \in \mathcal{I}$ can be rationalized by a risk aversion structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$. It is then straightforward to show that the structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ is not identified.

Under the exogeneity of the number of bidders, we assume that we observe at least two levels of potential bidders $I_1 < I_2$. Let $G_j^*(\cdot)$ be the truncated bid distribution on

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[15] To avoid infinite derivatives/limits at p_0, we can consider the bid transformation used in Guerre, Perrigne and Vuong (2000, Section 4), in which case rationalization and identification are based on the density of the transformed bids.

$[p_0, \overline{b}_j]$. Lemma 2 still holds with a reserve price as the latter simply reduces the shading relative to the case with no reserve price. In view of (8), $R_j(\alpha)$ becomes

$$R_j(\alpha) = \frac{1}{I_j - 1} \frac{1}{g_j^*[b_j^*(\alpha)]} \left(\alpha + \frac{F(p_0)}{1 - F(p_0)} \right),$$

for $j = 1, 2$. Note that $R_j(\alpha)$ differs from (6) by the additional term $F(p_0)/(1 - F(p_0))$. As before, the number of potential bidders I_j and $F(p_0)$ are identified from the Binomial distribution of the number of actual bidders. The problem reduces to identifying $\lambda^{-1}(\cdot)$ and $F(\cdot)$ on $[0, \overline{r}_1]$ and $[p_0, \overline{v}]$, respectively. A simple extension of Proposition 3 shows that $[\lambda^{-1}(\cdot), F(\cdot)]$ is nonparametrically identified on these intervals using the quantiles $b_j^*(\alpha)$ of $G_j^*(\cdot)$. Similarly, Lemma 3 and Corollary 2 can be readily adapted.

4.2 Affiliated Private Values

The vector $(v_1, \ldots, v_I)$ is distributed as $F(\cdot, \ldots, \cdot | I)$, which is exchangeable in its I arguments, affiliated and defined on $[\underline{v}(I), \overline{v}(I)]^I$. We follow the notations of Li, Perrigne and Vuong (2002). Let $G_{B_i|b_i}(b_i|b_i, I)$ be the probability that i has a bid larger than all his opponents conditional on his bid b_i with $B_i = \max_{k \neq i} b_k$ and $b_i = s(v_i)$. Without loss of generality, we can consider $G_{B_1|b_1}(\cdot|\cdot, I)$ as all bidders are symmetric. To simplify the notations, we omit the index 1. The inverse equilibrium strategy becomes

$$v = b + \lambda^{-1} \left(\frac{G_{B|b}(b|b, I)}{g_{B|b}(b|b, I)} \right) \equiv \xi(b, U, \mathbf{G}, I) \text{ for all } b \in [\underline{b}(I), \overline{b}(I)] \tag{9}$$

with the joint bid distribution $\mathbf{G}(\cdot, \ldots, \cdot | I)$. Definitions 1 and 2 remain the same except that $F(\cdot, \ldots, \cdot | I)$ is $R + I$ continuously differentiable following Li, Perrigne and Vuong (2000, 2002). Note that $G_{B|b}(\cdot| \cdot |I)/g_{B|b}(\cdot| \cdot |I) = G_{B \times b}(\cdot, \cdot | I)/g_{Bb}(\cdot, \cdot | I)$, where $G_{B \times b}(\cdot, \cdot | I) \equiv \partial G_{Bb}(\cdot, \cdot | I)/\partial b$ and $g_{Bb}(\cdot, \cdot | I)$ are the b-derivative of the joint c.d.f. and the joint density of (B, b), respectively. Let $\mathcal{G}_R$ be the set of exchangeable and affiliated distributions $\{\mathbf{G}(\cdot, \ldots, \cdot | I), I \in \mathcal{I}\}$ with R continuously differentiable densities such that $G_{B \times b}(b, b|I)/g_{Bb}(b, b|I)$ is $R + 1$ continuously differentiable in $b \in [\underline{b}(I), \overline{b}(I)]$ and strictly positive on $(\underline{b}(I), \overline{b}(I)]$.

Proposition 5: *Any structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ with affiliated values is not identified. On the other hand, the structure $[U, F]$ with the exclusion restriction $F(\cdot, \ldots, \cdot | I) = F(\cdot, \ldots, \cdot)$ is with $U(\cdot)$ identified on $[0, \max_{v \in [\underline{v}, \overline{v}]} v - s_1(v)]$ and $F(\cdot, \ldots, \cdot)$ identified on $[\underline{v}, \overline{v}]^I$.*

The joint bid distribution $\mathbf{G}(\cdot, \ldots, \cdot|\cdot)$ is rationalized if and only Lemma 1 is satisfied with $\mathcal{G}_R$ and $\xi(\cdot)$ as defined above. From this rationalization result, any $\mathbf{G}(\cdot, \ldots, \cdot|\cdot) \in \mathcal{G}_R$ can be rationalized by some risk aversion structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$. We can then show that the structure $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ is not identified using a similar argument as in Proposition 2, where $G(\cdot|I)/[(I-1)g(\cdot|I)]$ is replaced by $G_{B\times b}(\cdot, \cdot|I)/g_{Bb}(\cdot, \cdot|I)$ in view of (4) and (9).

Under two competitive environments $I_1 < I_2$ and exogenous bidders' participation, the vector $(v_1, \ldots, v_{I_j})$ is distributed as $F_j(\cdot, \ldots, \cdot)$, which is exchangeable and affiliated. Proposition 5 uses the property that $F_1(\cdot, \ldots, \cdot)$ and $F_2(\cdot, \ldots, \cdot)$ are related by $F_1(v_1, \ldots, v_{I_1}) = \int_{\underline{v}}^{\overline{v}} \ldots \int_{\underline{v}}^{\overline{v}} F_2(v_1, \ldots,\ v_{I_1}, v_{I_1+1}, \ldots, v_{I_2}) dv_{I_1+1} \ldots dv_{I_2}$, i.e. $F_1(\cdot, \ldots, \cdot)$ is the marginal of $F_2(\cdot, \ldots, \cdot)$. Hence, $F_j(\cdot, \ldots, \cdot)$ has support $[\underline{v}, \overline{v}]^j$. Assume that the structures $[U, F_j], j = 1, 2$ satisfy $s_1(v) < s_2(v)$, i.e. competition renders bidding more aggressive.[16] Here again, the exogeneity of the number of bidders allows one to identify $\lambda^{-1}(\cdot)$ on $[0, \overline{r}_1]$ by exploiting variations in the number of bidders, where $R_j(\cdot) = G^j_{B\times b}(b_j(\alpha), b_j(\alpha))/g^j_{B,b}(b_j(\alpha), b_j(\alpha))$ with $b_j(\alpha)$ the α-quantile of the marginal bid density $g_j(\cdot)$ associated with I_j bidders. Specifically, Proposition 3 and Lemma 3 extend to this case, while Corollary 2 requires that $F_1(\cdot, \ldots, \cdot|W, \epsilon)$ is the marginal of $F_2(\cdot, \ldots, \cdot|W, \epsilon)$.

4.3 Asymmetric Bidders

Asymmetry among bidders, which is known ex ante to all participants, can arise from two different sources, namely from (i) different distributions of private values and/or (ii) different utility functions. We consider these cases separately. Hereafter, it is assumed that bidders' identities are known.

ASYMMETRY IN PRIVATE VALUES

Given $I \in \mathcal{I}$, the joint private value distribution is $\mathbf{F}(\cdot, \ldots, \cdot|I) = \prod_i F_i(\cdot|I)$ with each $F_i(\cdot|I)$ satisfying Definition 2 on the support $[\underline{v}(I), \overline{v}(I)]$. To simplify, we assume that all $F_i(\cdot|I)$ have the same support. Let $\mathcal{F}_R$ be the set of such distributions $\mathbf{F}(\cdot, \ldots, \cdot|I)$ when $I \in \mathcal{I}$. Because of the boundary conditions $s_i(\underline{v}(I)) = \underline{v}(I)$ and $s_i(\overline{v}(I)) = s_j(\overline{v}(I))$, $j \neq i$, bidder i's distribution $G_i(\cdot|I)$ is defined on $[\underline{b}(I), \overline{b}(I)]$ for all $i = 1, \ldots, I$. Following

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[16] Section 4.4 relaxes this assumption. The competition effect is unclear with affiliated private values as some distributions $F(\cdot, \ldots, \cdot)$ may lead to bidding strategies decreasing in the number of bidders as shown by Pinkse and Tan (2005).

Campo, Perrigne and Vuong (2003), we have

$$v_i = b_i + \lambda^{-1}\left(\frac{1}{H_i(b_i|I)}\right) \equiv \xi_i(b_i, U, \mathbf{G}, I), \text{ where } H_i(\cdot|I) = \sum_{k\neq i} \frac{g_k(\cdot|I)}{G_k(\cdot|I)}, \qquad (10)$$

for $i = 1, \ldots, I$. Let $\mathcal{G}_R$ be the set of distributions $\mathbf{G}(\cdot, \ldots, \cdot|\cdot)$ such that each marginal distribution $G_i(\cdot|\cdot)$ satisfies Definition 3 with $G(b|I)/g(b|I)$ replaced by $1/H_i(b|I)$ in (v).

Proposition 6: *Any structure* $[U, \mathbf{F}] \in \mathcal{U}_R \times \mathcal{F}_R$ *with asymmetry in private values is not identified. On the other hand, the structure* $[U, \mathbf{F}]$ *with the exclusion restriction* $F_i(\cdot|I) = F_i(\cdot)$, $I \in \mathcal{I}$ *is partially identified with* $U(\cdot)$ *identified on* $[0, \max_{v\in[\underline{v},\overline{v}], i=1,\ldots,I_0} v - s_i(v)]$ *and* $F_i(\cdot), i = 1, \ldots, I_0$ *identified on* $[\underline{v}, \overline{v}]$*, where* I_0 *is the number of bidders participating to both auctions. For the remaining bidders,* $F_i(\cdot)$ *is identified for the quantiles satisfying* $R_{i1}(\alpha) \in [0, \max_{k=1,\ldots,I_0} \overline{r}_{k1}]$*, where* $R_{i1}(\alpha)$ *is defined below.*

The bid distribution $\mathbf{G}(\cdot, \ldots, \cdot|\cdot)$ is rationalized if and only if Lemma 1 is satisfied with $\mathcal{G}_R$ and $\xi_i(\cdot), i = 1, \ldots, I$, $I \in \mathcal{I}$ as defined above. Hence, any $\mathbf{G}(\cdot, \ldots, \cdot|\cdot) \in \mathcal{G}_R$ can be rationalized by a risk aversion structure $[U, \mathbf{F}] \in \mathcal{U}_R \times \mathcal{F}_R$. It is then straigthforward to show that any structure $[U, \mathbf{F}] \in \mathcal{U}_R \times \mathcal{F}_R$ is not identified.

Under two competitive environments $I_2 > I_1 \geq 2$ and exogenous bidders' participation, each bidder i has a private value distribution independent of I. Thus, all $F_i(\cdot)$s are defined on the same support $[\underline{v}, \overline{v}]$. Because our results under exclusion restrictions exploit the difference in bidding behavior under two competitive environments, it is crucial that at least one bidder participates in both auctions.[17] For instance, when $I_1 = 2$ and $I_2 = 3$, at least one of the bidders in the two bidders auction must participate in the three bidders auction. In the case of asymmetry, because of the complexity of the system of differential equations defining the equilibrium strategies, it is difficult if not impossible to prove that equilibrium strategies are increasing in competition. Nevertheless, because of the independence of private values, we postulate that equilibrium strategies are increasing in I due to the competition effect.

Let $s_{ij}(\cdot)$ denote the equilibrium strategy for bidder $i = 1, \ldots, I_j$, when the number of bidders is I_j, $j = 1, 2$. The boundary conditions are $s_{i1}(\underline{v}) = s_{i'2}(\underline{v}) = \underline{v}$ for $i = 1, \ldots, I_1$

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[17] In empirical studies, a few bidders' types are entertained as in Campo, Perrigne and Vuong (2003), Athey, Levin and Seira (2004) and Flambard and Perrigne (2006). In this case, it is crucial that at least one type is represented in both auctions.

and $i' = 1, \dots, I_2$, and $s_{ij}(\overline{v}) = s_{i'j}(\overline{v})$ for $i, i' = 1, \dots, I_j, j = 1, 2$. Bidders $1, \dots, I_0$ participate to both auctions, where $1 \leq I_0 \leq I_1$. Let $v_i(\alpha)$ and $b_{ij}(\alpha)$ be the α-quantiles of $F_i(\cdot)$ and $G_{ij}(\cdot) = G_i(\cdot|I_j)$, respectively. Instead of (4), we now have at the α-quantile

$$v_i(\alpha) = b_{ij}(\alpha) + \lambda^{-1}\left(R_{ij}(\alpha)\right), i = 1, \dots, I_j, j = 1, 2 \tag{11}$$

for $\alpha \in [0, 1]$, where $R_{ij}(\alpha) = 1/H_{ij}(b_{ij}(\alpha))$ takes values in the range $\mathcal{R}_{ij} = [0, \overline{r}_{ij}]$ with $H_{ij}(\cdot) = \sum_{k \neq i} g_{kj}(\cdot)/G_{kj}(\cdot)$. A straightforward extension of Proposition 3 shows that $\lambda^{-1}(\cdot)$ is identified on $[0, \max_{k=1,\dots,I_0} \overline{r}_{k1}]$, while $[F_1, \cdots, F_{I_0}]$ are identified on $[\underline{v}, \overline{v}]$. Because $\lambda^{-1}(\cdot)$ is identified on $[0, \max_{k=1,\dots,I_0} \overline{r}_{k1}]$, which may be a strict subset of $[0, \overline{r}_{i1}]$, where i refers to a remaining bidder, his private value distribution may not be identified everywhere justifying the partial identification result of Proposition 6. Lemma 3 also extends where the compatibility conditions (6) now hold for each of the I_0 bidders. Moreover, Corollary 2 applies provided the IV exclusion restrictions $F_i(\cdot|W, Z, \epsilon) = F_i(\cdot|W, \epsilon)$ for $i = 1, \dots, I_0$ hold.

Asymmetry in Preferences

We consider structures of the form $[\mathbf{U}, F] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$ with $\mathbf{U} = \{(U_1, \dots, U_I) \in \mathcal{U}_R^I \equiv \bigotimes_{i=1}^I \mathcal{U}_R, I \in \mathcal{I}\} \in \mathcal{U}_R^{\mathcal{I}}$. Given $I \in \mathcal{I}$, we obtain for $i = 1, \dots, I$

$$v_i = b_i + \lambda_i^{-1}\left(\frac{1}{H_i(b_i|I)}\right) \equiv \xi_i(b_i, U_i, \mathbf{G}, I), \tag{12}$$

where $\lambda_i(\cdot) = U_i(\cdot)/U_i'(\cdot)$ and $H_i(\cdot|I) = \sum_{k \neq i} g_k(\cdot|I)/G_k(\cdot|I)$. For each I, the boundary conditions $s_1(\underline{v}) = \dots = s_I(\underline{v}) = \underline{v}$ and $s_1(\overline{v}) = \dots = s_I(\overline{v})$ give a common support $[\underline{b}(I), \overline{b}(I)]$ for the bid distributions across bidders. Let $\mathcal{G}_R$ be the set of distributions $\mathbf{G}(\cdot, \dots, \cdot|\cdot)$ such that each marginal distribution $G_i(\cdot|\cdot)$ satisfies Definition 3 with $G(b|I)/g(b|I)$ replaced by $1/H_i(b|I)$ in (v).

Proposition 7: *Any structure $[\mathbf{U}, F] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$ with asymmetry in preferences satisfying $H_i'(\cdot|I) < 0, i = 1, \dots, I, I \in \mathcal{I}$ is not identified.*[18] *On the other hand, the structure $[\mathbf{U}, F]$ with the exclusion restriction $F(\cdot|I) = F(\cdot)$ is with $U_i(\cdot)$ identified on $[0, \max_{v \in [\underline{v}, \overline{v}]} v - s_i(v)]$ for $i = 1, \dots, I$ and $F(\cdot)$ identified on $[\underline{v}, \overline{v}]$.*

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[18] The assumption $H_i'(\cdot|I) < 0$ corresponds to an increasing markup $v_{i\alpha} - b_{i\alpha}$ in α from (12). If all the bid distributions $G_1, \dots, G_I$ are log-concave, this assumption is automatically satisfied. Our requirement is weaker as some bid distributions may not be log-concave. Log-concavity is usually verified on data.

Because the α-quantiles $(b_1(\alpha), \ldots, b_I(\alpha))$ all correspond to the same α-quantile $v(\alpha)$, (12) gives for an arbitrary pair (i, k) of bidders the compatibility conditions

$$b_k(\alpha) + \lambda_k^{-1}\left(\frac{1}{H_k(b_k(\alpha)|I)}\right) = b_i(\alpha) + \lambda_i^{-1}\left(\frac{1}{H_i(b_i(\alpha)|I)}\right) \tag{13}$$

for all $\alpha \in [0, 1]$. These conditions reduce the set of bid distributions that can be rationalized relative to the symmetric case and may help in identification. Specifically, the bid distribution $\mathbf{G}(\cdot, \ldots, \cdot|\cdot)$ is rationalized if and only if (i) Lemma 1 is satisfied with $\mathcal{G}_R$ and $\xi_i(\cdot), i = 1, \ldots, I$ as defined above and (ii) the compatibility conditions (13) are satisfied for any pair (i, k) of bidders.[19] Despite these compatibility conditions, the nonparametric model is still not identified. Because it is more involved than in previous cases, we provide a proof of such a result in the appendix.[20]

Under exogenous bidders' participation, we assume again that I_0 bidders participate to both auctions, where $I_0 \geq 1$ and that equilibrium strategies are increasing in competition. Equation (11) takes a similar form with $v(\alpha)$ and $\lambda_i^{-1}(\cdot)$ replacing $v_i(\alpha)$ and $\lambda^{-1}(\cdot)$. A similar argument as in Proposition 3 applies for identifying nonparametrically $\lambda_i^{-1}(\cdot)$ on $[0, \overline{r}_{i1}]$ for $i = 1, \ldots, I_0$ from which we can identify $F(\cdot)$ on $[\underline{v}, \overline{v}]$. Since $F(\cdot)$ is identified everywhere, following a similar argument as in Lu and Perrigne (2006), the $\lambda_j^{-1}(\cdot)$s for the remaining bidders are identified from (12). Specifically, because the $v(\alpha)$s are identified, we can recover the remaining $\lambda_j^{-1}(\cdot)$ on $[0, \overline{r}_j]$. Again Lemma 3 extends with the compatibility conditions (6) holding for each of the I_0 bidders and $\lambda_i^{-1}(\cdot)$ replacing $\lambda^{-1}(\cdot)$ to which we need to add the compatibility conditions (13) that must hold for any pair of bidders in each auction. Moreover, Corollary 2 applies.

ASYMMETRY IN BOTH PREFERENCES AND PRIVATE VALUES

This third case involves asymmetry in both private value distributions and preferences. Given the above results, the model is not identified. We then consider exogenous bidders' participation. Thus, (11) takes a similar form with $\lambda_i^{-1}(\cdot)$ replacing $\lambda^{-1}(\cdot)$. Despite the

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[19] Though similar in spirit, the compatibility conditions (13) apply within each auction, while the compatibility conditions (6) apply across auctions.

[20] On the other hand, if bidder 1 participates to all auctions and his utility $U_1(\cdot)$ is known, the nonparametric model $\mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$ becomes identified as (12) for $i = 1$ allows to identify $F(\cdot|\cdot)$. Thus, evaluated at the α-quantile, (12) for $i \neq 1$ allows to identify $\lambda_i(\cdot)$ on $[0, \max_\alpha(v_\alpha - b_{i\alpha})]$. This result is useful when bidders differ by their sizes and "large" ones, which participate to all auctions, are risk neutral.

complexity of this case, which has not been considered to our knowledge, it can be shown that the structure $[\lambda_i, F_i]$ is nonparametrically identified for the $I_0 \geq 1$ bidders who participate to both auctions. Specifically, we can apply Proposition 3 to these bidders to identify nonparametrically $\lambda_i^{-1}(\cdot)$ and $F_i(\cdot)$ on $[0, \overline{r}_{i1}]$ and $[\underline{v}, \overline{v}]$, respectively. On the other hand, we cannot identify the pair $[\lambda_i^{-1}(\cdot), F_i(\cdot)]$ for the other bidders. Again Lemma 3 extends with the compatibility conditions (6) holding for each of the common bidders where $\lambda_i^{-1}(\cdot)$ replaces $\lambda^{-1}(\cdot)$. Corollary 2 also applies.

4.4 Bidding Strategies Nonincreasing in Competition

So far we have assumed that equilibrium strategies are increasing in the number of bidders. This may not be always the case. For instance, affiliated private values may lead to equilibrium strategies that are decreasing in competition for some $\mathbf{F}(\cdot, \ldots, \cdot|\cdot)$. In this section, we discuss how our results extend when the equilibrium strategies are nonincreasing in competition. As before, let $I_1 < I_2$. We assume that for a bidder participating to both auctions his equilibrium strategies $s_1(\cdot)$ and $s_2(\cdot)$ intersect a finite number of times at most. This excludes the case where these strategies are identical on some open interval of private values. The nonparametric identification of the model is established through the following steps:

- Step 1: From the knowledge of $G_1(\cdot)$ and $G_2(\cdot)$, we can identify the positive values $0 < \alpha_1^* < \ldots < \alpha_K^* \leq 1$ at which the bid distributions and hence the bidder's equilibrium strategies $s_1(\cdot)$ and $s_2(\cdot)$ intersect, i.e. $b_1(\alpha_k^*) = b_2(\alpha_k^*)$.
- Step 2: Let $s_j(\cdot) < s_{j'}(\cdot)$ on $(\underline{v}, v(\alpha_1^*)), j, j' = 1, 2$. By Proposition 3, for any $\alpha_0 \in (0, \alpha_1^*)$, we can identify $\lambda^{-1}(u_0)$ as $\sum_{t=0}^{+\infty} |\Delta b(\alpha_t)|$, where $u_0 = R_j(\alpha_0)$. By continuity of $\lambda^{-1}(\cdot)$, we can also identify $\lambda^{-1}(R_j(\alpha_1^*))$ which is also equal to $\lambda^{-1}(R_{j'}(\alpha_1^*))$ by the compatibility condition (6). Hence, $\lambda^{-1}(\cdot)$ is identified on $[0, \max_{\alpha \in [0,\alpha_1^*]} R_j(\alpha)] = [0, \max\{\max_{\alpha \in [0,\alpha_1^*]} R_j(\alpha), \max_{\alpha \in [0,\alpha_1^*]} R_{j'}(\alpha)\}]$.
- Step 3: We have $s_{j'}(\cdot) < s_j(\cdot)$ on $(v(\alpha_1^*), v(\alpha_2^*))$. For any $\alpha_0 \in (\alpha_1^*, \alpha_2^*)$, we let $u_0 = R_{j'}(\alpha_0)$ and by Proposition 3 we construct recursively α_{t+1} from the equation $R_{j'}(\alpha_{t+1}) = R_j(\alpha_t)$ subject to $\alpha_{t+1} < \alpha_t$. There are two possibilities:

(i) If $\alpha_{t+1} \in [0, \alpha_1^*]$, we stop this sequence as we switch to values for which $s_j(\cdot) \le s_{j'}(\cdot)$. We have $\lambda^{-1}(u_0) = \lambda^{-1}(R_{j'}(\alpha_{t+1})) + \sum_{s=0}^{t} |\Delta b(\alpha_s)|$. But $\lambda^{-1}(R_j(\alpha_{t+1})) = \lambda^{-1}(R_{j'}(\alpha_{t+1})) + |\Delta b(\alpha_{t+1})|$, where $\lambda^{-1}(R_j(\alpha_{t+1}))$ is identified from Step 1 and hence equal to $\sum_{r=0}^{+\infty} |\Delta b(\alpha'_r)|$ for some decreasing sequence $\{\alpha'_r\}$ with $\alpha'_0 = \alpha_{t+1}$. Thus $\lambda^{-1}(u_0) = \sum_{r=1}^{+\infty} |\Delta b(\alpha'_r)| + \sum_{s=0}^{t} |\Delta b(\alpha_s)|$, which gives $\lambda^{-1}(u_0) = \sum_{t=0}^{+\infty} |\Delta b(\alpha_t)|$ by letting $\alpha'_r \equiv \alpha_{t+r}$. Figure 2 illustrates this case, where $j = 1$, $j' = 2$ and $t + 1 = 2$.

(ii) If α_{t+1} remains in (α_1^*, α_2^*) for all t, the sequence $\{\alpha_t\}$ will converge to α_1^*. Taking the limit gives $\lambda^{-1}(u_0) = \lambda^{-1}(R_{j'}(\alpha_1^*)) + \sum_{s=0}^{+\infty} |\Delta b(\alpha_s)|$. But $\lambda^{-1}(R_{j'}(\alpha_1^*)) = \lambda^{-1}(R_j(\alpha_1^*))$, where the latter is identified from Step 2. Figure 3 illustrates this case with $j = 1$ and $j' = 2$.

By continuity of $\lambda^{-1}(\cdot)$, we can also identify $\lambda^{-1}(R_{j'}(\alpha_2^*))$ which is also equal to $\lambda^{-1}(R_j(\alpha_2^*))$ by the compatibility condition (6). Hence, at the end of Step 3, $\lambda^{-1}(\cdot)$ is identified on $[\ \min\{\min_{\alpha\in[\alpha_1^*,\alpha_2^*]} R_j(\alpha), \min_{\alpha\in[\alpha_1^*,\alpha_2^*]} R_{j'}(\alpha)\}, \max\{\max_{\alpha\in[\alpha_1^*,\alpha_2^*]} R_j(\alpha), \max_{\alpha\in[\alpha_1^*,\alpha_2^*]} R_{j'}(\alpha)\}\ ]$. By combining Step 2 and Step 3, $\lambda^{-1}(\cdot)$ is identified on $[0, \max\{\max_{\alpha\in[0,\alpha_2^*]} R_j(\alpha), \max_{\alpha\in[0,\alpha_2^*]} R_{j'}(\alpha)\}]$.

- Step 4: For any $k \ge 2$ and $\alpha_0 \in (\alpha_k^*, \alpha_{k+1}^*)$, we repeat Step 3 and its two possibilities. If $\alpha_{t+1} \in [0, \alpha_k^*]$, we stop the sequence and we have $\lambda^{-1}(u_0) = \lambda^{-1}(R_{j'}(\alpha_{t+1})) + \sum_{s=0}^{t} |\Delta b(\alpha_s)|$, where $\lambda^{-1}(R_{j'}(\alpha_{t+1}))$ is identified from previous steps. Thus $\lambda^{-1}(u_0) = \sum_{t=0}^{+\infty} |\Delta b_{\alpha_t}|$ as in Step 3-(i). If α_{t+1} remains in $(\alpha_k^*, \alpha_{k+1}^*)$, Step 3-(ii) applies and $\lambda^{-1}(u_0)$ is identified. As before, by continuity, $\lambda^{-1}[R_j(\alpha_{k+1}^*)] = \lambda^{-1}[R_{j'}(\alpha_{k+1}^*)]$ is identified. Applying a similar argument, the combination of the various steps allows one to identify $\lambda^{-1}(\cdot)$ on $[0, \max\{\max_{\alpha\in[0,\alpha_{k+1}^*]} R_j(\alpha), \max_{\alpha\in[0,\alpha_{k+1}^*]} R_{j'}(\alpha)\}]$ and hence on $[0, \max\{\max_{\alpha\in[0,1]} R_j(\alpha), \max_{\alpha\in[0,1]} R_{j'}(\alpha)\}]$ when $k + 1 = K$.

The identification of the bidder's private value distribution $F(\cdot)$ follows as in Proposition 3. The above procedure shows that assuming bidding strategies increasing in competition is not necessary to identify nonparametrically the model under exclusion restrictions. The construction of the sequence $\{\alpha_t\}$ is more involved.

5 Conclusion

This paper addresses the problem of nonparametric identification of bidders' utility function(s). We show that the auction model with risk aversion is not identified and that it imposes weak restrictions on observables. This implies that the auction model with risk averse biddders is not testable in view of bids only. In view of these results, we exploit exclusion restrictions to identify nonparametrically the bidders' utility function and their private value distribution. The exclusion restrictions take the form of either an exogenous bidders' participation or the availability of instruments when bidders' participation is endogenous. The results are general as they extend to a reserve price, affiliated private values and asymmetric bidders. We also provide some conditions that must be verified by the data and can be used for model testing under exclusion restrictions. More generally, identifying risk aversion is an important issue in the analysis of microeconomic data. In this respect, it would be interesting to investigate how similar ideas can be exploited to identify nonparametrically agents' utility function in other economic contexts such as in insurance. Based on our results, d'Haultfoeuille and Fevrier (2007) apply a similar recursive construction of a quantile sequence to identify the principal's objective function, the agent's cost function and the type distribution under exclusion restrictions in a Principal-Agent model.

A nonparametric estimation method clearly needs to be developed. A first and natural strategy would be to rely on the construction of the sequence of quantiles $\{\alpha_t\}$ in Proposition 3. Specifically, we define $\hat{R}_j(\alpha)$ as the estimate of $R_j(\alpha)$, where $g_j(b_j(\alpha))$ is replaced by its estimate $\hat{g}_j(\hat{b}_j(\alpha))$ with $\hat{g}_j(\cdot)$ a nonparametric estimate of the bid density $g_j(\cdot)$ and $\hat{b}_j(\alpha)$ a nonparametric estimate of the bid α-quantile $b_j(\alpha)$. For any $u_0 \in \mathcal{R}_1 \backslash \{0\}$, we estimate $(\hat{\alpha}_0, \ldots, \hat{\alpha}_t)$ by solving recursively $\hat{R}_1(\alpha_0) = u_0$, $\hat{R}_1(\alpha_1) = \hat{R}_2(\hat{\alpha}_0)$,…, $\hat{R}_1(\hat{\alpha}_t) = \hat{R}_2(\hat{\alpha}_{t-1})$. An estimator of $\lambda^{-1}(u_0)$ is

$$\hat{\lambda}^{-1}(u_0) = \sum_{t=0}^{t_L} \Delta \hat{b}(\hat{\alpha}_t),$$

where t_L increases appropriately with the number L of auctions in the sample. Repeating this procedure for a large number of values u_0 and upon inversion of $\hat{\lambda}^{-1}(\cdot)$, an estimator of $U(\cdot)$ is

$$\hat{U}(x) = \exp\left(\int_1^x \frac{1}{\hat{\lambda}(t}dt\right),$$

for $x \in [0, \max_{u_0 \in \hat{\mathcal{R}}_1} \hat{\lambda}^{-1}(u_0)]$.[21] The asymptotic properties of such an estimator are yet unknown. The difficulty relies in the serial correlation of the sequence $\{\hat{\alpha}_t\}$ and in the accumulation of the errors in the estimation of $\hat{\lambda}^{-1}(u_0)$. The private value density $f(\cdot)$ can be estimated as in Guerre, Perrigne and Vuong (2000) using (4) with $\lambda^{-1}(\cdot)$ replaced by its estimate $\hat{\lambda}^{-1}(\cdot)$. The latter may then slow down the rate of convergence of $\hat{f}(\cdot)$.

Another possibility would be to exploit the compatibility conditions (6) in Lemma 3 to estimate nonparamerically $\lambda^{-1}(\cdot)$. Given some chosen quantiles $\alpha_1, \ldots, \alpha_{n_L}$ with n_L increasing appropriately with L, we estimate nonparametrically the quantiles $\hat{b}_j(\alpha_n)$ and the bid densities at those quantiles $\hat{g}_j(\hat{b}(\alpha_n))$ for $j = 1, 2$. We can then estimate $\lambda^{-1}(\cdot)$ by the method of sieves by minimizing the objective function

$$\sum_{n=1}^{n_L} \left[\hat{b}_2(\alpha_n) + \lambda^{-1}\left(\frac{1}{I_2 - 1}\frac{\alpha_n}{\hat{g}_2(\hat{b}_2(\alpha_n))}\right) - \hat{b}_1(\alpha_n) - \lambda^{-1}\left(\frac{1}{I_1 - 1}\frac{\alpha_n}{\hat{g}_1(\hat{b}_1(\alpha_n))}\right)\right]^2$$

subject to $\lambda^{-1}(\cdot) \in \Lambda_L^{-1}$, where Λ_L^{-1} is an appropriate set of increasing and smooth functions. See Chen (2007) for a survey on sieve estimation. Estimation of $\hat{f}(\cdot)$ is obtained as in the previous method. Establishing the asymptotic properties of such an estimation procedure is left for future research. In addition to having nonparametric estimates in the objective function, the main difficulty relies in controlling the increasing rate of the number n_L of prescribed quantiles.

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[21] Because estimation is performed on a finite number of values u_0, the integral should be approximated by a sum.

Appendix

This appendix contains the proofs of Lemmas 1–3, Propositions 1–3, the first part of Proposition 7 and Corollary 1.

Proof of Lemma 1: First, we prove that (i) and (ii) are necessary. Because $b_i = s(v_i, U, F, I)$ and the v_is are i.i.d., the b_is are also i.i.d. The fact that $G(\cdot|\cdot) \in \mathcal{G}_R$ follows from Theorem 1, Definitions 1-2 and (3). To prove that (ii) is also necessary, consider (4), where $\lambda(\cdot) \equiv U(\cdot)/U'(\cdot)$. Thus $\lambda(\cdot)$ is defined from $\mathbb{R}_+$ to $\mathbb{R}_+$ because $\lambda(0) = \lim_{x \downarrow 0} \lambda(x) = 0$, as noted after Definition 1. As $U(\cdot)$ admits $R+2$ continuous derivatives on $(0, +\infty)$ with $U'(\cdot) > 0$, and $\lim_{x \downarrow 0} \lambda^{(r)}$ is finite for $r = 1, \ldots, R+1$, then $\lambda(\cdot)$ has $R+1$ continuous derivatives on $[0, +\infty)$. As $\lambda'(\cdot) = 1 - \lambda(\cdot)U''(\cdot)/U'(\cdot)$, we have $\lambda'(\cdot) \geq 1$ because $\lambda(\cdot) \geq 0$, $U'(\cdot) > 0$ and $U''(\cdot) \leq 0$. It remains to show that $\xi'(\cdot) > 0$. The equilibrium strategy must solve the differential equation (2). As (4) follows from (2), $s(\cdot)$ must satisfy $\xi[s(v), U, G, I] = v$ for all $v \in [\underline{v}, \overline{v}]$. We then obtain $\xi(b, U, G, I) = s^{-1}(b, U, F, I)$. This implies $\xi'(\cdot) = [s^{-1}(\cdot)]' > 0$ using Theorem 1.

Second, we show that (i) and (ii) are together sufficient. First, we construct a pair $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$. Let $U(\cdot)$ be such that $\lambda(\cdot) = U(\cdot)/U'(\cdot)$ or $U'(\cdot)/U(\cdot) = 1/\lambda(\cdot)$. Integrating with the normalization $U(1) = 1$ gives $U(x) = \exp \int_1^x 1/\lambda(t)dt$. We verify that $U(\cdot) \in \mathcal{U}_R$. Because $\lambda(\cdot)$ admits $R+1$ continuous derivatives on $[0, +\infty)$, then Definition 1-(iii) is clearly satisfied. Moreover, in the neighborhood of zero, $\lambda(t) \sim \lambda'(0)t$ with $1 \leq \lambda'(0) < \infty$. Thus $\int_x^1 1/\lambda(t)dt$ diverges to infinity, which implies that $U(x)$ tends to zero as $x \downarrow 0$. Define $U(0) = 0$ so that $U(\cdot)$ is continuous on $[0, +\infty)$. Because $U'(x) = \exp \int_1^x 1/\lambda(t)dt/\lambda(x)$, where $\lambda(\cdot) > 0$ on $(0, +\infty)$, we have $U'(\cdot) > 0$ on $(0, +\infty)$. The second-order derivative gives $U''(x) = [-\lambda'(x) + 1] \exp \int_1^x 1/\lambda(t)dt/\lambda^2(x)$. Since $\lambda'(x) \geq 1$, $U''(\cdot) \leq 0$ on $(0, +\infty)$. It remains to show that $U(\cdot)$ admits $R+2$ continuous derivatives on $(0, +\infty)$. By assumption, $\lambda(\cdot)$ has $R+1$ continuous derivatives on $[0, +\infty)$. It follows that $U(\cdot)$ admits $R+2$ continuous derivatives on $(0, +\infty)$.

Let $F(\cdot|I)$ be the distribution of $X = b + \lambda^{-1}[G(b|I)/(I-1)g(b|I)]$ conditional on I, where $b \sim G(\cdot|I)$. We verify that $F(\cdot|\cdot) \in \mathcal{F}_R$. We have $F(x|I) = \Pr(X \leq x|I) = \Pr[\xi(b) \leq x|I] = \Pr[b \leq \xi^{-1}(x)|I] = G[\xi^{-1}(x)|I]$, because $\xi'(\cdot) > 0$ by assumption. This implies $F(\cdot|I) = G[\xi^{-1}(\cdot)|I]$ on $[\underline{v}(I), \overline{v}(I)]$, where $\underline{v}(I) \equiv \xi(\underline{b}(I)) = \underline{b}(I)$ and $\overline{v}(I) \equiv \xi(\overline{b}(I)) < \infty$ by continuity of $\xi(\cdot)$. Because $\xi(\cdot)$ and $G(\cdot|I)$ are strictly increasing, $F(\cdot|I)$ is strictly increasing on its support $[\underline{v}(I), \overline{v}(I)]$. Moreover, $\xi(\cdot)$ is $R+1$ continuously differentiable on $[\underline{b}(I), \overline{b}(I)]$. This follows from the definition of $\xi(\cdot)$, the $R+1$ continuous differentiability of $\lambda^{-1}(\cdot)$ on $[0, +\infty)$, and the $R+1$ continuous differentiability of $G(\cdot|I)/g(\cdot|I)$ on $[\underline{b}(I), \overline{b}(I)]$, which follows from Definition 3-(iv,v). Thus $F(\cdot|I) = G[\xi^{-1}(\cdot)|I]$ admits $R+1$ continuous derivatives on $[\underline{v}(I), \overline{v}(I)]$ because $G(\cdot|I)$ has

$R+1$ continuous derivatives on $[\underline{b}(I), \overline{b}(I)]$. It remains to show that $f(\cdot|I) > 0$ on $[\underline{v}(I), \overline{v}(I)]$. This follows from $f(\cdot|I) = g[\xi^{-1}(\cdot)|I]/\xi'[\xi^{-1}(\cdot)]$, where $g(\cdot|I) > 0$ from Definition 3 and $\xi'(\cdot)$ is finite on $[\underline{b}(I), \overline{b}(I)]$.

Lastly, we show that the pair $[U, F]$ rationalizes $G(\cdot|\cdot)$, i.e. that $G(\cdot|I) = F[s^{-1}(\cdot, U, F, I)|I]$ on $[\underline{b}(I), \overline{b}(I)]$, where $s(\cdot, U, F, I)$ solves (2) with the boundary condition $s(\underline{v}(I), U, F, I) = \underline{v}(I)$. By construction of $F(\cdot|I)$, $G(\cdot|I) = F[\xi(\cdot)|I]$. Thus, it suffices to show that $\xi^{-1}(\cdot)$ solves (2) with the boundary condition $\xi^{-1}(\underline{v}(I)) = \underline{v}(I)$. The boundary condition is straightforward as $\xi(\underline{b}(I)) = \underline{b}(I) = \underline{v}(I)$. By construction of $F(\cdot|I)$, $f(\cdot|I)/F(\cdot|I) = [\xi^{-1}(\cdot)]'g[\xi^{-1}(\cdot)|I]/G[\xi^{-1}(\cdot)|I]$. Thus $\xi^{-1}(\cdot)$ solves (2) if $1 = \{(I-1)g[\xi^{-1}(v)|I]\lambda[v - \xi^{-1}(v)]\}/G[\xi^{-1}(v)|I]$ for all $v \in [\underline{v}(I), \overline{v}(I)]$. Making the change of variable $v = \xi(b)$ and noting that $\xi(b) - b = \lambda^{-1}[G(b|I)/(I-1)g(b|I)]$ from the definition of $\xi(\cdot)$, it follows that $\xi^{-1}(\cdot)$ solves (2) with boundary condition $\xi^{-1}(\underline{v}(I)) = \underline{v}(I)$. $\square$

Proof of Proposition 1: In view of Lemma 1, it suffices to prove sufficiency. Specifically, it suffices to find a function $\lambda(\cdot)$ satisfying Lemma 1-(ii), where $G(\cdot|\cdot) \in \mathcal{G}_R$. Denote $G(b|I)/[(I-1)g(b|I)]$ by $\psi(b)$. Let $\min_{b \in [\underline{b}(I), \overline{b}(I)]} \psi'(b) = \underline{\psi}'$, which is finite from Definition 3. If $\underline{\psi}' > 0$, any strictly increasing function $\lambda(\cdot)$ will satisfy $\xi'(\cdot) \geq 0$, where $\xi(b) = b + \lambda^{-1}(\psi(b))$. It then suffices to take a $\lambda(\cdot)$ function that admits $R+1$ continuous derivatives on $[0, +\infty)$ with $\lambda(0) = 0$ and $\lambda'(\cdot) \geq 1$. If $\underline{\psi}' < 0$, we must find a strictly increasing and differentiable function $\lambda(\cdot)$ such that $\min_{b \in [\underline{b}(I), \overline{b}(I)]} \{(1/\lambda'[\lambda^{-1}(\psi(b))]) \times \psi'(b)\} > -1$ to satisfy $\xi'(\cdot) > 0$ on $[\underline{b}(I), \overline{b}(I)]$. To satisfy the latter inequality, it suffices that the function $\lambda(\cdot)$ satisfies $\lambda'(\cdot) \geq 1$ and $\underline{\psi}' \max_{b \in [\underline{b}(I), \overline{b}(I)]} 1/\lambda'[\lambda^{-1} (\psi(b))] > -1$, where the latter inequality is equivalent to $\underline{\psi}' > -1/[\max_{b \in [\underline{b}(I), \overline{b}(I)]} 1/\lambda'[\lambda^{-1}(\psi(b))]$. But $\max_{b \in [\underline{b}(I), \overline{b}(I)]} 1/\lambda'[\lambda^{-1}(\psi(b))] = \max_{x \in [0, \overline{x}]} 1/\lambda'(x)$ because $x \equiv \lambda^{-1}(\psi(b))$ takes its value between $\lambda^{-1}[\psi(\underline{b}(I))] = 0$ and $\overline{x} =\equiv \lambda^{-1}[\max_{b \in [\underline{b}(I), \overline{b}(I)]} \psi(b(I))] < +\infty$. Moreover, $\max_{x \in [0, \overline{x}]} 1/ \lambda'(x) = 1/\min_{x \in [0, \overline{x}]} \lambda'(x) \equiv 1/\underline{\lambda}'$, where $\underline{\lambda}' \geq 1$. Hence $\underline{\psi}' > -\underline{\lambda}'$, i.e. $0 < -\underline{\psi}' < \underline{\lambda}'$. Thus, $\lambda(\cdot)$ must have a sifficiently steep slope. To complete the proof, it suffices to take a $\lambda(\cdot)$ function that admits $R+1$ continuous derivatives on $[0, +\infty)$ with $\lambda(0) = 0$.$\square$

Proof of Proposition 2: Let $[U, F] \in \mathcal{U}_R \times \mathcal{F}_R$ with bid distribution $G(\cdot|\cdot) \in \mathcal{G}_R$ by Lemma 1. Let $[\tilde{U}, \tilde{F}]$ be such that $\tilde{U}(\cdot) = [U(\cdot/\delta)/U(1/\delta)]^\delta$, with $\delta \in (0,1)$ and $\tilde{F}(\cdot|I)$ be the conditional distribution given I of

$$\tilde{\xi}(b, \tilde{U}, G, I) = b + \tilde{\lambda}^{-1}\left(\frac{1}{I-1}\frac{G(b|I)}{g(b|I)}\right) = b + \delta\lambda^{-1}\left(\frac{1}{I-1}\frac{G(b|I)}{g(b|I)}\right) = (1-\delta)b + \delta\xi(b, U, G, I),$$

where $b \sim G(\cdot|I)$. It is easy to check that $[\tilde{U}, \tilde{F}] \in \mathcal{U}_R \times \mathcal{F}_R$. Because $\tilde{\xi}(\cdot)$ is the weighted sum of two strictly increasing functions in b, then $\tilde{\xi}(\cdot)$ is strictly increasing. Hence, from Lemma 1 the structures $[U, F]$ and $[\tilde{U}, \tilde{F}]$ are observationally equivalent, and $[U, F]$ is not identified. $\Box$

Proof of Lemma 2: (i) We first prove that $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$. We have $s_2(\underline{v}) = s_1(\underline{v}) = \underline{v}$. Moreover, from Theorem 1-(ii) we have

$$0 < s_j'(\underline{v}) = \frac{(I_j - 1)\lambda'(0)}{(I_j - 1)\lambda'(0) + 1} = 1 - \frac{1}{(I_j - 1)\lambda'(0) + 1} < 1 \tag{A.1}$$

where $\lambda'(0) \geq 1$. Thus $0 < s_1'(\underline{v}) < s_2'(\underline{v}) < 1$. In particular, by continuity of $s_j(\cdot)$ it follows that $\underline{v} < s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \epsilon)$ for some ϵ satisfying $\underline{v} < \epsilon \leq \overline{v}$. The proof is now by contradiction. Suppose that $s_1(v) \geq s_2(v)$ for some $v \in [\epsilon, \overline{v}]$. Because $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \epsilon)$, the continuity of $s_j(\cdot)$ would imply the existence of some $v_0 \in [\epsilon, \overline{v}]$ such that $s_1(v_0) = s_2(v_0)$. Moreover, for (at least) one of such v_0 denoted v_0^*, the strategy $s_1(\cdot)$ must intersect the strategy $s_2(\cdot)$ from below, i.e. $s_1'(v_o^*) \geq s_2'(v_o^*)$. From (2), we have

$$s_j'(v_0^*) = (I_j - 1)\frac{f(v_0^*)}{F(v_0^*)}\lambda\left(v_0^* - s_j(v_0^*)\right)$$

for $j = 1, 2$, where $f(v_0^*) > 0$ and $F(v_0^*) > 0$ since $v_0^* \in (\underline{v}, \overline{v}]$, while $\lambda(v_0^* - s_j(v_0^*)) > 0$ since $v_0^* > s_j(v_0^*)$ by Theorem 1-(i) and $\lambda(\cdot) > 0$ on $(0, +\infty)$. By construction $s_1(v_0^*) = s_2(v_0^*)$. The previous equation then implies $s_1'(v_0^*) < s_2'(v_0^*)$, contradicting $s_1'(v_0^*) \geq s_2'(v_0^*)$.

(ii) Next, we prove the first inequality, which implies the third inequality after immediate algebra. For each $j = 1, 2$, (2) gives

$$\frac{s_j'(v)}{I_j - 1} = \frac{f(v)}{F(v)}\lambda\left(v - s_j(v)\right) \tag{A.2}$$

for any $v \in [\underline{v}, \overline{v}]$. From (i), $v - s_2(v) < v - s_1(v)$ for any $v \in (\underline{v}, \overline{v}]$. Because $0 < v - s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$ by Theorem 1-(i), and $\lambda(\cdot)$ is strictly increasing with $\lambda(\cdot) > 0$ on $(0, +\infty)$, then $0 < \lambda(v - s_2(v)) < \lambda(v - s_1(v))$ for any $v \in (\underline{v}, \overline{v}]$. Hence, because $f(\cdot) > 0$ and $F(\cdot) > 0$ on $(\underline{v}, \overline{v}]$, it follows from (A.2) that

$$s_2'(v)/(I_2 - 1) < s_1'(v)/(I_1 - 1) \tag{A.3}$$

for any $v \in (\underline{v}, \overline{v}]$.[22] Integrating (A.3) from $\underline{v}$ to $v > \underline{v}$ and using $s_j(\underline{v}) = \underline{b}$ give

$$\frac{s_2(v) - \underline{b}}{I_2 - 1} < \frac{s_1(v) - \underline{b}}{I_1 - 1} \tag{A.4}$$

for any $v \in (\underline{v}, \overline{v}]$. The desired result follows after immediate algebra. $\Box$

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[22] Equation (A.1) shows that $s_2'(v)/(I_2 - 1) < s_1'(v)/(I_1 - 1)$ also holds at $v = \underline{v}$.

Proof of Corollary 1: The desired result holds when $I = \underline{I}$ and $I = \overline{I}$ by Lemma 2. Let $I \in (\underline{I}, \overline{I})$. With $I = I_1 < I_2$ in (5), the first inequality in (5) gives

$$(I-1)\frac{b_2(\alpha) - \underline{b}}{I_2 - 1} + \underline{b} < b_I(\alpha),$$

for any $\alpha \in (0,1]$ and any $I_2 > I$. Equation (A.4) applied to an arbitrary pair (I_2, I_2') with $I_2 < I_2'$ shows that the LHS in the above inequality is strictly decreasing in I_2. Hence, the most stringent inequality is obtained when I_2 is the smallest, i.e. when $I_2 = I + 1$. Similarly, with $I = I_2 > I_1$ in (5), the third inequality in (5) gives

$$b_I(\alpha) < (I-1)\frac{b_1(\alpha) - \underline{b}}{I_1 - 1} + \underline{b}$$

for any $\alpha \in (0,1]$ and any $I_1 < I$. The RHS in the above inequality is strictly decreasing in I_1 from (A.4). Hence, the most stringent inequality is obtained when I_1 is the largest, i.e. when $I_1 = I - 1$. Combining these two results gives

$$\frac{I-1}{I} b_{I+1}(\alpha) + \frac{1}{I}\underline{b} < b_I(\alpha) < \frac{I-1}{I-2} b_{I-1}(\alpha) - \frac{1}{I-2}\underline{b}, \tag{A.5}$$

for any $\alpha \in (0,1]$. On the other hand, the middle inequality of (5) gives

$$b_{I-1}(\alpha) < b_I(\alpha) < b_{I+1}(\alpha), \tag{A.6}$$

for any $\alpha \in (0,1]$ and any $I \in [\underline{I}, \overline{I}]$. The desired result follows by combining (A.5) and (A.6).

The second part of the corollary follows by noting that b_I is a strictly increasing function of v, namely $b_I = s_I(v)$ fo each I. Hence, the random variables $\max\{b_{I-1}, [(I-1)b_{I+1} + \underline{b}]/I\}$ and $\min\{b_{I+1}, [(I-1)b_{I-1} - \underline{b}]/(I-2)\}$ are also strictly increasing functions of v. It follows that the α-quantiles of their corresponding distributions $\underline{G}_I(\cdot)$ and $\overline{G}_I(\cdot)$ are equal to these functions evaluated at $v(\alpha)$. Thus, they are equal to the first term and third term of the two inequalities displayed in Corollary 1, respectively since $b_I(\alpha) = s_I[v(\alpha)]$. The stochastic dominance assertion then follows from these two inequalities. □

Proof of Proposition 3: From $b_j(\alpha) = s_j[v(\alpha)]$ and (4) evaluated at $v = v(\alpha)$, we obtain the crucial relation

$$v(\alpha) = b_j(\alpha) + \lambda^{-1}\left(\frac{1}{I_j - 1}\frac{\alpha}{g_j[b_j(\alpha)]}\right) \tag{A.7}$$

for $j = 1, 2$ and any $\alpha \in [0,1]$. Hence, using (6) we obtain the nonlinear relation

$$\lambda^{-1}[R_1(\alpha)] = \lambda^{-1}[R_2(\alpha)] + \Delta b(\alpha) \tag{A.8}$$

for any $\alpha \in [0,1]$. For future use, we note that $\Delta b(0) = 0$ as $s_1(\underline{v}) = s_2(\underline{v}) = \underline{b}$. Moreover, from Lemma 2, we know that $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$, which implies $b_1(\alpha) < b_2(\alpha)$ for any $\alpha \in (0,1]$. Hence, $\Delta b(\alpha) > 0$ for any $\alpha \in (0,1]$. Because $\lambda^{-1}(\cdot)$ is strictly increasing and $R_j(\alpha) > 0$ for any $\alpha \in (0,1]$, it follows from (A.8) that $R_1(\alpha) > R_2(\alpha) > 0$ for any $\alpha \in (0,1]$. In particular, because $R_j(\cdot)$ is continuous on $[0,1]$ and $R_j(0) = 0$ for $j = 1,2$, the range $\mathcal{R}_j$ of $R_j(\cdot)$ must be of the form $[0, \overline{r}_j]$ with $0 < \overline{r}_j < \infty$ and $\overline{r}_1 > \overline{r}_2$, as claimed in the text.

Now, by assumption u_0 belongs to $\mathcal{R}_1 = [0, \overline{r}_1]$. If $u_0 = 0$, then $\lambda^{-1}(0) = 0$. Next, consider the general case $u_0 \in (0, \overline{r}_1]$. Thus, there exists some $\alpha_0 \in (0,1]$ such that $u_0 = R_1(\alpha_0)$. In particular, we have $u_0 = R_1(\alpha_0) > R_2(\alpha_0) > 0 = R_1(0)$ because $R_1(\cdot) > R_2(\cdot) > 0$ on $(0,1]$. Moreover, because $R_1(\cdot)$ is continuous on $[0,1]$, there exists some α_1 satisfying $\alpha_0 > \alpha_1 > 0$ and $R_1(\alpha_1) = R_2(\alpha_0)$. Continuing such a construction, we have $R_1(\alpha_1) > R_2(\alpha_1) > 0 = R_1(0)$, which implies that there exists some α_2 satisfying $\alpha_1 > \alpha_2 > 0$ and $R_1(\alpha_2) = R_2(\alpha_1)$. Thus, we have constructed a sequence, which is not necessarily unique such that $1 \geq \alpha_0 > \alpha_1 > \ldots > \alpha_t > \ldots > 0$ with $u_0 = R_1(\alpha_0) > R_2(\alpha_0) = R_1(\alpha_1) > R_2(\alpha_1) = R_1(\alpha_2) > \ldots > R_2(\alpha_{t-1}) = R_1(\alpha_t) > \ldots > 0$, as indicated in the text.[23] Because the sequence $\{\alpha_t\}$ is strictly decreasing and is in $(0,1]$, it must converge to some finite limit $\alpha_\infty \in [0,1]$. Because $R_j(\cdot)$ is continuous on $[0,1]$, then $\lim_{t\to+\infty} R_j(\alpha_t) = R_j(\alpha_\infty)$ for $j = 1,2$. But $R_2(\alpha_{t-1}) = R_1(\alpha_t)$ by construction, implying that $R_2(\alpha_\infty) = R_1(\alpha_\infty)$. Because $R_2(\alpha) = R_1(\alpha)$ only for $\alpha = 0$, this implies that $\alpha_\infty = 0$, and consequently $\lim_{t\to+\infty} R_j(\alpha_t) = 0$ for $j = 1,2$.

We now iterate (A.8). Specifically, for any $u_0 \in \mathcal{R}_1 \setminus \{0\}$ and any corresponding sequence $\{\alpha_t\}$ as constructed above, we must have the nonlinear dynamic relation

$$\begin{aligned}
\lambda^{-1}(u_0) &= \lambda^{-1}[R_2(\alpha_0)] + \Delta b(\alpha_0) \\
&= \lambda^{-1}[R_1(\alpha_1)] + \Delta b(\alpha_0) \\
&= \lambda^{-1}[R_2(\alpha_1)] + \Delta b(\alpha_0) + \Delta b(\alpha_1) \\
&\vdots \\
&= \lambda^{-1}[R_2(\alpha_t)] + \Delta b(\alpha_0) + \ldots \Delta b(\alpha_t).
\end{aligned}$$

See Figure 1 for an illustration. Because $\lambda^{-1}(\cdot)$ is continuous on $[0,+\infty)$ with $\lambda^{-1}(0) = 0$ and $\lim_{t\to+\infty} R_2(\alpha_t) = 0$, as shown above, then $\lim_{t\to+\infty} \lambda^{-1}[R_2(\alpha_t)] = 0$. Because $\lambda^{-1}(u_0)$ is finite, it follows from the above equation that $\lim_{t\to+\infty} \sum_{\tau=0}^{t} \Delta b(\alpha_\tau)$ must exist and that it is equal

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[23] When $R_1(\cdot)$ is strictly increasing, or equivalently by (4) when the bidder's rent is strictly increasing in v, then $\alpha_0 = R_1^{-1}(u_0)$ and $\alpha_t = [R_1^{-1} \circ R_2]^t(\alpha_0)$, for $t = 1,2,\ldots$, where $\circ$ denotes the composition of two functions and $[R_1^{-1} \circ R_2]^t$ denotes the t-composition of $R_1^{-1} \circ R_2$. Thus the sequence $\{\alpha_t\}$ is unique.

to $\lambda^{-1}(u_0)$, i.e. $\lambda^{-1}(u_0) = \sum_{\tau=0}^{+\infty} \Delta b(\alpha_\tau)$ as desired. Note that this must be so irrespective of the sequence $\{\alpha_t\}$, whether such a sequence is unique. Moreover, because $\Delta b(\alpha_\tau)$ depends only on $b_j(\cdot)$ and $R_j(\cdot)$, which depend only on the distributions $G_j(\cdot)$, the latter equality shows that $\lambda^{-1}(\cdot)$ is identified nonparametrically on $\mathcal{R}_1$ from observed equilibrium bids.

The nonparametric identification of $F(\cdot)$ follows immediately from $F(\cdot) = G_j[\xi_j^{-1}(\cdot)]$. For, the nonparametric identification of $\lambda^{-1}(\cdot)$ on $\mathcal{R}_1$ and hence on $\mathcal{R}_2 \subset \mathcal{R}_1$ implies the nonparametric identification of $\xi_j(\cdot)$ on $[\underline{b}, \overline{b}_j]$ for $j = 1, 2$ by (4) and (6). The latter implies the nonparametric identification of $\xi_j^{-1}(\cdot) = s_j(\cdot)$ on $[\underline{v}, \overline{v}]$. Alternatively, pick an arbitrary $\alpha_0 \in [0, 1]$. From (6) and (A.7) for (say) $j = 1$, we have $v(\alpha_0) = b_1(\alpha_0) + \lambda^{-1}[R_1(\alpha_0)]$. Thus, the above explicit expression for $\lambda^{-1}(u_0)$ with $u_0 = R_1(\alpha_0)$ gives

$$v(\alpha_0) = b_1(\alpha_0) + \sum_{t=0}^{+\infty} \Delta b(\alpha_t) \tag{A.9}$$

showing that the α_0-quantile of $F(\cdot)$ is identified. Because α_0 is arbitrary in $[0, 1]$, it follows that $F(\cdot)$ is identified on $[\underline{v}, \overline{v}]$. □

Proof of Lemma 3: First, we prove that (i), (ii) and (iii) are necessary. We use a double index (i, j) with i indexing bidder i among the I_j bidders and $j = 1, 2$ indicating the level of competition. Because $b_{ij} = s_j(v_i, U, F, I_j)$ and the v_{ij}s are i.i.d., the b_{ij}s are also i.i.d. given $I_j, j = 1, 2$. The fact that $G_j(\cdot) \in \mathcal{G}_R, j = 1, 2$ follows from Lemma 1. This establishes (i). Because $s_1(v) < s_2(v)$ for any $v \in (\underline{v}, \overline{v}]$ from Lemma 2 and noting that $b_j = s_j(v)$ with $s_j(\cdot)$ strictly increasing, it follows that $b_j(\alpha) = s_j[v(\alpha)]$. Hence, $b_1(\alpha) < b_2(\alpha)$ for any $\alpha \in (0, 1]$ or equivalently $G_1(\cdot) \prec_{\underline{b}} G_2(\cdot)$. This establishes (ii). Lastly, because $\lambda(\cdot) = U(\cdot)/U'(\cdot)$ and $U(\cdot)$ satisfies Definition 1, then $\lambda(\cdot)$ is defined from $I\!R_+$ to $I\!R_+$ with $\lambda(0) = 0$, $\lambda'(\cdot) \geq 1$ and $\lambda(\cdot)$ is continuously differentiable on $[0, \infty)$. Because $F(\cdot)$ is invariant in I, its quantiles are also invariant in I. Thus, considering (4) for two values I_1 and I_2 at any α-quantile with $\alpha \in [0, 1]$ and $I_1 \neq I_2$ leads to (7). It remains to show that $\xi_j'(\cdot) > 0, j = 1, 2$. The equilibrium strategy $s_j(\cdot)$ must satisfy $\xi_j[s_j(v), U, G, I_j] = v$ for any $v \in [\underline{v}, \overline{v}]$ and $j = 1, 2$. We then obtain $\xi_j(b, U, G, I_j) = s_j^{-1}(b, U, F, I)$. This implies $\xi_j'(\cdot) = [s^{-1}(\cdot)]' > 0$. This establishes (iii).

Conversely, we show that (i), (ii) and (iii) are together sufficient. We construct a pair $[U, F]$ that satisfies Definitions 1 and 2 and is independent of I. Let $U(\cdot)$ be such that $\lambda(\cdot) = U(\cdot)/U'(\cdot)$ or $1/\lambda(\cdot) = U'(\cdot)/U(\cdot)$. Integration of the latter with the normalization $U(1) = 1$ gives $U(x) = \exp \int_1^x 1/\lambda(t)dt$. We need to verify that $U(\cdot)$ satisfies Definition 1. This follows from the proof of Lemma 1. Let $F_j(\cdot)$ be the distribution of $X_j = b + \lambda^{-1}[G_j(b)/((I_j - 1)g_j(b))]$ given I_j, where $b \sim G_j(\cdot), j = 1, 2$. Note that $F_j(\cdot)$ satisfies Definition 2 by the proof of Lemma

1. Moreover, because the compatibility condition is satisfied for any $\alpha \in [0,1]$, it implies that the corresponding α-quantile of $F_j(\cdot)$ does not depend on I_j. Hence $F_1(\cdot) = F_2(\cdot) \equiv F(\cdot)$, which thereby satisfies Definition 2. Lastly, we show that the pair $[U,F]$ can be rationalized by $G_1(\cdot)$ and $G_2(\cdot)$ with $I_2 > I_1$, i.e. that $G_j(\cdot) = F[s_j^{-1}(\cdot, U, F, I_j)], j = 1,2$, where $s_j(\cdot, U, F, I_j)$ solves the first-order differential equation defining the equilibrium strategy with the boundary condition $s_j(\underline{v}, U, F, I_j) = \underline{v}$. By construction, $G_j(\cdot) = F[\xi_j(\cdot)]$. Thus it suffices to show that $\xi_j^{-1}(\cdot), j = 1,2$ solves the differential equation (2). This proof can be found in Lemma 1, which shows that $\xi_j^{-1}(\cdot), j =, 1, 2$ solves the differential equation with $I = I_j$ under the boundary condition $\xi_j^{-1}(\underline{v}) = \underline{v}$. □

Proof of Proposition 7: We prove the first part only. Let $[\mathbf{U}, F] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$. This structure generates $\mathbf{G}(\cdot, \ldots, \cdot|\cdot) \in \mathcal{G}_R$ whose marginal distributions satisfy Definition 3 and the compatibility condition (13). We show that there exists another structure $[\tilde{\mathbf{U}}, \tilde{F}] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$ rationalizing $\mathbf{G}(\cdot, \ldots, \cdot|\cdot)$. The proof is in four steps and is done for every fixed $I \in \mathcal{I}$.

Step 1: *Construction of* $[\tilde{U}_1, \ldots, \tilde{U}_I, \tilde{F}(\cdot|I)]$. Let $\tilde{U}_1(\cdot) = [U_1(\cdot/\delta)/U_1(1/\delta)]^\delta$ with $\delta \in (0,1)$. Thus, $\tilde{\lambda}_1(\cdot) = \lambda_1(\cdot/\delta)$ and $\tilde{\lambda}_1^{-1}(\cdot) = \delta\lambda_1^{-1}(\cdot)$. For $i = 2, \ldots, I$, let $\tilde{U}_i(x) = \exp\left[\int_1^x 1/\tilde{\lambda}_i(t)dt\right]$ so that $\tilde{\lambda}_i(\cdot) = \tilde{U}_i(\cdot)/\tilde{U}_i'(\cdot)$, where $\tilde{\lambda}_i(\cdot)$ is such that $\tilde{\lambda}_i^{-1}\left[1/H_i(b_{i\alpha}|I)\right] = \tilde{\lambda}_1^{-1}\left[1/H_1(b_{1\alpha}|I)\right] + b_{1\alpha} - b_{i\alpha}$, for all $\alpha \in [0,1]$. The latter well-defines $\tilde{\lambda}_i^{-1}(\cdot)$ because $1/H_i(b_{i\alpha}|I)$ strictly increases as α increases given $H_i'(\cdot|I) < 0$ by assumption. Moreover, $\tilde{\lambda}_i(\cdot)$ is strictly increasing as shown in Step 3. Note that the compatibility condition (13) is satisfied by construction. We then let $\tilde{F}(\cdot|I)$ be the conditional distribution given I of $\tilde{v}_i \equiv b_i + \tilde{\lambda}_i^{-1}[1/H_i(b_i|I)] \equiv \tilde{\xi}_i(b_i)$ for an arbitrary i, where $b_i \sim G_i(\cdot|I)$. Using $\tilde{\lambda}_1^{-1}(\cdot) = \delta\lambda_1^{-1}(\cdot)$, we obtain $\tilde{\lambda}_i^{-1}\left[1/H_i(b_{i\alpha}|I)\right] = \delta\lambda_1^{-1}\left[1/H_1(b_{1\alpha}|I)\right] + b_{1\alpha} - b_{i\alpha}$. Thus, (13) with $j = 1$ gives

$$\tilde{\lambda}_i^{-1}\left(\frac{1}{H_i(b_{i\alpha}|I)}\right) = \delta\lambda_i^{-1}\left(\frac{1}{H_i(b_{i\alpha}|I)}\right) + (1-\delta)(b_{1\alpha} - b_{i\alpha}). \tag{A.10}$$

Equivalently, $\tilde{\lambda}_i^{-1}[1/H_i(b_{i\alpha}|I)] = \lambda_i^{-1}[1/H_i(b_{i\alpha}|I)] - (1-\delta)\lambda_1^{-1}[1/H_1(b_{1\alpha}|I)]$. In particular, since $\lambda_i^{-1}(\cdot)$ is bidder's i shading, the shading under $[\tilde{U}_1, \ldots, \tilde{U}_I, \tilde{F}]$ is smaller than under $[U_1, \ldots, U_I, F]$, i.e. bidders bid more aggressively under the former than under the latter.

Step 2: $\tilde{\lambda}_i(0) = 0$ *and* $\tilde{\xi}_i'(\cdot) > 0$ *on* $[\underline{b}, \overline{b}]$. Because $[\mathbf{U}, F] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$ so that $\mathbf{G}(\cdot, \ldots, \cdot|\cdot) \in \mathcal{G}_R$, we have $\lambda_i^{-1}(0) = 0$ and $\lim_{b \downarrow \underline{b}} 1/H_i(b|I) = 0$ for $I \in \mathcal{I}$. Thus, (A.10) with the boundary conditions $\underline{b}_1 = \ldots = \underline{b}_I \equiv \underline{b} = \underline{v}$ imply $\tilde{\lambda}_i^{-1}(0) = 0$ and hence $\tilde{\lambda}_i(0) = 0$. Regarding $\tilde{\xi}_i'(\cdot) > 0$, we note that $\tilde{\xi}_i(b_{i\alpha}) = (1-\delta)b_{1\alpha} + \delta\xi_i(b_{i\alpha})$ from (A.10) and (12). Noting that $b_{1\alpha} = G_1^{-1}[G_i(b_{i\alpha})] \equiv B_i(b_{i\alpha})$ and letting $b_{i\alpha} = b$, we obtain $\tilde{\xi}_i'(b) = (1-\delta)B_i'(b) + \delta\xi_i'(b)$, where $B_i'(b) = g_i(b)/g_1[B(b)]$. Hence, $\tilde{\xi}_i'(b) > 0$ since $B_i'(b) > 0$ and $\xi_i'(b) > 0$.

Step 3: $\tilde{\lambda}_i'(\cdot) \geq 1$. From (A.10) and (12), $\tilde{\lambda}_i^{-1}[1/H_i(b_{i\alpha}|I)] = \delta\xi_i(b_{i\alpha}) + (1-\delta)b_{1\alpha} - b_{i\alpha}$, i.e. $1/H_i(b_{i\alpha}|I) = \tilde{\lambda}_i[\delta\xi_i(b_{i\alpha}) + (1-\delta)b_{1\alpha} - b_{i\alpha}]$. From the structure $[\mathbf{U}, F]$, we have $1/H_i(b_{i\alpha}|I) = \lambda_i[\xi_i(b_{i\alpha}) - b_{i\alpha}]$. Thus, $\lambda_i[\xi_i(b_{i\alpha}) - b_{i\alpha}] = \tilde{\lambda}_i[\delta\xi_i(b_{i\alpha}) + (1-\delta)b_{1\alpha} - b_{i\alpha}]$. Differentiating with respect to $b = b_{i\alpha}$ and noting that $b_{1\alpha} = G_1^{-1}[G_i(b_{i\alpha})] \equiv B_i(b_{i\alpha})$ gives

$$\tilde{\lambda}_i'(**) = \frac{\xi_i'(b) - 1}{\delta\xi_i'(b) + (1-\delta)B_i'(b) - 1}\lambda_i'(*) \equiv R_i(b)\lambda_i'(*), \tag{A.11}$$

where the different arguments of $\lambda_i'(\cdot)$ and $\tilde{\lambda}_i'(\cdot)$ are indicated by * and **, respectively. Thus, it suffices to show that $R_i(\cdot) \geq 1$ since $\lambda_i'(\cdot) \geq 1$. We note that $\xi_1(b_{1\alpha}) = \xi_i(b_{i\alpha}) = v_\alpha$ for all $\alpha \in [0,1]$ from the compatibility condition. Using $b_{1\alpha} = B_i(b_{i\alpha})$, this gives $\xi_1[B_i(b)] = \xi_i(b)$ for all $b \in [\underline{b}, \overline{b}]$. Differentiating gives $\xi_1'[B_i(b)]B_i'(b) = \xi_i'(b)$, i.e. $B_i'(b) = \xi_i'(b)/\xi_1'[B_i(b)]$. Hence,

$$R_i(b) = \frac{\xi_i'(b) - 1}{\delta\xi_i'(b) - 1 + (1-\delta)\frac{\xi_i'(b)}{\xi_1'[B_i(b)]}} = 1 + \frac{(1-\delta)\xi_i'(b)\{\xi_1'[B_i(b)] - 1\}}{\delta\xi_i'(b)\{\xi_1'[B_i(b)] - 1\} - \{\xi_1'[B_i(b)] - \xi_i'(b)\}}, \tag{A.12}$$

for $b \in [\underline{b}, \overline{b}]$. Note that $\xi_i'(\cdot) > 1$ on $(\underline{b}, \overline{b}]$ for every $i = 1, \ldots, I$ since differentiating (12) gives $\xi_i'(b) = 1 - \lambda_i^{-1'}[1/H_i(b|I)][H_i'(b|I)/H_i^2(b|I)]$, where $\lambda_i^{-1'}(\cdot) > 0$ and $H_i'(\cdot|I) < 0$ by assumption. Hence, $\xi_i'(\cdot) \geq 1$ on $[\underline{b}, \overline{b}]$ by continuity. Since $1 - \delta > 0$ and $\xi_i'(\cdot) > 0$, it suffices to show that the denominator $D_i(b)$ (say) in the RHS is strictly positive for all $b \in [\underline{b}, \overline{b}]$ and some $\delta \in [\delta^*, 1]$.

To study the sign of $D_i(\cdot)$ on $[\underline{b}, \overline{b}]$, we note that

$$\frac{g_j(b)}{G_j(b)} = \frac{g_j(\underline{b}) + o(1)}{g_j(\underline{b})(b - \underline{b}) + o(b - \underline{b})} = \frac{1}{b - \underline{b}}\frac{g_j(\underline{b}) + o(1)}{g_j(\underline{b}) + o(1)} = \frac{1}{b - \underline{b}}(1 + o(1)).$$

Thus, a Taylor expansion of $1 = \lambda_i[\xi_i(b) - b]\sum_{j \neq i}[g_j(b)/G_j(b)]$ from (12) gives

$$1 = \{\lambda_i'(0)[\xi_i'(\underline{b}) - 1](b - \underline{b}) + o(b - \underline{b})\}\frac{I - 1}{b - \underline{b}}(1 + o(1)) = \{\lambda_i'(0)[\xi_i'(\underline{b}) - 1](I - 1)\} + o(1).$$

Hence, $\xi_i'(\underline{b}) = 1 + \{1/[(I-1)\lambda_i'(0)]\} > 1$ as $\lambda_i'(\cdot) \geq 1$. Thus, because $\xi_i'(\cdot) > 1$ on $[\underline{b}, \overline{b}]$, then $D_i(\cdot) > 0$ on $[\underline{b}, \overline{b}]$ if and only if $\delta > \delta^* \equiv \max_{b \in [\underline{b}, \overline{b}]} \underline{\delta}(b)$, where $\underline{\delta}(\cdot)$ is continuous on $[\underline{b}, \overline{b}]$ with

$$\underline{\delta}(b) \equiv \frac{\xi_1'[B_i(b)] - \xi_i'(b)}{\xi_i'(b)\{\xi_1'[B_i(b)] - 1\}} = \frac{1}{\xi_i'(b)}\left[1 - \frac{\xi_i'(b) - 1}{\xi_1'[B_i(b)] - 1}\right].$$

It remains to show that $\underline{\delta}(\cdot) < 1$ on $[\underline{b}, \overline{b}]$ so that $\delta^* < 1$. Clearly, $\underline{\delta}(\cdot) < 1$ on $(\underline{b}, \overline{b}]$ as $\xi_i'(\cdot) > 1$ on $(\underline{b}, \overline{b}]$. Moreover, at $b = \underline{b}$, we have $\underline{\delta}(\underline{b}) = [1/\xi_i'(\underline{b})]\{1 - [\lambda_i'(0)/\lambda_1'(0)]\} < 1$.

Step 4: $[\tilde{\mathbf{U}}, \tilde{F}] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$. From the previous steps and the rationalization result given after (13), it follows that $[\tilde{\mathbf{U}}, F]$ rationalizes $\mathbf{G}(\cdot, \ldots, \cdot|\cdot)$. It remains to show that $[\tilde{\mathbf{U}}, \tilde{F}] \in \mathcal{U}_R^{\mathcal{I}} \times \mathcal{F}_R$. From the proof of Lemma 1, it sufffices to show that $\tilde{\lambda}_i(\cdot)$ is $R+1$ continuously differentiable on $[0, \infty)$ for $i = 1, \ldots, I$. This follows from (A.11)–(A.12) and the $R+1$ continuous differentiablity of $\lambda_i(\cdot)$ and $\xi_i(\cdot)$ as $\mathbf{G}(\cdot, \ldots, \cdot|\cdot) \in \mathcal{G}_R$. $\square$

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Figure 1: Identification with $s(\cdot)$ increasing in competition

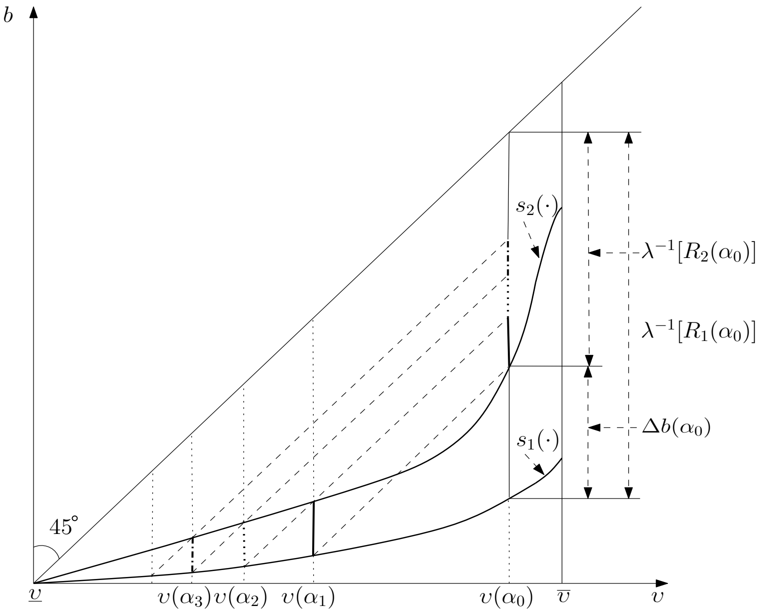

Figure 2: Identification with $s(\cdot)$ nonincreasing in competition, case (i)

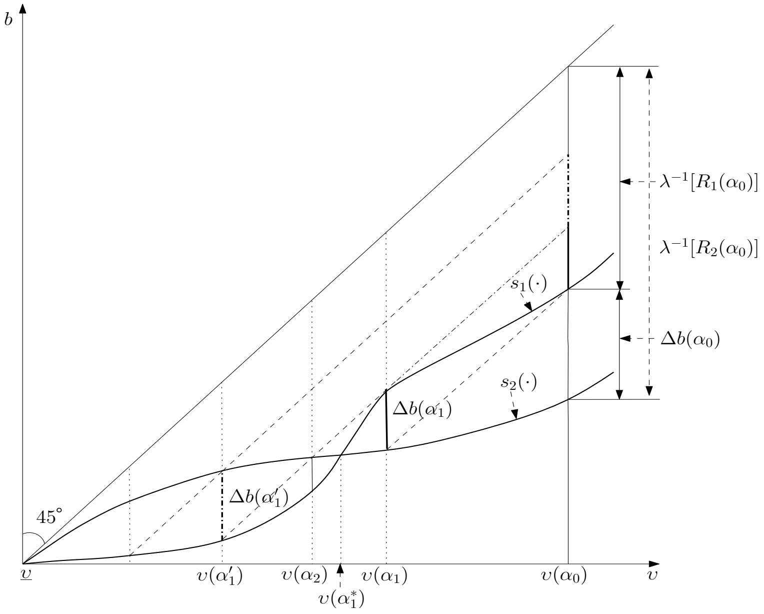

Figure 3: Identification with $s(\cdot)$ nonincreasing in competition, case (ii)