# Making Efficient Public Good Decisions using an Augmented Ausubel Auction.

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**Abstract**

We provide the appropriate generalization of Ausubel's 2004 ascending bid auction to environments where the goods are non-rival and non-excludable. Like its private good counterpart, we show that the public good Ausubel auction encourages truthful revelation of preferences, is privacy preserving, and yields an equilibrium allocation that is outcome equivalent to the public good Vickrey sealed-bid auction. We then generalize this auction to fit a broader set of public good environments.

## 1 Introduction

When a good is *rivalrous* in consumption and *excludable* it is called a private good. On the other hand, if the good is *non-rivalrous* in consumption and *non-excludable* we call it a public good. A simple example of a public good is the amount of insecticide sprayed in a neighborhood. All people in the neighborhood enjoy the benefits of the spray, and it is not possible to exclude one neighbor from enjoying the benefits of the spray because he did not help pay for it. This and similar public good environments are plagued by the "free riding problem" – the incentive to enjoy the public good's benefits while not sharing in its cost. Due to the possibility to free ride, it is not clear how an institution could be arranged to make efficient allocation decisions. One possibility for making such a decision might be some sort of elicitation of agents' preferences and using that elicited information appropriately, but unless this process is carefully constructed, a designer can not be sure of having elicited the correct preferences from the agents. In other words, there may incentive for different economic agents to misrepresent their preferences. The nature of this problem suggests that mechanism design theory should be well suited for finding an institution to overcome the strategic difficulties of this preference revelation problem. In fact, there is already an extensive literature in mechanism design trying find ways of aligning individual incentives with those of the group in both private and public good settings. In this paper we look to recent contributions in dynamic auctions and experimental economics to give us insight into a new, and hopefully improved, way of overcoming the preference revelation problem in a public good setting.

It is well known that that the earliest successin preference revelation mechanisms for a private good setting came from the path breaking auctions paper of Vickrey (1961). Among the results in the paper, Vickrey established a multi-unit sealed bid auction for efficiently allocating multiple units of a homogeneous good to the bidders. Moreover, this auction was particularly nice because it encouraged truthful revelation of preferences as a dominant strategy and is relatively easy to study analytically. His mechanism was later, and independently, generalized by Clarke (1971) and Groves (1973) and is now commonly called the VCG mechanism. Loeb (1977) surveyed these three papers as well as Tideman and Tullock (1975) and Groves and Loeb (1976) and, more importantly, demonstrated how a multi-unit sealed bid Vickrey auction can be re-defined to make efficient decisions in public good settings.[1] Unfortunately, despite the nice properties of the VCG mechanism, Rothkompf (2008) and others, have pointed out that VCG mechanisms are far from being practical. He lists thirteen different arguments for why they may be difficult to implement. Among these reasons are that the VCG mechanisms make computations difficult for bidders and do not preserve the privacy of the bidders. For example, when bidding on $K$ units (public or private) in an auction, each bidder is required to submit $K$ bids, which, in equilibrium, is the true valuation schedule of that bidder for each of the units. Rothkompf argues that in a private good setting this lack of privacy allows for dishonest practices by the seller (i.e., incentive for introducing a 'shill' bidder to increase prices) and that buyers typically feel uncomfortable revealing their whole schedule of values.

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[1] While this class of mechanisms (often referred to as Vickrey-Clarke-Groves (VCG) mechanisms) has been well studied, in this paper, we focus on the specific formulation of the public good Vickrey mechanism as defined by Loeb.

One potential solution to this privacy preservation critique is to look for a dynamic auction that maintains some of the nice properties of the Vickrey auction– i.e., truth telling being an equilibrium. The advantage of looking at a dynamic auction is that the auction typically stops before bidders can reveal their whole valuation schedule. In this spirit, Ausubel (2004) introduced a dynamic auction procedure that retains the nice properties of the private good sealed-bid Vickrey auction (and is in fact outcome equivalent to the private good Vickrey auction) while, in general, preserving the privacy of some of the bidders. The privacy preservation concern also extends to situations involving public goods, where consumers may be hesitant to reveal their full valuation schedule to a central authority. An additional reason why we might want to look at a dynamic auction for a public good setting is that it may be more "behaviorally" successful than its sealed bid counterpart. In this regard, the private good Ausubel auction has been quite successful in a laboratory environment. Kagel and Levine (2001) find that bidders in the private good Ausubel auction do, in general, bid truthfully when compared with bidders in uniform price sealed bid auctions. In a follow up paper, Kagel, Kinross, and Levin (2001) compare the private good, multi-unit Vickrey auction against the private good Ausubel auction with different information feedback treatments. They find that the Ausubel auction with information feedback (i.e., when bidders drop out) outperforms the sealed bid multi-unit Vickrey auction in a laboratory environment. The authors attribute this success to the observation that subjects in the Ausubel auction seem to benefit from the relative transparency of mechanism. Thus, the privacy preservation, the relative transparency, and the behavioral success are all appealing reasons why we might want to translate a similar auction to be applied in a public good setting.

In this paper, we show that: first, Ausubel's 2004 auction can be naturally re-defined to fit a public good context such that truthful revelation of valuations, depending on the information feedback conditions, is a very robust refinement of a Nash equilibrium; second, the equilibrium of the induced auction and yields an outcome equivalent to the public good sealed bid Vickrey auction; third, through an example we show that, in general, the public good Ausubel auction preserves the valuation privacy of all of the participants; finally, we generalize the public good Ausubel auction to include so called "public bads" and compare this outcome to outcomes yielded by a family of Groves mechanisms

# 2 The Public Good Economy

The setting consists of $N$ bidders that participate in an auction to determine the production level of some good $x$ that is both non-rival and non-excludable. This good can be produced at a constant marginal cost of $c \in \mathbb{R}_+$ and there is a maximum production level of $\bar{x}$. The set of all production levels is therefore $X = \{0, 1, ..., \bar{x}\}$, where for each *positive* unit $x$, there is some *positive* reservation price $r_{ix} \geq 0$ that indexes consumer $i$'s maximum willingness to pay for that unit. Reservation prices for each consumer $i$ are weakly decreasing in the public good, i.e., $r_{i1} \geq r_{i2} \geq \cdots \geq r_{i\bar{x}}$ and where $r_{i\bar{x}} < c$. We denote an arbitrary profile of $\bar{x}$ reservation prices for consumer $i$ by $r_i = (r_{i1}, ..., r_{i\bar{x}})$. The set of all such profiles for bidder $i$ is denoted $R_i$, and $R = \times_{i=1}^{N} R_i$ is the set of all possible bidder reservation profiles. Types are drawn before the auction begins (i.e., at stage 0) according to the pdf $f^i(\cdot)$, where each player observes only his own type profile. The the joint pdf is denoted by $f$.

Consumers care only about the level of the public good $x \in X$ produced and the amount of money $\tau^i \in \mathbb{R}_+$they have to pay toward production. We assume these preferences can be represented by a quasi-linear payoff function $u^i : X \times\mathbb{R}_+ \times R \rightarrow \mathbb{R}$ which is equal to his private value, $v^i(x)$, for the quantity $x$ of the public good produced less the total payment $\tau^i$ – i.e., $u^i(x, \tau^i, r_i) = v^i(x) - \tau^i$. The value $v^i(x)$ is assumed to be equal to $i$'s total surplus,

$$v^i(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \sum_{j=1}^{x} r_{ij} & \text{otherwise} \end{cases} .$$

An allocation determines the level of the public good produced and a total payment for each consumer, i.e., $(x, \tau^1, ..., \tau^N) \in X \times \mathbb{R}_+^N$. For this paper we are interested in allocations with desirable social welfare properties such as Pareto optimality and efficiency. The following definitions formalize these two concepts.

**Definition 1** *A public good allocation is **Pareto optimal** if: the amount of the public good maximizes social surplus (i.e., $\hat{x} \in \arg\max_{x \in X} [\sum_{i=1}^{N} v^i(x) - cx]$); and the payments collected from the consumers add up to the total cost of producing $\hat{x}$ (i.e., $\sum_{i=1}^{N} \hat{\tau}^i = c\hat{x}$).*

**Definition 2** *The public good decision $x$ is **efficient** if and only if it maximizes social surplus.*

In the next section, we analyze to the public good counterparts to the Vickrey and Ausubel auctions.

# 3 The Public Good Vickrey Auction

Loeb (1977) is the first paper to explicitly take a sealed bid private good auction and convert it to a public good environment. Specifically, he showed that it is straightforward to adapt the multi-unit private good Vickrey auction into a public good setting.

## 3.1 Rules of the Public Good Vickrey Mechanism:

A sealed bid auction in a public good context needs to specify an allocation $(x, \tau_1, ..., \tau_N)$ to distribute to all of the consumers. In this section, we outline how the public good Vickrey auction takes bids submitted by consumers and maps them into an allocation.

For each element $x \in X$, each consumer $i$ sends a bid $b_{ix}$ to the government. The total report is the profile $b_i = (b_{i1}, ..., b_{i\bar{x}})$. These bids can be viewed as the consumer's reported maximum willingness to pay of each of the possible units of the public good. The government takes these bids from each consumer and constructs reported valuation functions $\tilde{v}^i : X \to \mathbb{R}$, where

$$\tilde{v}^i(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \sum_{j=1}^{x} b_{ij} & \text{otherwise} \end{cases}.$$

In an analogous manner, the government can calculate reported residual supply $\hat{s}^{-i} : X \to \mathbb{R}$, where for each $x \in X$, $\hat{s}^{-i}(x) = \max\{0, c - \sum_{j \neq i} b_{jx}\}$. Using the reported valuation and the reported residual supply functions the government calculates the production quantity $x^*$ and each consumer's payment $\tau^i$ (i.e. the allocation received by each consumer). Specifically, $x^*$ is chosen to maximize reported social benefit – i.e., chooses

$$x^* \in \arg\max_{x \in X}[\sum_i \tilde{v}^i(x) - cx].$$

Each consumer $i$ is charged a tax equal to the reported residual supply for each positive unit that is produced and zero if no units are produced–i.e.,

$$\tau^i(x^*) = \begin{cases} 0 & \text{if } x^* = 0 \\ \sum_{j=1}^{x^*} \hat{s}^{-i}(j) & \text{otherwise} \end{cases}.$$

Loeb (1977) has demonstrated that each consumer has a weakly dominant strategy to bid truthfully in this auction. As this result is relatively well known we refer the interested reader to Loeb for the proof of this theorem. However, in order to provide some intuition for some later results, it is useful to llustrate the workings of the public good Vickrey Auction through the following example.

**Example 1** *Suppose there are* 3 *consumers in the economy and the government is trying to decide how much of public good to produce at a marginal cost of* \$10 *per unit. Further suppose that each of the consumers value the public good according to the following benefit schedule*

| | *Unit* 1 | *Unit* 2 | *Unit* 3 |
|---|---|---|---|
| *Consumer A* | 14 | 4 | 2 |
| *Consumer B* | 4 | 3 | 2 |
| *Consumer C* | 3 | 2 | 1 |

*The government decides to make its production decision by using the Vickrey Auction.*

The production for a unit of the public good is efficient if and only if the sum of the individual benefits for that unit exceeds marginal cost. In the example, the only efficient quantity to produce is one. However, without knowledge of the individual benefit functions it is not clear how the authority could figure this out. He could try surveying the consumers, but it is well known that consumers in these types of problems have incentives to misrepresent their preferences. Given that the government uses a Vickrey mechanism to make their production decision, Consumers A, B, and C will have a dominant strategy to submit bid vectors of $b_A = (14, 4, 2)$, $b_B = (4, 3, 2)$, and $b_C = (3, 2, 1)$ respectively. Adding the bids up vertically we have 21 for the first unit, 9 for the second unit, and 6 for the third unit. The public good Vickrey auction therefore specifies that one unit be produced.

As for the Vickrey Tax, consider Consumer A whose reported residual supply vector is equal to

$$(s^{-A}(1), s^{-A}(2), s^{-A}(3)) = (3, 5, 7).$$

Since $b_{A1} > s^{-A}(1)$, $b_{A2} < s^{-A}(2)$, and $b_{A3} < s^{-A}(3)$, Consumer $A$'s Vickrey Tax is equal to $s^{-A}(1) = 3$ (or the shaded area under his residual supply).

Consumer $A$'s reported demand and residual supply are illustrated in figure 1. In a similar manner, we calculate that both consumer's B and C pay a Vickrey Tax of zero. The total tax revenue is 3. Thus, in this example, the public good Vickrey mechanism runs a budget deficit.

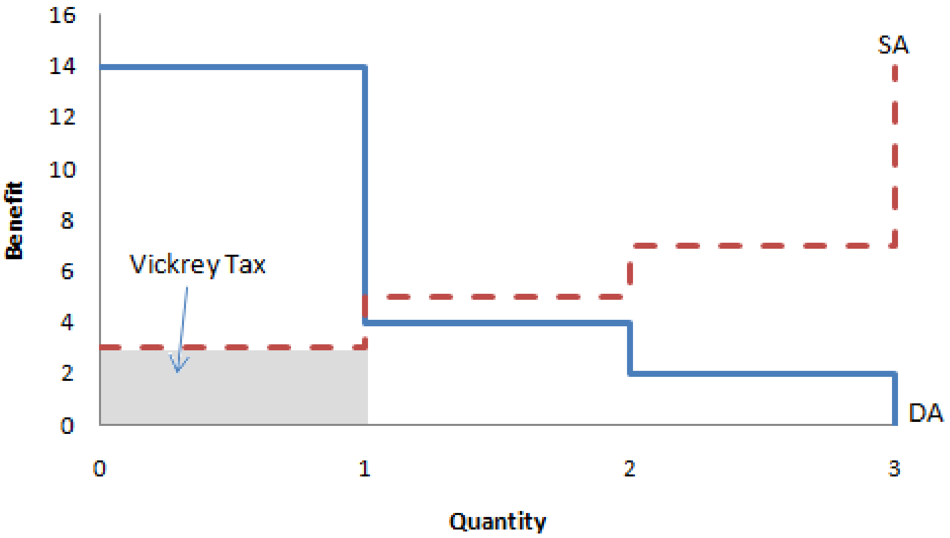

Figure 1: Revealed Bids and Tax for a Public Good Vickrey Auction

## 4 The Public Good Ausubel Auction

The Ausubel auction in the private good case is outcome equivalent to the multi-unit Vickrey auction. We now investigate whether the private good Ausubel auction can be redefined to fit the public good case and, if this is the case, whether or not the new auction remains outcome equivalent to the public good Vickrey.

Recall that in the private good Ausubel setting, all bidders must take the auctioneer's price as given and respond with potentially different quantity bids. From the dual nature of the problem, it seems natural to re-define the private good Ausubel auction by using an ascending 'quantity' auction instead of an ascending 'price' auction. Specifically, having an auctioneer start by calling out a low quantity and have individuals respond by submitting bids. If the bids exceed the marginal cost, the auction continues and the auctioneer increases the quantity. This process would continue until there is no longer “excess demand” for the public good. The trick to making truth telling an equilibrium in the auction is to ‘disentangle what the bidders say and what they pay.’ A variation of Ausubel’s clinching rule is a natural candidate.

## 4.1 Rules of the Ascending Quantity Ausubel Auction:

In this section, we outline the process in which the public good Ausubel auction takes consumers’ bids in each stage of the auction and converts them into an allocation $(x, \tau_1, ..., \tau_N)$.

The auction begins at round 1 where the initial level of the public good is set to be 1. At each round $t = 1, ..., \bar{x}$, the level of the public good announced is $x_t = t$ (i.e., the level of the public good increases in increments of 1 unit per round) and each consumer $i$ is required to submit a bid $b_{it}$ to the auctioneer. If $\sum_i b_{it} \geq c$, the auction continues $(x_{t+1} = t + 1)$, and stops otherwise. Denote the round the auction stops by $L$. The quantity produced at the end of the auction is

$$x^* = L - 1.$$

Bidders do not pay their bids, but rather some function of the bids of the other players. Specifically, let $s^{-i} : X \rightarrow \mathbb{R}$ be $i$’s reported residual supply function, where $s^{-i}(x) = \max\{0, c - \sum_{j \neq i} b_{jx}\}$. At each $t$ where $b_{it} \geq s^{-i}(t)$, consumer $i$ pays a tax $\tau^i(t) = s^{-i}(t)$ . In words, consumer $i$ pays in round $t$ if without his bid the auction would have stopped and is payment is equal then to the pivotal amount (i.e. $c - \sum_{j \neq i} b_{it}$) and total payment is

$$\tau^i(L) = \begin{cases} 0 & \text{if } L = 1 \\ \sum_{t=1}^{L-1} s^{-i}(t) & \text{otherwise} \end{cases} .$$

## 4.2 The Ascending Quantity Ausubel Auction as a Game

The rules of the Ausubel auction define how consumers’ bids during the auction are mapped into an allocation. This is a game form. Once preferences over these allocations are added we have a game. In this section we formalize the AQ-AA as a game.

The AQ-AA is a dynamic auction which takes place in stages, where $T$ is the set of possible stages. These stages progress in ascending order – i.e., $t = 1, ..., T$. Nature moves at stage 0 and draws $r \in R$ according to $f$. Player observe their own type and in each subsequent stage make simultaneous bids.

The set of available bids start with an initial set $B_i^1$ for each player $i$. At each stage $t$, each player $i$ observes some function of the past bidding histories of the other players, denote this set by $H_i^t$, where $H_i^1 = \emptyset$. For the first result in the paper, we assume that bidders have *full bid information* – i.e. at each stage, each bidder has a complete history of each bidders past bidding behavior. Bids may be restricted by past bidding behavior. Specifically, for any $h_i^t \in H_i^t$, the set of bids available to $i$ is $B_i^t(h_i^t)$. If the bids in stage $t$ do not cover the marginal cost of production (i.e., $\sum b_{it} < c$), the auction ends. We denote the collection of auction ending bidding histories at stage $t$ as $Z^t$ and $Z = \cup_t Z^t$ as the set of all auction ending bidding histories. Since the end of the auction determines an allocation $(x, \tau^1, ..., \tau^N)$, a player's payoff function $u^i$ is defined on terminal histories and type – i.e.,

$$u^i : Z \times R \rightarrow \mathbb{R}$$

Thus, the dynamic game $\Gamma^e$ induced by the Ascending Quantity Ausubel Auction (AQ-AA) is a *multi-stage game of incomplete information* defined by

$$\Gamma^e = (N, T, f, Z, (R_i, (H_i^t, B_i^t(\cdot))_{t=1}^T, u_i)_{i \in I}).$$

**Definition 3** *An pure strategy $\beta_i$ for player i in the AQ-AA is a collection of T functions (1 for each stage), where each function maps observable histories at that stage and the player's type into a feasible action–i.e., for each t*

$$\beta_i^t : H_i^t \times R_i \rightarrow B_i^t,$$

*where*

$$B_i^t = \cup_{h_i^t \in H_i^t} B_i^t(h_i^t)$$

*such that for each $h_i^t \in H_i^t$ and $r_i \in R_i$ that $\beta_i^t(h_i^t, r_i) \in \beta_i^t(h_i^t)$.*

**Definition 4** *An ex-post pure strategy $\beta_{i(r_i)}$ for player i in the AQ-AA is a collection of T functions (1 for each stage), where, conditioning on observed type $r_i$, each function maps observable histories at that stage into a feasible action–i.e., for each t*

$$\beta_{i(r_i)} : H_i^t \rightarrow B_i^t,$$

*where*

$$B_i^t = \cup_{h_i^t \in H_i^t} B_i^t(h_i^t)$$

*such that for each $h_i^t \in H_i^t$, we have $\beta_{i(r_i)}^t(h_i^t) \in B_i^t(h_i^t)$.*

**Definition 5** *A strategy for bidder i is "truth telling" if for every time period t, bidder type* $r_i \in R_i$*, and information set* $h_i^t \in H_i^t$*, bidder i reports the "t"th component of their type profile –i.e,* $\beta_i^t(h_i^t, r_{it}) = r_{it}$*.*

Since strategies are complete plans of action they yield terminal histories when played out in a game. As such, we can alternatively define payoff functions mapping strategies into the real line. This is the convention we adopt in this paper.

Finally, if the realization of types in the AQ-AA were common knowledge to all of the bidders, then the dynamic game of incomplete information would reduce to a game of complete information – for any $\hat{r} \in R$, the realized game of complete information $\Gamma^e(\hat{r})$ is defined by

$$\Gamma^e(\hat{r}) = (N, T, Z_E, (\hat{r}_i, (H_i^t, B_i^t(\cdot))_{t=1}^T, u_i)_{i \in I}).$$

The objects in this game are defined exactly as above. Notice, that if $\Gamma^e(\hat{r})$ were to be played the appropriate equilibrium concept would be subgame perfection. In our dynamic game of incomplete information, one could define beliefs at each information set and use equilibrium concepts akin to perfect Bayesian. However, we desire a stronger equilibrium concept. Specifically an equilibrium concept that is 'regret-proof' – i.e., if at every stage a player were to find out the types of the other players, he would not want to change his strategy. This concept is *ex-post* perfect Nash equilibrium and is formally defined below.

**Definition 6** *A strategy profile* $\bar{\beta}$ *is an ex post perfect Nash equilibrium of* $\Gamma^e$ *if, for each* $r \in R$*, the ex post pure strategy profile* $\bar{\beta}_{(r)} = (\bar{\beta}_{1(r_1)}, ..., \bar{\beta}_{N(r_N)})$ *is a subgame perfect Nash equilibrium of the game* $\Gamma^e(r)$*.*

A casual inspection of the rules of the AQ-AA reveals that if bidders bid truthfully in every round the outcome of the auction is equivalent to the Vickrey auction. The next theorem shows that truthful bidding in the AQ-AA is indeed an ex post perfect equilibrium.

**Theorem 1** *At each time period in the full bid information AQ-AA, truthful revelation of bidding type for each consumer is an ex-post perfect Nash equilibrium. Moreover, this equilibrium allocation is outcome equivalent to the Public Good Vickrey Auction.*

**Proof.** Suppose that the profile of truthful bidding strategies $\beta = (\beta_1, ..., \beta_N)$ is not an ex-post perfect Nash equilibrium, then there exists some realization of types $\bar{r} \in R$ such that the strategy profile $\beta_{(\bar{r})}$ applied to the resulting multi-stage game with observed actions $\Gamma^e(\bar{r})$ is not subgame perfect. Since $\Gamma^e(\bar{r})$ is a finite horizon game (Since $\bar{x} < \infty$), the fact that $\beta_{(\bar{r})}$ is not subgame perfect implies the strategy does not satisfy the "one-stage-deviation principle" for finite horizon games (see Fudenberg & Tirole p. 109). This means that there is some player $i$ and strategy $\tilde{\beta}_{i(\bar{r}_i)}$ that agrees with $\beta_{i(\bar{r}_i)}$ except at a single $t$ and $h^t$, where $\tilde{\beta}_{i(\bar{r}_i)}$ is a better response to $\beta_{-i(\bar{r}_i)}$ than $\beta_{i(\bar{r}_i)}$ conditional on history $h^t$ being reached.

Suppose the auction ends at stage $L$ (i.e., $x^* = L - 1$) if all bidders report truthfully. Given the truthful reports of the other players, a 1 stage deviation in the AQ-AA can only result in three relevant outcomes: Case 1, the deviation doesn't change the outcome (i.e., the auction ends at $L$); Case 2, the auction ends earlier in some round $E < L$; Case 3, the auction ends in round $L + 1$.

Case 1 is obviously not a profitable deviation since the auction ends at the same round and bidder $i$'s payment is independent of his own action.

Suppose Case 2 is true, then the one stage deviation causes the auction to end earlier than $L$ say round $E$ ($x_E = E - 1$). Player $i$'s payoff is

$$v^i(x_E) - \tau^i(E).$$

The payoff from truth telling is

$$v^i(x_E) - \tau^i(E) + \sum_{k=E+1}^{L-1} [r_{i(x_k)} - s^{-i}(k)].$$

For each $k \in \{E + 1, ..., L - 1\}$ it must be true that $\sum_{j=1}^{N} r_{j(x_k)} \geq c$, since otherwise the auction would have ended earlier when everyone was truthfully reporting. However, this implies that $r_{i(x_k)} > c - \sum_{j \neq i} r_{j(x_k)}$. Since $r_{i(x_k)} \geq 0$ by assumption, then we also have $r_{i(x_k)} \geq \max\{0, c - \sum_{j=1}^{N} r_{j(x_k)}\} = s^{-i}(k)$. Thus, for each $k \in \{E + 1, ..., L - 1\}$ we have that $r_{i(x_k)} \geq s^{-i}(k)$ which in turn implies $\sum_{k=E+1}^{L-1} [r_{i(x_k)} - s^{-i}(k)] \geq 0$. Therefore ending the auction before $L$ cannot lead to a profitable deviation.

Suppose Case 3 is true, then the one stage deviation causes the auction to end at round $L+1$ (i.e., $x_{L+1} = L$ being produced).[2] Player $i$'s payoff from this deviation is

$$v^i(x^*) - \tau^i(L) + r_{ix_{(L+1)}} - s^{-i}(L+1).$$

Player $i$'s payoff from truthful reporting is

$$v^i(x^*) - \tau^i(L).$$

The single stage deviation is profitable only if and only if $r_{ix_{(L+1)}} - s^{-i}(L+1) > 0$. However, when bidders were bidding truthfully the auction stopped at $L$. From the continuation rule of the AQ-AA, it must be that $\sum_i r_{ix_{(L+1)}} < c$ or $r_{ix_{(L+1)}} < c - \sum_{j \neq i} r_{jx_{(L+1)}}$. It is easy to see that this observation implies that

$$r_{ix_{(L+1)}} - s^{-i}(L+1) < 0.$$

There is therefore no single stage deviation that yields a higher payoff. This contradicts our assumption that truthful revelation was not subgame perfect in the realized game. Thus, $\beta$ is ex-post perfect. Furthermore, in equilibrium, a unit is produced if and only if it is efficient to do so and each bidder pays the area under their residual supply for each unit produced. The outcome, therefore, is equivalent to the Vickrey outcome. ■

It is possible to strengthen the solution concept above weakly dominance if we impose some additional restrictions. A bidding In other words, truthful bidding is a weakly dominant strategy if we restrict bids to be monotonically decreasing and give no bid information to agents in each round.

Formally, bids satisfy the monotonic bidding constraint if for any bid $b_{i(t-1)}$ and any subsequent $h_i^t \in H_i^t$ that contains $b_{i(t-1)}$, the set of bids available to $i$ is $B_i^t(h_i^t) = [0, b_{i(t-1)}]$. A bidder receives no bid information if, at each stage, he is only told whether the auction is going to continue or not– i.e.,

$$h_i^t = \begin{cases} 1 \text{ if } \sum_{j=1}^{N} b_{j(t-1)} \geq c \\ 0 \text{ otherwise} \end{cases}.$$

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[2] That the deviation has to come at this stage follows from the assumption of weakly decreasing marginal valuations and the fact that $\tilde{b}_i$ deviates from $b_i$ in only one stage.

**Definition 7** *A bid vector* $b_i$ *is weakly dominated by* $\hat{b}_i$ *if*

$$\begin{array}{llcll} (*) & u_i(\hat{b}_i, b_{-i}) & \geq & u_i(b_i, b_{-i}) & \textit{for all } b_{-i} \\ & & & \textit{and} & \\ (**) & u_i(\hat{b}_i, \tilde{b}_{-i}) & > & u_i(b_i, \tilde{b}_{-i}) & \textit{for at least one } \tilde{b}_{-i}. \end{array}$$

**Definition 8** *A bid vector* $\hat{b}_i$ *is weakly dominant if it weakly dominates all other bid vectors* $b_i$*.*

We now formally state the result and its proof.

**Theorem 2** *If both the "No Bid Information" and the "Monotonic Bidding" constraints are imposed, truthful revelation of bidding type in each round* $b_i^T$ *is a weakly dominant strategy.*

**Proof.** The proof will proceed as follows: first, we show $(*)$ for arbitrary $b_i$ and $b_{-i}$; second, we show $(**)$ by taking any $b_i \neq b_i^T$ and constructing a $\tilde{b}_{-i}$ such that $u_i(b_i^T, \tilde{b}_{-i}) > u_i(b_i, \tilde{b}_{-i})$.

Suppose $b_{-i}$ is the strategy profile followed by bidders other than $i$. It specifies a bid for each consumer other than $i$ at each round where the auction continues. Since this is all of the information each bidder knows at each information set where the auction continues, no bidder can distinguish between 'auction continuing' strategies of their rivals. It also means that at any information set where the auction is continued, no bidder knows whether or not they were pivotal any specific rounds.

Consider the truthful bidding strategy $b_i^T$. Suppose that given $(b_i^T, b_{-i})$ the auction ends at round $L$ yielding $i$ a payoff of $u_i(b_i^T, b_{-i})$. Now consider the consequences of deviating to an alternative strategy $b_i \neq b_i^T$.

First, if $b_i$ also ends the auction in round $L$, then it gives $i$ the same payoff – i.e., $u_i(b_i^T, b_{-i}) = u_i(\tilde{b}_i, b_{-i})$. This is since whether or not $i$ is pivotal in any given round is independent of $i$'s action in a round; and by the "No Bid Information condition," bidders other than $i$ cannot distinguish between $b_i$ and $b_i^T$ for rounds 1 to $L$ and therefore cannot respond to the change.

Second, if $b_i$ ends the auction earlier than truthful bidding in round $E < L$. Bidder $i$'s surplus for the first $E - 1$ rounds is exactly the same as when he was bidding truthfully so there are no gains. Furthermore, since truthtelling guarantees non-negative surplus at each round, bidder $i$ is potentially foregoing positive payoffs from round $E$ to $L - 1$.

Last, if $b_i$ ends the auction later than truthful bidding in round $M > L$, then $i$'s surplus for the first $L - 1$ rounds is exactly the same as before due to the "No Bid Information" constraint and the fact that $i$'s tax is independent of own action. Since the auction ended in round $L$ when $i$ was bidding truthfully, $i$ must be pivotal in round $L$. Since the auction ended in round $L$ when $i$ was bidding truthfully, then it must have been that $\sum_{k=1}^{N} b_{kL} < c$ or $r_{iL} < C - \sum_{j \neq i} b_{jL}$. Therefore the tax for changing $i$'s bid in round $L$ is bigger than the gains from having the auction continuing for that round. It may be the case that gains in later rounds may offset the losses from this round. However this cannot happen due to the "Monotonic Bidding Constraint" which guarantees $C - \sum_{j \neq i} b_{jt}$ is a weakly increasing function in $t$; additionally, since marginal valuations are assumed to be weakly decreasing, $i$'s payoff from continuing past round $L$ is strictly decreasing– i.e., $u_i(b_i, b_{-i}) < u_i(b_i^T, b_{-i})$. Thus, for all $b_{-i}$, $u_i(b_i^T, b_{-i}) \geq u_i(b_i, b_{-i})$.

The second part of the proof requires us to show $(**)$.

Let $b_i \neq b_i^T$ and $t$ be the first round such that either $b_{it} < b_{it}^T$ or $b_{it} > b_{it}^T$. If $b_{it} < b_{it}^T$, define strategy $b_{-i}$ such that $\sum_{j \neq i} b_{jt'} = c$ for all $t' < t$ and $b_{it}^T + \sum_{j \neq i} b_{jt} = c + \epsilon$, where $\epsilon > 0$ is small enough that $b_{it} + \sum_{j \neq i} b_{jt} < c$. Given $b_{-i}$, the auction ends in round $t$ under $b_i$ (i.e., $u_{it}(b_i, b_{-i}) = 0$). However, by ending the auction early, $i$ foregoes a positive surplus he could have obtained in round $t$ if he had followed $b_{it}^T$. This is since in round $t$

$$\begin{aligned} u_{it}(b_i^T, b_{-i}) &= r_{it} - \max(0, c - (c + \epsilon - r_{it})) \\ &= \epsilon > 0. \end{aligned}$$

Alternatively, if $b_{it} > b_{it}^T$, define strategy $b_{-i}$ such that $\sum_{j \neq i} b_{jt'} = c$ for all $t' < t$ and $b_{it}^T + \sum_{j \neq i} b_{jt} = c - \epsilon$, for $\epsilon$ small and $b_{it} + \sum_{j \neq i} b_{jt} \geq c$. Given $b_{-i}$, the auction continues in round $t$ under $b_i$. Moreover, by ending the auction later, $i$ incurs negative surplus he could have avoided in round $t$ if he had followed $b_{it}^T$ since

$$\begin{aligned} u_{it}(b_i^T, b_{-i}) &= 0 \\ &\quad \text{and} \\ u_{it}(b_i, b_{-i}) &= r_{it} - \max(0, c - (c - \epsilon - r_{it})) \\ &= -\epsilon < 0. \end{aligned}$$

■

An immediate consequence of using the AQ-AA is that unless the auction continues to last period (i.e., $t = \bar{x}$), then bidders will **not** have to report all of their values to the auctioneer. The AQ-AA is, in general, privacy preserving for all of the participants.[3] This is in contrast to the private good Ausubel auction which is, in general, only privacy preserving for participants with the highest marginal valuations. However, unlike the private good Ausubel auction, which at times preserves the "highest" marginal valuations of participants, the AQ-AA preserves the lowest marginal valuations of all of the bidders. This is unfortunate since the high values are the ones bidders would prefer not to have to report. Another issue with this auction is that the tax revenue doesn't always cover the cost of production. This property is shared by the Vickrey mechanism.

To illustrate the workings of the AQ-AA consider the following example where, for between mechanism comparison, the marginal valuations will be the same as in Example 1.

**Example 2** *Suppose there are* 3 *consumers in the economy and the government is trying to decide how much of public good to produce at a marginal cost of* \$10 *per unit. Further suppose that each of the consumers value the public good according to the following benefit schedule*

| | *Unit* 1 | *Unit* 2 | *Unit* 3 |
|---|---|---|---|
| *Consumer A* | 14 | 4 | 2 |
| *Consumer B* | 4 | 3 | 2 |
| *Consumer C* | 3 | 2 | 1 |

*The government decides to make its production decision by using the AQ-AA.*

Given that the government uses an AQ-AA to make their production decision, it is an ex post perfect equilibrium for consumers A, B, and C to each follow their truthful bidding strategy. In round 1, consumers A, B, and C submit the bids $(b_{A1}, b_{B1}, b_{C1}) = (14, 4, 3)$. Since $14+4+3 > 10$ the auction continues. Bidder $A$ pays $s^{-A}(1) = 3$. In round 2 the quantity is increased to 2 and consumers A, B, and C submit the bids $(b_{A2}, b_{B2}, b_{C2}) = (4, 3, 2)$. Since $4 + 3 + 2 < 10$, the auction stops. None of the consumers pay a tax in the second round and the final quantity produced is $2 - 1 = 1$. The total tax

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[3] If the auction ends at $t < \bar{x} - 1$, then the AQ-AA is privacy preserving for all participants.

revenue is 3 (paid completely by Consumer $A$). Notice that this is exactly the outcome we arrived at in Example 1. Figure 2 illustrates Consumer $A$'s revealed demand and residual supply. In particular, note that values for unit 3 were not revealed.

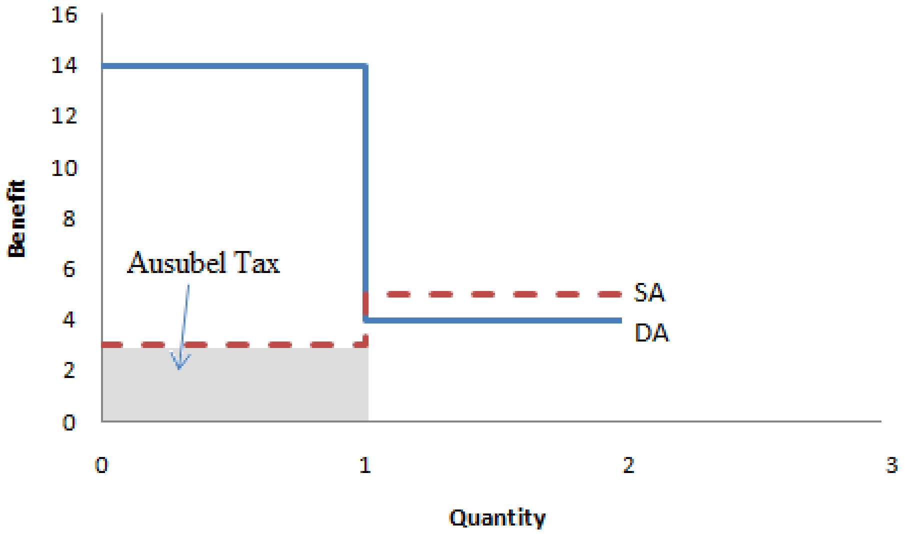

Figure 2: Revealed Bids and Tax for an Ascending Quantity Ausubel Auction

The example showed that the AQ-AA preserves the privacy of all three bidders and chooses the efficient level of the public good. Moreover, the Ausubel Tax paid by each consumer is exactly the tax they would have paid under the public good Vickrey scheme. This demonstrates the outcome equivalence the two procedures and completes the exercise that this paper began. As a useful summary, the following table illustrates the relationship between the four auctions so far discussed in this paper.

| | | Type of Good | | |
|---|---|---|---|---|
| | | *Private* | | *Public* |
| Format | *Sealed-Bid* | Vickrey (1961) | ⟷ | Loeb (1977) |
| | | ↕ | | ↕ |
| | | Outcome Equivalent | | Outcome Equivalent |
| | | ↕ | | ↕ |
| | *Dynamic* | Ausubel (2004) | ⟷ | AQ-AA (This Paper) |

# 5 Extending the AQ-AA to Include Public "Bads"

In the previous section, we were concerned with public good environments with the property that every consumer's marginal valuation for each good was non-negative. This obviously limits the number of applicable environments as there are many examples of goods where this property may not hold. In this section, we generalize the AQ-AA to include preference environments where some consumers may have a negative marginal valuations over some range of their type profile – i.e., the setting is exactly the same as before, except for each *positive* unit $x \in X$, there is some reservation price $r_{ix}$ that indexes consumer $i$'s maximum willingness to pay for that unit. This number may be positive or negative, but we maintain the assumption of weakly decreasing marginal valuations. This generalization comes with a cost. In the previous environments, each consumer only ever paid a tax. There was never any subsidies. In order to include public bads it becomes necessary to add subsidies into the mix.

## 5.1 Rules of the Generalized Ascending Quantity Ausubel Auction:

In this section, we outline the process in which the generalized public good Ausubel auction takes consumers' bids in each stage of the auction and converts them into an allocation $(x, \tau_1, ..., \tau_N)$. The main difference between this auction and the one introduced in the last section is the potential usage of subsidies in addition to taxes at every round.

The auction begins at round 1 where the initial level of the public good is set to be 1. At each round $t = 1, ..., \bar{x}$, the level of the public good announced is $x_t = t$ (i.e., the level of the public good increases in increments of 1 unit per round) and each consumer $i$ is required to submit a bid $b_{it}$ to the auctioneer. If $\sum_i b_{it} \geq c$, the auction continues ($x_{t+1} = t + 1$), and stops otherwise. Denote the round the auction stops by $L$. The quantity produced at the end of the auction is

$$x^* = L - 1.$$

Bidders do not pay their bids, but rather some function of the bids of the other players. Specifically, let $s^{-i} : X \to \mathbb{R}$ be $i$'s reported residual supply function, where $s^{-i}(x) = c - \sum_{j \neq i} b_{jx}$. At each $t$ where $\sum b_{it} \geq c$, consumer

$i$ pays or receives a subsidy equal to $\tau^i(t) = s^{-i}(t)$ . In words, consumer $i$ pays in round $t$ if without his bid the auction would have stopped and receives a subsidy if without his bid the auction would not have stopped is payment (subsidy) is then equal to the pivotal amount (non-pivitol amount) (i.e. $c - \sum_{j \neq i} b_{it}$) and total payment is

$$\tau^i(L) = \begin{cases} 0 & \text{if } L = 1 \\ \sum_{t=1}^{L-1} s^{-i}(t) & \text{otherwise} \end{cases} .$$

**Theorem 3** *At each time period in the full information generalized AQ-AA, truthful revelation of bidding type for each consumer is an ex-post perfect Nash equilibrium. Furthermore, the equilibrium outcome is efficient.*

**Proof.** Note, the proof is almost identical to the proof of Theorem 1 with the exception of Case 2. For space considerations, only Case 2 is considered.

Suppose Case 2 is true, then the one stage deviation causes the auction to end earlier than $L$ say round $E$ ($x_E = E - 1$). Player $i$'s payoff is

$$v^i(x_E) - \tau^i(E).$$

The payoff from truth telling is

$$v^i(x_E) - \tau^i(E) + \sum_{k=E+1}^{L-1} [r_{i(x_k)} - s^{-i}(k)].$$

For each $k \in \{E+1, ..., L-1\}$ it must be true that $\sum_{j=1}^{N} r_{j(x_k)} \geq c$, since otherwise the auction would have ended earlier when everyone was truthfully reporting. However, this implies that $r_{i(x_k)} > c - \sum_{j \neq i} r_{j(x_k)}$. For all $r_{i(x_k)}$, we also have $r_{i(x_k)} \geq c - \sum_{j=1}^{N} r_{j(x_k)} = s^{-i}(k)$. Thus, for each $k \in \{E+1, ..., L-1\}$ we have that $r_{i(x_k)} \geq s^{-i}(k)$ which in turn implies $\sum_{k=E+1}^{L-1} [r_{i(x_k)} - s^{-i}(k)] \geq 0$. Therefore ending the auction before $L$ cannot lead to a profitable deviation.

Cases 1,3 follow for the same reasons as in the proof to Theorem 1. There is therefore no single stage deviation that yields a higher payoff. This contradicts our assumption that truthful revelation was not subgame perfect in the realized game. Thus, $\beta$ is ex-post perfect. In equilibrium, a unit is produced if and only if it is efficient to do so. The outcome, therefore, is efficient. ■

Again, an immediate consequence of using the extended AQ-AA is that unless the auction continues to last period (i.e., $t = \bar{x}$), then bidders will **not** have to report all of their values to the auctioneer. The extended AQ-AA is thus, in general, privacy preserving for all of the participants.[4]

Given that the Vickrey mechanism belongs to a larger class of strategy-proof Groves mechanisms, it should therefore not be surprising that the extended AQ-AA bears a strong relation to another standard Groves mechanism.[5]

Specifically the Groves mechanism where for each element $x \in X$, each consumer $i$ sends a bid $b_{ix}$ to the government. The total report is the profile $b_i = (b_{i1}, ..., b_{i\bar{x}})$. As in the Vickrey mechanism, the government takes these bids from each consumer and constructs reported valuation functions and a slightly different reported residual supply function, where for each $x \in X$, $\hat{s}^{-i}(x) = c - \sum_{j \neq i} b_{jx}$. Using the reported valuation and the reported residual supply functions the government calculates the production quantity $x^*$ and each consumer's payment $\tau^i$ (i.e. the allocation received by each consumer). Specifically, $x^*$ is chosen to maximize reported social benefit – i.e., chooses

$$x^* \in \arg\max_{x \in X} [\sum_i \tilde{v}^i(x) - cx].$$

Each consumer $i$ is charged a tax equal to the reported residual supply for each positive unit that is produced and zero if no units are produced–i.e.,

$$\tau^i(x^*) = \begin{cases} 0 & \text{if } x^* = 0 \\ \sum_{j=1}^{x^*} \hat{s}^{-i}(j) & \text{otherwise} \end{cases}.$$

# 6 Conclusion

This paper introduces a set of dynamic auctions which make efficient decisions in a public good context. The auction seems relatively simple and scales well to increases in the number of consumers. Moreover, each bidder has a 'regret-proof' equilibrium strategy to report their true valuation for the public good at each stage in the auction. Despite these nice properties, these auctions has several obvious shortcomings that leave room for further research.

First, since, in general, neither procedure covers the costs of production, it would be interesting if one could generate a dynamic auction that at least

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[4] If the auction ends at $t < \bar{x} - 1$, then the AQ-AA is privacy preserving for all participants.

[5] see, for example, Varian p.427, where $c = 0$.

always guarantee a surplus. This issue was attacked for sealed bid VCG mechanisms by Clarke (1971) and Groves and Loeb (1975), but the same techniques applied there will not work for this particular context.

Both the AQ-AA and the generalized AQ-AA procedures work because truthful revelation over the course of the auction is always individually rational. In other words, if production is efficient a bidder's profit is weakly increasing. If one were to assign fixed shares of the cost for each unit (as in Clarke's original paper) and then apply the same procedure, there would typically be incentives for people to end the auction earlier than they otherwise would to avoid paying this tax.[6] In other words, adding fixed cost shares may eliminate the individual rationality property from a truth-telling strategy.

The second concern is that while the auction is, in general, privacy preserving for each consumer, the auction only preserves the lowest marginal valuations of the participants. One plausible solution might be to formulate a decreasing quantity rule that makes efficient decisions. If the quantity is decreasing then when the auction ends, it is the highest valuations of the consumers that are protected. Another option is to define increasing bid dynamic auction that accomplishes the same sort of properties. Neither of these directions have been pursued by this author as there are some initial hurdles that must be cleared before these options are viable.

# 7 Resources

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