On Bargaining Problems Admitting a Unique Nash Bargaining Solution

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Abstract

We completely characterize a class of bargaining problems allowing for non-convexity, on which three of Nash axioms uniquely characterize the Nash bargaining solution up to specifications of bargaining powers. This class consists of bargaining problems whose strictly individually rational portion of the choice set can be transformed under a logarithmic function into a convex set, which does not contain line segments with strictly positive normal vectors in the Pareto frontier. We show that this class includes non-convex bargaining problems arising from duopolies with asymmetric constant marginal costs. We compare Nash bargaining solution for these duopoly bargaining problems with some extended Nash bargaining solutions that appeared in the literature.

1 Introduction

The model of bargaining introduced in seminal papers of Nash (1950, 1953) postulates that a group of players choose a payoff allocation from a set of feasible payoff allocations. The implementation of a payoff allocation requires unanimous agreement among the players. In the case of disagreement, the players end up getting some pre-assigned payoff allocation, known as the *status quo* or the *threat point*. A

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bargaining problem in the sense of Nash is thus represented by a pair consisting of a choice set and a threat point.

Given a class of bargaining problems, a bargaining solution is a rule that specifies a payoff allocation in the choice set of each bargaining problem in the class. Nash considered an axiomatic approach to resolve the bargaining problem:

> One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely (Nash 1953, p. 129).

Nash postulated six axioms that are deemed to be natural for a bargaining solution to satisfy. They are equivalent to four axioms for the class of bargaining problems with each having a compact convex choice set and a strictly Pareto dominated threat point.[1] It is remarkable that for this class of bargaining problems, the axioms characterize a unique solution, which has to be called the (symmetric) *Nash bargaining solution* (see Nash 1950, 1953, and also Roth 1979). Nash's bargaining model has become one of the most fruitful paradigms in cooperative game theory.[2]

One of the four axioms is that of symmetry. When it is removed, Kalai (1977) showed that for the same class of problems as in Nash (1953), a bargaining solution *unique up to* specifications of solutions for the "normalized problem" is characterized by the remaining three axioms. This unique solution is known as the asymmetric Nash bargaining solution. A solution for the normalized problem has been customarily interpreted as representing players' bargaining powers. The symmetry axiom helps to uniquely determine the solution for the normalized problem.

In most applications, both the choice set and threat point are not directly given. They are derived from more primitive data. As a result, the choice set needs not to be convex. Indeed, the non-convexity of the choice set of feasible profit shares arising from duopolies with asymmetric constant marginal costs has been well recognized in literature.[3]

The convexity of the choice sets in the axiomatic theory of bargaining is usually justified by allowing for randomized payoff allocations and by assuming that players

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[1] This is the same class of bargaining problems as in Nash (1953) except for the requirement that the threat points be strictly Pareto dominated.

[2] An alternative approach to bargaining is to use sequential, non-cooperative models of negotiation. Nash (1953, p. 129) argued that the two appraoches "are complementary; each helps to justify and clarify the other." See Binmore, Rubinstein and Wolinsky (1986) for establishing formally the relationship between Nash's axiomatic bargaining theory and the sequential non-cooperative approach to bargaining.

[3] See, for example, Bishop (1960), Schmalensee (1987), and Tirole (1988, p. 242, 271).

are expected utility maximizers. The expected utility maximization is, however, not an automatically valid behavioral hypothesis, while randomized allocations can cause conflicts of interest at the post-realization stage and may not be realistic in some applications. It is therefore desirable to analyze the extent to which Nash bargaining solution and its uniqueness depend on the convexity of the choice sets.

Zhou (1997) considered bargaining problems with non-convexity.[4] He showed that the same characterization of bargaining solutions satisfying Nash's three axioms as in Kalai (1977) carries over to any single-valued solution on a broad class of bargaining problems with choice sets being closed, comprehensive, and bounded from above and with threat points being strictly Pareto dominated. The Nash bargaining solutions are, however, not unique on this broad class. The non-uniqueness limits the applicability of the Nash bargaining solution.

In this paper, we contribute to the axiomatic approach by completely characterizing a class of bargaining problems with choice sets being closed, comprehensive, and bounded from above, on which three of Nash axioms uniquely characterize the Nash bargaining solution up to specifications of bargaining powers.[5] We show that this class consists of bargaining problems whose *strictly individually rational* portions of the choice sets can be transformed under a logarithmic function into convex sets, which do not contain segments with strictly positive normal vectors in the Pareto frontiers. The bargaining problems arising from duopolies with asymmetric constant marginal costs are non-convex and are included in this class. We compare Nash bargaining solution for these duopoly bargaining problems with the Conley and Wilkie's (1996) extended Nash bargaining solution.

The rest of the paper is organized as follows. The next section briefly reviews Nash axioms, Nash bargaining solution, and extensions without convexification. Section 3 provides a complete characterization of bargaining problems with a unique Nash bargaining solution. Section 4 applies our characterization to duopoly bargaining problems and compares the Nash bargaining solution for these problems with the Conley and Wilkie (1996) extended Nash bargaining solutions. Section 5

[4] Conley and Wilkie (1996) took a different approach to extended the Nash bargaining solution to bargaining problems with non-convexity. Specifically, their approach works in two steps. First, convexify the choice set via randomized allocations and apply the Nash bargaining solution to the convexified problem. Second, use the intersection point of the segment between the threat point and the Nash bargaining solution for the convexified problem with the original Pareto frontiers as the bargaining solution. In section 4, we compare the Nash bargaining solution with their extended solution. Herrero (1989) considered another different approach to bargaining problems with non-convexity. She imposed a set of axioms which includes those of Nash. These axioms imply multi-valued bargaining solutions for certain non-convex bargaining problems in her class.

[5] We require the choice set to be comprehensive with respect to the threat point.

concludes.

2 Nash Bargaining Solution with Convexity

The theory of bargaining pioneered in seminal papers of Nash (1950, 1953) takes a bargaining problem as primitive. A two-person bargaining problem is composed of a choice set $S \subset \Re^2$ of payoff allocations the players can jointly achieve with agreement, and a threat point d the players end up getting in case of disagreement. A bargaining solution on a class $\mathcal{B}$ of bargaining problems is a rule f such that it assigns an allocation $f(S,d) = (f_1(S,d),\ f_2(S,d)) \in S$ to each bargaining problem $(S,d) \in \mathcal{B}$. Nash imposed the following well-known axioms on bargaining solutions.[6]

Strict Individual Rationality (SIR): For any $(S,d) \in \mathcal{B}$, $f_i(S,d) > d_i$, $i = 1,2$.

Symmetry (S): For any $(S,d) \in \mathcal{B}$ with $d_1 = d_2$ and $(u_2,u_1) \in S$ whenever $(u_1,u_2) \in S$, $f_1(S,d) = f_2(S,d)$.

Invariance to Equivalent Utility Representations (INV): For any $(S,d) \in \mathcal{B}$ and for any positive affine transformation $\tau : \Re^2 \longrightarrow \Re^2$, $f(\tau(S),\tau(d)) = \tau(f(S,d))$.

Independence of Irrelevant Alternatives (IIA): For any $(S,d),(S',d) \in \mathcal{B}$ with $S \subseteq S'$, $f(S',d) \in S$ implies $f(S',d) = f(S,d)$.

It is remarkable that the preceding four axioms uniquely characterize a bargaining solution, known as the Nash bargaining solution, when $\mathcal{B}$ is composed of all compact convex bargaining problems with strictly Pareto dominated threat points. Moreover, the Nash bargaining solution assigns the payoff allocation to $(S,d) \in \mathcal{B}$ determined by

$$\max_{u \in S, u \geq d} (u_1 - d_1)(u_2 - d_2). \tag{1}$$

See Nash (1953) and Roth (1979) for details.

When the symmetry axiom is removed, Kalai (1977) showed that for the same class of bargaining problems as considered by Nash, the bargaining solution, which assigns the payoff allocation given by

$$\max_{u \in S, u \geq d} (u_1 - d_1)^{\alpha}(u_2 - d_2)^{1-\alpha} \tag{2}$$

[6] Roth (1977) showed that a bargaining solution is (strictly) Pareto optimal whenever the solution satisfies SIR, INV, and IIA. Thus, with INV and IIA, Pareto optimality as Nash originally considered can be replaced by SIR.

to $(S, d) \in \mathcal{B}$, is the unique bargaining solution characterized by the remaining three axioms together with the payoff allocation $(\alpha, 1-\alpha)$ as the solution for the normalized bargaining problem (S°, d°), where

$$S^{\circ} = \left\{ u \in \Re^2 | u_1 + u_2 \leq 1 \right\}, \text{ and } d^{\circ} = (0, 0)$$

The payoffs α and $(1-\alpha)$ are usually interpreted as representing players' relative bargaining powers. This interpretation can be traced to Shubik (1959, p. 50). The unique solution is known as the Nash bargaining solution with bargaining powers α for player 1 and $1-\alpha$ for player 2. Thus, another way to state Kalai's result is that the Nash bargaining solution on the class $\mathcal{B}$ of compact convex bargaining problems is uniquely characterized by SIR, INV, and IIA up to specifications of bargaining powers.

3 Nash Bargaining Solution without Convexity

Consider the bargaining problem between two firms supplying a homogeneous product. The firms have constant marginal costs c_1 and c_2, respectively, and the market (inverse) demand for the product is given by $P(Q)$ with $Q = q_1 + q_2$ representing total supply from the two firms. Assume *side-payments* are not feasible.

Let $\pi = (\pi_1, \pi_2)$ be a profit distribution. Then, π is feasible if and only if there exists a pair $q = (q_1, q_2) \geq 0$ of quantities (or output quotas) such that

$$\pi_1 = [P(Q) - c_1]q_1 \quad \text{and} \quad \pi_2 = [P(Q) - c_2]q_2. \tag{3}$$

Set $p = P(Q)$ and $Q = D(p)$. From (3),

$$\frac{\pi_1}{p - c_1} + \frac{\pi_2}{p - c_2} = D(p). \tag{4}$$

We defer a general discussion of the non-convexity of duopoly bargaining problems to Section 4. Below we provide a simple illustration of the Pareto profit frontier with an example.

Example 1: Let $D(p) = \max\{20 - p, 0\}$, $c_1 = 0$, and $c_2 = 10$. Notice that the monopoly profit for firm 2 is $\pi_2 = 25$. Given price $10 < p < 15$ and given firm 2's profit $\pi_2 \leq 25$, (4) implies that firm 1's profit is given by

$$h(p, \pi_2) = p(20 - p) - \frac{p}{p - 10}\pi_2.$$

Thus, fixing firm 2's profit at $\pi_2 \leq 25$, firm 1's maximum profit is determined by choosing p to maximize $h(p, \pi_2)$. The first-order condition yields

$$2(10 - p) + \frac{10\pi_2}{(p - 10)^2} = 0.$$

It follows that

$$p(\pi_2) = 10 + (5\pi_2)^{\frac{1}{3}}$$

is the price that results in firm 1's maximum profit given firm 2's profit π_2. Hence,

$$\pi_1 = h(p(\pi_2), \pi_2) = 100 - \pi_2 - 3(5\pi_2)^{\frac{2}{3}}, \ \pi_2 \leq 25$$

characterizes the Pareto profit frontier for the firms. Simple calculation shows that the frontier is a convex curve, which is also illustrated in Figure 1. Consequently, the feasible profit choice set is not convex.

When convexity is not imposed, Zhou (1997) showed that the same characterization as in Kalai (1977) of bargaining solutions satisfying SIR, INV, and IIA on a large class of bargaining problems allowing for non-convexity carries over to single-valued bargaining solutions.[7] Specifically, he showed that if a single-valued bargaining solution f satisfies SIR, INV, and IIA on the class $\mathcal{B}$ of bargaining problems (S, d) such that S is closed, comprehensive, bounded from above, and d is strictly Pareto dominated, then f is determined by (2) for some $\alpha \in (0, 1)$ such that $f(S^\circ, d^\circ) = (\alpha, 1 - \alpha)$. However, on this enlarged class of bargaining problems, the bargaining solutions are not unique up to choices of bargaining powers under SIR, INV, and IIA. The following example provides an illustration.

Example 2: Let a and b be positive numbers such that $a + b < 1$. Consider bargaining problem (S, d) where $d = (0, 0)$ and S consists of payoff allocations $u = (u_1, u_2)$ bounded from above by the following two segments. One segment connects points $(1, 0)$ and (a, b) and the other connects (a, b) and $(0, 1)$, as illustrated in Figure 2. Note that S is not convex although it is closed, comprehensive, and bounded from above. Simple calculation shows that there exists $\alpha \in (0, 1)$ with $a < \alpha < 1 - b$ such that both

$$(\alpha, \frac{b}{1 - a}(1 - \alpha))$$

[7] The INV axiom is imposed only on the subclass of convex bargaining problems in his paper.

and

$$(\frac{\alpha}{1-b}a, 1-\alpha)$$

are solutions to the maximization problem (2). We cannot determine which of the two allocations is the Nash bargaining solution on the class of bargaining problems with choice sets being closed, comprehensive, and bounded from above and with disagreement point being strictly Pareto dominated.[8]

3.1 A Logarithmic Transformation of Bargaining Problems

Fix a bargaining problem (S, d) and take a logarithmic transformation of the strictly individually rational portion of the choice set S to get

$$V(S,d) = \left\{ v \in \Re^2 \middle| \exists u \in S : \begin{array}{l} u_1 > d_1, u_2 > d_2, \\ v_1 \leq \ln(u_1 - d_1), \\ v_2 \leq \ln(u_2 - d_2). \end{array} \right\}. \tag{5}$$

Then, the maximization problem (2) is equivalent to

$$\max_{v \in V(S,d)} \alpha v_1 + (1-\alpha) v_2. \tag{6}$$

Since $\ln(u_i - d_i)$ is increasing and concave over $u_i \in (d_i, \infty)$ for $i = 1, 2$, the convexity of S implies the convexity of $V(S, d)$. But, the converse does not necessarily hold. Indeed, for any strictly Pareto dominated threat point d in the choice set S of the bargaining problem in Example 1, $V(S, d)$ is strictly convex even though S itself is not.

We assume bargaining problems considered in this paper satisfy the following basic properties:

(i) S is closed, bounded above, and $d-$comprehensive (i.e. $u \in S$ whenever $d \leq u \leq u'$ for some $u' \in S$); and

(ii) d is strictly Pareto dominated.

We show next that the Nash bargaining solution on the class $\mathcal{B}$ of bargaining problems satisfying (i) and (ii) is uniquely characterized by SIR, INV, and IIA up to choices of bargaining powers if and only if all $(S, d) \in \mathcal{B}$ satisfy the following crucial property:

[8] In this example, it can be easily illustrated that for any given $\alpha \in (0, 1)$, there is a continuum of pairs of (a, b) such that the maximization problem (2) has two solutions.

(iii) $V(S,d)$ *is closed, convex, and its boundary does not contain any segment with a normal vector* $(\alpha, 1-\alpha)$ *for any* $\alpha \in (0,1)$.

Before presenting our characterization result, we provide the following remark on property (iii) which will be useful in the proof of Theorem 2 below. Let (S,d) and (S',d) be two bargaining problems with the same threat point d that satisfy (i)-(ii). Then $V(S \cap S', d) = V(S,d) \cap V(S',d)$, which implies that $(S \cap S', d)$ satisfies (iii) whenever both (S,d) and (S',d) satisfy (iii). Indeed, it is straightforward to verify that $(S \cap S', d)$ also satisfies (i) and (ii) and $V(S \cap S', d) \subseteq V(S,d) \cap V(S',d)$. On the other hand, take any $v \in V(S,d) \cap V(S',d)$. It follows from (5) that there exist $u \in S$ and $u' \in S'$ such that $u_i > d_i$, $u'_i > d_i$, $v_i \leq \ln(u_i - d_i)$, and $v_i \leq \ln(u'_i - d_i)$ for $i = 1, 2$. Set $\underline{u}_i = \min\{u_i, u'_i\}$. Then, $v_i \leq \ln(\underline{u}_i - d_i)$. Since $d << \underline{u} \leq u, u'$ and since S and S' satisfy (i), it follows that $\underline{u} \in S \cap S'$, implying $v \in V(S \cap S', d)$. This shows $V(S \cap S', d) \supseteq V(S,d) \cap V(S',d)$.

The uniqueness of the Nash bargaining solution requires that the maximization problem (2) or equivalently the maximization problem (6) have a unique solution. It turns out that this imposes enough restrictions on $V(S,d)$ to make it satisfy (iii).

Theorem 1 *Let* (S,d) *be a bargaining problem satisfying (i)-(ii). Then, (6) has a unique solution for any* $\alpha \in (0,1)$ *if and only if* $V(S,d)$ *satisfies (iii).*

Proof. See the Appendix. ■

3.2 Existence and Uniqueness

We are now ready to present our characterization of the class of bargaining problems on which the Nash bargaining solution is uniquely characterized by SIR, INV, and IIA up to choices of bargaining powers.

Theorem 2 *Let* $\mathcal{B}$ *be the class of all bargaining problems satisfying (i)-(iii). Then SIR, INV, and IIA uniquely characterize the Nash bargaining solution on* $\mathcal{B}$ *up to choices of bargaining powers. Furthermore, if the solution assigns* $(\alpha, 1-\alpha)$ *to* (S°, d°)*, then it is determined by problem (6).*

Conversely let $\mathcal{B}$ *be a class of bargaining problems satisfying (i)-(ii). If SIR, INV, and IIA uniquely characterize the Nash bargaining solution on* $\mathcal{B}$ *up to choices of bargaining powers, then* (S,d) *satisfies (iii) for all* $(S,d) \in \mathcal{B}$.

Proof. The proof of the first part is by contradiction. Let $\mathcal{B}$ be the class of all bargaining problems satisfying (i)-(iii). Suppose that g is a bargaining solution satisfying SIR, INV, IIA, and $g(S^\circ, d^\circ) = (\alpha, 1-\alpha)$ for some $\alpha \in (0,1)$ but g is not determined by (6). Denote by f the bargaining solution determined by maximization problem (6) with the same α. Now consider the following affine transformation:

$$\tau(u) = (\tau_1(u_1), \tau_2(u_2)) = \left(\frac{\alpha(u_1 - d_1)}{f_1(S,d) - d_1}, \frac{(1-\alpha)(u_2 - d_2)}{f_2(S,d) - d_2} \right), \ u \in \Re^2. \tag{7}$$

We have $(\tau(S), \tau(d)) \in \mathcal{B}$ and

$$\tau(f(S,d)) = (\alpha, 1-\alpha), \ \tau(d) = (0,0). \tag{8}$$

Since $x \equiv \tau(f(S,d)) \neq y \equiv \tau(g(S,d))$, it follows from (iii) that

$$x_1^\alpha x_2^{1-\alpha} > y_1^\alpha y_2^{1-\alpha}. \tag{9}$$

Without loss of generality, assume (the other case can be analogously proved)

$$x_1 < y_1. \tag{10}$$

Since $x, \ y \in \tau(S)$, it follows from (i) and (iii) that [9]

$$z_t \equiv (y_1^t x_1^{1-t}, y_2^t x_2^{1-t}) \in \tau(S), \ \forall t \in [0,1]. \tag{11}$$

Notice for $t \in (0,1)$, the absolute value of the slope of the tangent line of Nash product $u_1^\alpha u_2^{1-\alpha}$ at z_t is

$$\frac{\alpha y_2^t x_2^{1-t}}{(1-\alpha) y_1^t x_1^{1-t}}$$

and that of the segment between z_t and y is

$$\frac{y_2^t x_2^{1-t} - y_2}{y_1 - y_1^t x_1^{1-t}}.$$

Thus, the tangent line is flatter than the segment if and only if

$$\frac{\alpha y_2^t x_2^{1-t}}{(1-\alpha) y_1^t x_1^{1-t}} < \frac{y_2^t x_2^{1-t} - y_2}{y_1 - y_1^t x_1^{1-t}} \Leftrightarrow \frac{\alpha x_2^{1-t}}{(1-\alpha) x_1^{1-t}} < \frac{x_2^{1-t} - y_2^{1-t}}{y_1^{1-t} - x_1^{1-t}} \tag{12}$$

[9] To see this, notice that since x and y are in $\tau(S)$, $t(\ln y_1, \ln y_2) + (1-t)(\ln x_1, \ln x_2) = (\ln y_1^t x_1^{1-t}, \ln y_2^t x_2^{1-t}) \in V(\tau(S), \tau(d))$ for $t \in [0,1]$. Thus, by construction, there is $u(t) \in \tau(S)$ such that $(\ln y_1^t x_1^{1-t}, \ln y_2^t x_2^{1-t}) \leq (\ln u_1(t), \ln u_2(t))$. It follows $(y_1^t x_1^{1-t}, y_2^t x_2^{1-t}) \leq (u_1(t), u_2(t))$. By (ii), $(y_1^t x_1^{1-t}, y_2^t x_2^{1-t}) \in \tau(S)$.

By L'Hôpital's rule, (12) holds as $t \longrightarrow 1$ if

$$\frac{\alpha}{1-\alpha} < \frac{\ln x_2 - \ln y_2}{\ln y_1 - \ln x_1}. \tag{13}$$

By (10), $\ln y_1 > \ln x_1$. Hence, (13) holds if and only if

$$\alpha(\ln y_1 - \ln x_1) < (1-\alpha)(\ln x_2 - \ln y_2) \Leftrightarrow y_1^{\alpha} y_2^{1-\alpha} < x_1^{\alpha} x_2^{1-\alpha}$$

which holds due to (9). Hence, when t is close enough to 1, the tangent line of Nash product $u_1^{\alpha} u_2^{1-\alpha}$ at z_t is flatter than the segment between z_t and y.

By (10), $y_1^t x_1^{1-t} < y_1$ which implies $y_2^t x_2^{1-t} > y_2$.[10] It follows that when t is close enough to 1 so as to make (12) satisfied, y is below the tangent line. Fix such a number $t \in (0,1)$ and let S_t denote the choice set formed by the tangent line together with the two axis. Clearly, S_t satisfies (i)-(iii). Thus, by INV and $g(S^{\circ}, d^{\circ}) = (\alpha, 1-\alpha)$,

$$g(S_t, d^{\circ}) = z_t. \tag{14}$$

By (11),

$$z_t \in S_t \cap \tau(S). \tag{15}$$

Since $S_t \cap \tau(S)$ satisfies (i)-(iii), it follows from IIA and, (8), (14), and (15) that

$$g(S_t \cap \tau(S), d^{\circ}) = z_t.$$

On the other hand, since $y = \tau(g(S,d)) = g(\tau(S), \tau(d)) \in \tau(S)$, we also have

$$g(S_t \cap \tau(S)), d^{\circ}) = y.$$

Thus, it must be $y = z_t$. Since $t < 1$, this results in the desired contradiction.

Now let $\mathcal{B}$ be a class of bargaining problems satisfying (i)-(ii). Suppose SIR, INV, and IIA uniquely characterize the Nash bargaining solution on $\mathcal{B}$ up to choices of bargaining powers. Then, for any $\alpha \in (0,1)$, $f(S^{\circ}, d^{\circ}) = (\alpha, 1-\alpha)$ implies that $f(S,d)$ is determined by (6) for all $(S,d) \in \mathcal{B}$. Thus, for any $(S,d) \in \mathcal{B}$ and for any $\alpha \in (0,1)$, (6) has a unique solution. By Theorem 1, (S,d) satisfies (iii) ■

Condition (iii) provides a way to check if the Nash bargaining solution on a class of bargaining problems satisfying (i) and (ii) can be uniquely characterized by SIR, INV, and IIA up to choices of bargaining powers. It can be easily verified that the bargaining problem in Example 2 satisfies (i) and (ii), but fails to satisfy (iii). Thus, the Nash bargaining solution is not unique up to choices of bargaining powers on any class of bargaining problems containing the problem in Example 2.

[10] Notice $x_1 < y_1$ necessarily implies $x_2 > y_2$ due to the Pareto optimality of y and x.

4 Bargaining Problems Arising in Duopolies

As Example 1 illustrates, the profit choice set of the duopoly bargaining problem is not convex. In this section, we show that for a general demand function and asymmetric constant marginal costs, the duopoly bargaining problem without side-payments belongs to the class of bargaining problems characterized in Theorem 2. We show that, for any strictly Pareto dominated threat point d, the set $V(S,d)$ is strictly convex.

Denote by $\bar{\pi}_i(p) = (p - c_i)D(p)$ the monopoly profit and $\bar{p}_i$ the monopoly price for firm $i = 1, 2$. Assume that $c_1 < c_2$, $D(c_2) > 0$, and side-payments between the firms are not feasible. To make bargaining non-trivial, we further assume $c_2 < \bar{p}_1$. Let $\pi = (\pi_1, \pi_2)$ be a profit distribution between the firms. By (3) and (4), the profit choice set for the two firms is given by

$$\Pi = \left\{ \pi \middle| \exists p : \quad \begin{array}{c} c_2 < p < D^{-1}(0) \\ \frac{\pi_1}{(p-c_1)} + \frac{\pi_2}{(p-c_2)} = D(p) \end{array} \right\}. \tag{16}$$

The following lemma characterizes Pareto optimal prices for the firms.

Lemma 1 *Suppose $\bar{\pi}_1(p)$ and $\bar{\pi}_2(p)$ are concave. Then, p is Pareto optimal if and only if $p \in [\bar{p}_1, \bar{p}_2]$.*

Proof. Notice that the concavity of the profit functions and $c_1 < c_2$ together imply $\bar{p}_1 < \bar{p}_2$. Suppose first $p < \bar{p}_1$. Then, since both $\bar{\pi}_1$ and $\bar{\pi}_2$ are monotonically increasing over $[p, \bar{p}_1]$, it follows that $\bar{\pi}_1(p) < \bar{\pi}_1(\bar{p}_1)$ and $\bar{\pi}_2(p) < \bar{\pi}_2(\bar{p}_1)$. This means that the profit-share line in (4) at $\bar{p}_1$ is everywhere above the profit-share line at p. Consequently, p cannot be Pareto optimal. Suppose now $p > \bar{p}_2$. Then, since both $\bar{\pi}_1$ and $\bar{\pi}_2$ are monotonically decreasing over $[\bar{p}_2, p]$, repeating the preceding argument shows that p cannot be Pareto optimal either. Suppose finally $p \in [\bar{p}_1, \bar{p}_2]$. Then, since Π_2^m is monotonically increasing over $[\bar{p}_1, \bar{p}_2]$ while $\bar{\pi}_1$ is monotonically decreasing over that interval, no price in $[\bar{p}_1, \bar{p}_2]$ can Pareto dominate price p. ■

Given price $p > c_2$ and firm 2's profit π_2, it follows from (4) that

$$h(p, \pi_2) = \bar{\pi}_1(p) - \frac{\pi_2(p - c_1)}{p - c_2} \tag{17}$$

is the profit for firm 1 should they agree to let firm 2 receive profit π_2. Thus, by Lemma 1, a pair $\pi \in \Pi$ is on the Pareto frontier if and only if

$$\pi_1 = \max_{\bar{p}_1 \le p \le \bar{p}_2} h(p, \pi_2).$$

Thus, the supporting price, $p(\pi_2)$, is determined by

$$h_p(p, \pi_2) = \frac{\partial \bar{\pi}_1(p)}{\partial p} + \frac{(c_2 - c_1)\pi_2}{(p - c_2)^2} = 0 \tag{18}$$

and the Pareto frontier is given by $\pi_1 = h(p(\pi_2), \pi_2)$ for $\pi_2 \in [0, \bar{\pi}_2(\bar{p}_2)]$. Lemma 2 below describes the behavior of the Pareto frontier and the supporting price.

Lemma 2 *(Schmalensee 1987, Tirole 1988) Suppose $\bar{\pi}_1(p)$ and $\bar{\pi}_2(p)$ are concave. Then, $p'(\pi_2) > 0$ and $\pi_1 = h(p(\pi_2), \pi_2)$ is strictly decreasing and strictly convex in $\pi_2 \in [0, \bar{\pi}_2(\bar{p}_2)]$.*

Lemma 2 implies that the choice set in (16) is not convex under the assumed conditions. We now present a result showing that (Π, d) satisfies (i)-(iii) for all non-negative strictly Pareto dominated threat point in Π, despite the non-convexity of the choice set as established in Lemma 2.

Theorem 3 *Assume $\bar{\pi}_i(p)$ is twice continuously differentiable and concave over the finite range of prices at which demand is positive. Then, (Π, d) satisfies (i)-(iii) for all threat points $d \geq 0$ that are strictly Pareto dominated in Π.*

Proof. It suffices to show that $V(\Pi, d)$ is strictly convex for all threat point $d \geq 0$ that are strictly Pareto dominated in Π. To this end, notice first that from Lemma 2, the Pareto frontier of Π is given by the function $\pi_1 = h(p(\pi_2), \pi_2))$ over $[0, \bar{\pi}_2(\bar{p}_2)]$. It follows

$$V(\Pi, d) = \left\{ v \middle| \exists \pi_2 : \quad \begin{array}{l} \pi_2 > d_2, h(p(\pi_2), \pi_2) > d_1, \\ v_2 \leq \ln(\pi_2 - d_2), \\ v_1 \leq \ln(h(p(\pi_2), \pi_2) - d_1). \end{array} \right\}$$

and the Pareto frontier of $V(\Pi, d)$ is characterized by relations $v_2 = \ln(\pi_2 - d_2)$ and $v_1 = \ln(h(p(\pi_2), \pi_2) - d_1)$ for $\pi_2 > d_2$ and $h(p(\pi_2), \pi_2) > d_1$. Consequently, along the Pareto frontier

$$\frac{dv_2}{d\pi_2} = \frac{1}{\pi_2 - d_2}, \quad \frac{dv_1}{d\pi_2} = \frac{h_{\pi_2}}{h - d_1}.$$

Denote by $k(\pi_2) = dv_1/dv_2$ the slope of the Pareto frontier in (v_2, v_1) space. Since $v_2 = \ln(\pi_2 - d_2)$ alone the strictly individually rational portion of the Pareto frontier, to show the Pareto frontier of $V(\Pi, d)$ is strictly concave it suffices to show $k(\pi_2)$ is strictly decreasing in π_2. To this end, notice from the preceding two equations,

$$k(\pi_2) = \frac{\pi_2 - d_2}{h - d_1} h_{\pi_2}. \tag{19}$$

Differentiating both sides of (19) with respect to π_2 yields

$$k'(\pi_2) = \left(\frac{dv_1}{dv_2}\right)' = \frac{(h-d_1)[h_{\pi_2} + (\pi_2 - d_2)h_{\pi_2 p}p'] - (\pi_2 - d_2)(h_{\pi_2})^2}{(h-d_1)^2}. \tag{20}$$

Since $h > d_1$ and $\pi_2 > d_2$, it suffices to show $h_{\pi_2} + (\pi_2 - d_2)h_{\pi_2 p}p' < 0$. By (17)

$$h_{\pi_2}(p, \pi_2) = -\frac{p - c_1}{p - c_2}, \;\; h_{\pi_2 p}(p, \pi_2) = \frac{c_2 - c_1}{(p - c_2)^2} > 0, \tag{21}$$

and by (18),

$$h_{pp} = \frac{\partial^2 \bar{\pi}_1(p)}{\partial p^2} - \frac{2(c_2 - c_1)\pi_2}{(p - c_2)^3}, \;\; p'(\pi_2) = -\frac{h_{\pi_2 p}(p, \pi_2)}{h_{pp}(p, \pi_2)}. \tag{22}$$

Since $\partial^2 \bar{\pi}_1(p)/\partial p^2 < 0$, it follows from (21) and (22) that

$$h_{\pi_2 p}p' = -\frac{\frac{(c_2 - c_1)^2}{(p - c_2)^4}}{\frac{\partial^2 \bar{\pi}_1(p)}{\partial p^2}(p) - \frac{2(c_2 - c_1)\pi_2}{(p - c_2)^3}} \leq \frac{1}{\pi_2}\frac{c_2 - c_1}{2(p - c_2)}. \tag{23}$$

By Lemma 1, $p(\pi_2) \geq \bar{p}_1$. This together with $c_2 < \bar{p}_1$, $d_2 \geq 0$, and (23) implies

$$h_{\pi_2} + (\pi_2 - d_2)h_{\pi_2 p}p' = \frac{-2p + c_1 + c_2}{2(p - c_2)} - \frac{d_2}{\pi_2}\frac{c_2 - c_1}{2(p - c_2)} < 0. \tag{24}$$

The claim follows. ■

Theorem 3 implies that duopoly bargaining problems with constant marginal costs are included in the class as characterized in Theorem 2, even though they are not convex when the firms' marginal costs differ.[11]

In the context of duopoly bargaining, it would be interesting to compare the Nash bargaining solution with other bargaining solutions proposed in the literature. For instance, Conley and Wilkie (1996) use lotteries to convexify the choice set and then apply the Nash bargaining solution to the convexified set. They propose a bargaining solution as the intersection between the Pareto frontier of the original choice set and the line segment linking the threat point and the Nash bargaining solution on the

[11] Duopoly bargaining problems are convex whenever the firms' marginal costs are identical, since the Pareto profit frontiers are straight lines.

convexified set. Clearly, when the choice set is convex, their proposed solution coincides with the Nash bargaining solution. We compare the Nash bargaining solution with their extension for the class of duopoly bargaining problems analyzed above. For simplicity, we assume equal bargaining powers for the firms and $(0, 0)$ as the threat point.

Theorem 4 *In the duopoly bargaining problem with marginal costs $c_1 < c_2$ and concave profit functions $\bar{\pi}_1(p)$ and $\bar{\pi}_2(p)$, the firm with the higher marginal cost receives more profit under the Nash bargaining solution with equal bargaining powers and threat point (0, 0) than it does under the Conley-Wilkie's extended Nash bargaining solution with the same bargaining powers and threat point.*

Proof. With $\alpha = 1/2$ and the analysis of the Pareto frontier of Π in the previous section, the Nash bargaining solution solves

$$\begin{array}{l} \max\limits_{\pi_1 \geq 0, \pi_2 \geq 0} \pi_1 \pi_2 \\ \textit{subject to } \pi_1 = h(p(\pi_2), \pi_2) \end{array}$$

where $p(\pi_2) \in (\bar{p}_1, \bar{p}_2)$ is determined by (18). From the first-order condition,

$$\frac{\pi_2}{\pi_1} = \frac{p(\pi_2) - c_2}{p(\pi_2) - c_1}.$$

On the other hand, from (7) and (18),

$$\pi_i = \left(\frac{(p(\pi_2) - c_i)^2}{c_j - c_i} \right) \frac{\partial \bar{\pi}_j(p(\pi_2))}{\partial p}, i \neq j.$$

Consequently,

$$\frac{\pi_2}{\pi_1} = -\frac{\partial \bar{\pi}_2(p(\pi_2))/\partial p}{\partial \bar{\pi}_1(p(\pi_2))/\partial p}$$

and

$$\phi(p(\pi_2)) \equiv (p(\pi_2) - c_1)\frac{\partial \bar{\pi}_2(p(\pi_2))}{\partial p} + (p(\pi_2) - c_2)\frac{\partial \bar{\pi}_1(p(\pi_2))}{\partial p} = 0.$$

Next, the Pareto frontier of the convex hull of Π is given by the line segment between $(\bar{\pi}_1(\bar{p}_1), 0)$ and $(0, \bar{\pi}_2(\bar{p}_2))$. In this case, the Conley-Wilkie extended Nash bargaining solution is given by the intersection of this segment and the line represented by

$$\pi_2 = \frac{\bar{\pi}_2(\bar{p}_2)}{\bar{\pi}_1(\bar{p}_1)} \pi_1.$$

It thus suffices to show

$$-\frac{\partial\bar{\pi}_2(p^*)/\partial p}{\partial\bar{\pi}_1(p^*)/\partial p} > \frac{\bar{\pi}_2(\bar{p}_2)}{\bar{\pi}_1(\bar{p}_1)},$$

where p^* is the price that corresponds to the Nash bargaining solution with equal bargaining power. To this end, define

$$g(p) = (\bar{p}_1 - c_1)\frac{\partial\bar{\pi}_2(p)}{\partial p} + \frac{D(\bar{p}_2)}{D(\bar{p}_1)}(\bar{p}_2 - c_2)\frac{\partial\bar{\pi}_1(p)}{\partial p}$$

and

$$\Delta(p) = g(p) - \phi(p).$$

Note that

$$\Delta(\bar{p}_1) = 0$$

and

$$\Delta'(p) = \left(\frac{\bar{\pi}_2(\bar{p}_2)}{D(\bar{p}_1)} - p + c_2\right)\frac{\partial^2\bar{\pi}_1(p)}{\partial p^2} + (\bar{p}_1 - p)\frac{\partial^2\bar{\pi}_2(p)}{\partial p^2} - \frac{\partial\bar{\pi}_1(p)}{\partial p} - \frac{\partial\bar{\pi}_2(p)}{\partial p}. \quad (25)$$

We want to show $\Delta(p) > 0$ for all $p \in (\bar{p}_1, \bar{p}_2)$ which holds whenever $\Delta'(p) > 0$ for $p \in (\bar{p}_1, \bar{p}_2)$ such that $\Delta(p) = 0$. Indeed, $\Delta(p) = 0$ implies

$$\left(\frac{\bar{\pi}_2(\bar{p}_2)}{D(\bar{p}_1)} - p + c_2\right)\frac{\partial\bar{\pi}_1(p)}{\partial p} + (\bar{p}_1 - p)\frac{\partial\bar{\pi}_2(p)}{\partial p} = 0. \quad (26)$$

Since $\partial\bar{\pi}_1(p)/\partial p < 0$ and $\partial\bar{\pi}_2(p)/\partial p > 0$ on $(\bar{p}_1, \bar{p}_2)$, it must be from (26)

$$\frac{\bar{\pi}_2(\bar{p}_2)}{D(\bar{p}_1)} - p + c_2 < 0. \quad (27)$$

Furthermore, (26) implies

$$-\frac{\partial\bar{\pi}_1(p)}{\partial p} - \frac{\partial\bar{\pi}_2(p)}{\partial p} = \left(\frac{(\bar{p}_1 - c_2)D(\bar{p}_1) - \bar{\pi}_2}{(p - \bar{p}_1)D(\bar{p}_1)}\right)\frac{\partial\bar{\pi}_1(p)}{\partial p} \quad (28)$$

Putting (25), (27), and (28) together with the concavity of the monopoly profit functions,

$$\begin{aligned} \Delta'(p) = & \left(\frac{\bar{\pi}_2(\bar{p}_2)}{D(\bar{p}_1)} - p + c_2\right)\frac{\partial^2\bar{\pi}_1(p)}{\partial p^2} + (\bar{p}_1 - p)\frac{\partial^2\bar{\pi}_2(p)}{\partial p^2} \\ & + \left(\frac{(\bar{p}_1 - c_2)D(\bar{p}_1) - \bar{\pi}_2(\bar{p}_2)}{(p - \bar{p}_1)D(\bar{p}_1)}\right)\frac{\partial\bar{\pi}_1(p)}{\partial p} \\ > & \ 0. \end{aligned}$$

The claim follows. ■

Theorem 4 shows that in the presence of asymmetric non-convex choice sets and equal bargaining powers, the Nash bargaining solution (without convexification) favors the less efficient firm as compared to Conley-Wilkie extended Nash bargaining solution involving convexification via lotteries. In another word, the symmetric Nash bargaining solution without convexification tends to be more egalitarian (favoring the less efficient firm) than the one with convexification.

Another factor affecting the comparison of the two bargaining solutions is the relative bargaining powers. Fix a bargaining problem (S, d) with $d = (0, 0)$ and the Pareto frontier of the choice set being strictly decreasing and convex. Suppose that the Pareto frontier has a bigger intercept with player 1's payoff axis as in the duopoly bargaining problem. If $V(S, d)$ satisfies (iii), then it can be shown that there exists $\hat{\alpha} \in (0, 1)$ such that player 1 is better off under the Nash bargaining solution without convexification than under the Conley-Wilkie's extended Nash bargaining solution if and only if player 1's bargaining power is greater than $\hat{\alpha}$, and that the two solutions coincide when player 1's bargaining power is equal to $\hat{\alpha}$. Therefore, the Nash bargaining solution without convexification is more positively responsive to bargaining powers than the Conley-Wilkie's extended Nash bargaining solution.

5 Conclusion

Bargaining between duopolists results in bargaining problems with non-convexity. We have shown that these problems are included in the class, on which the Nash bargaining solution is uniquely characterized up to choices of bargaining powers by all Nash's axioms except for that of symmetry. We have shown that condition (iii) is the one that selects the largest class of bargaining problems satisfying basic conditions (i) and (ii) for which the unique characterization of Nash bargaining solution is valid.

Our characterization makes it possible to apply the Nash bargaining solution to a range of non-convex bargaining problems, including duopoly bargaining problems as special cases, without convexifying the choice sets via lotteries. For duopoly bargaining problems with asymmetric constant marginal costs, the Conley and Wilkie's (1996) extended Nash bargaining solution via lotteries is shown to favor more the efficient firm as compared with the Nash bargaining solution without convexifying the choices sets. Moreover, our characterization can be extended to the case of n-person bargaining problems.

Appendix: Proof of Theorem 1

Before preceding with the proof, we make the following observations. First, the logarithmic transformation $(v_1, v_2) = (\ln(u_1 - d_1), \ln(u_2 - d_2))$ is a homeomorphism from $S \cap \{(u_1, u_2) \,|\, u_1 > d_1, u_2 > d_2\}$ onto $V(S, d)$. It follows that the boundary $\partial V(S, d)$ of $V(S, d)$ must be the homeomorphic image of that of $S \cap \{(u_1, u_2) \,|\, u_1 > d_1, u_2 > d_2\}$; that is,

$$\partial V(S,d) = \left\{ \big(\ln(u_1 - d_1), \ln(u_2 - d_2)\big) \middle| u \in \partial S : \quad \begin{array}{l} u_1 > d_1, \\ u_2 > d_2. \end{array} \right\}.$$

Thus, $V(S, d)$ is closed. Second, because S is bounded above, so is $V(S, d)$, implying $\partial V(S, d) \neq \emptyset$. Third, for any $(u_1, u_2) \in S \cap \{(u_1, u_2) \,|\, u_1 > d_1, u_2 > d_2\}$, the intervals $((d_1, u_2), (u_1, u_2)]$ and $((u_1, d_2), (u_1, u_2)]$ are transformed into $((-\infty, \ln(u_2 - d_2)), (\ln(u_1 - d_1), \ln(u_2 - d_2))]$ and $((\ln(u_1 - d_1), -\infty), (\ln(u_1 - d_1), \ln(u_2 - d_2))]$ in $V(S, d)$, respectively. As a result, $V(S, d) = V(S, d) - \Re^2_+$. Thus, by moving the origin properly, we can assume without loss of generality that for some constant $h > 0$,

$$V(S, d) \subset -(h, h) - \Re^2_+, \qquad (A1)$$

so that the closed cone generated by $V(S, d)$ and any line passing through the origin with a normal vector in $\Re^2_+$ have no point in common other than the origin.

Lemma 3 *Let $U \subseteq \Re^2$ be closed, convex, and comprehensive (i.e. $U = U - \Re^2_+$). If the boundary ∂U of U is non-empty, then $U = \partial U - \Re^2_+$.*

Proof. Notice $\partial U - \Re^2_+ \subseteq U - \Re^2_+ = U$ is automatic. Conversely, let u be any interior point of U. Then, there exists a number $\epsilon > 0$ such that $(u_i + \delta, u_{-i}) \in U$ for all i and for all $\delta \in [0, \epsilon)$.[12] Set $\delta_i = \sup\{\delta | (u_i + \delta, u_{-i}) \in U\}$ for $i = 1, 2$. Since U is convex and comprehensive, $\delta_i < \infty$ for at least one i, for otherwise $U = \Re^2$ which contradicts $\partial U \neq \emptyset$. Assume without loss of generality $\delta_1 < \infty$. Then, $u^1 = (u_1 + \delta_1, u_{-1}) \in \partial U$ because U is closed. It follows that $u = u^1 - (\delta_1, 0_{-1}) \in \partial U - \Re^2_+$. This shows $U \subseteq \partial U - \Re^2_+$. ■

Proof of Theorem 1: The sufficiency of Theorem 1 is trivial. To prove the necessity, we only need to show that $V(S, d)$ (which we simply denote by V) is closed convex because the necessity of the last part of the condition is obvious. To

[12] Here $(u_i + \delta, u_{-i})$ denotes the vector in $\Re^n$ whose j-th coordinate is u_j for $j \neq i$ and i-th coordinate is $u_i + \delta$.

this end, we first show that the representation of the elements in the convex hull of V by elements in V as guaranteed by the Carathéodory theorem can be extended to the closure, $\tilde{V}$, of the convex hull under the assumed conditions. This in turn implies $V = \tilde{V}$. Since the closure of a convex set is convex, it follows from the preceding equality that V is closed convex.

Notice that $\tilde{V}$ also satisfies $\tilde{V} = \tilde{V} - \Re^2_+$, and from Lemma 3, $\tilde{V} = \partial\tilde{V} - \Re^2_+$. For any boundary point $\tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in \partial\tilde{V}$ of $\tilde{V}$, it follows from the separation theorem that there exists $a \in \Re^2$ such that

$$a \cdot \tilde{v} = \max_{v \in \tilde{V}} a \cdot v. \tag{A2}$$

Since V is comprehensive, $a \in \Re^2_+$. Without loss of generality, we may take $a = (\alpha, 1-\alpha)$ for some $\alpha \in [0,1]$.

There exist sequences $\{v^k(n)\}_n$ in V, $k = 1,2,3$, $\{\lambda(n)\}_n$ in $\Re^3_+$, and $\{\epsilon(n)\}_n$ in $\Re$ such that[13]

$$\tilde{v} = \lambda_1(n)v^1(n) + \lambda_2(n)v^2(n) + \lambda_3(n)v^3(n) + \varepsilon(n), \tag{A3}$$

$$\lambda_1(n) + \lambda_2(n) + \lambda_3(n) = 1, \tag{A4}$$

$$\varepsilon(n) \to 0. \tag{A5}$$

By $(A4)$, we may assume the sequence

$$\lambda(n) \to \bar{\lambda} \in \Re^3_+ \Rightarrow \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 = 1. \tag{A6}$$

By $(A1)$, the sequences $\{\lambda_k(n)v^k(n)\}_n$, $k = 1,2,3$, are bounded above, and by $(A3)$, $(A5)$, and $(A6)$, they are also bounded below. Hence, we may assume

$$\lambda_k(n)v^k(n) \to \bar{w}^k \in \Re^2,\ k = 1,2,3 \Rightarrow \tilde{v} = \bar{w}^1 + \bar{w}^2 + \bar{w}^3. \tag{A7}$$

Suppose first $\bar{\lambda}_k > 0$. Then $\{v^k(n)\}_n$ converges to its limit in V because V is closed. In this case, from $(A3)$, $(A5)$, and $(A6)$, we have

$$\tilde{v} = \bar{\lambda}_1\bar{v}^1 + \bar{\lambda}_2\bar{v}^2 + \bar{\lambda}_3\bar{v}^3,\ v^k(n) \to \bar{v}^k \in V(S,d),\ k = 1,2,3, \tag{A8}$$

[13] The reason is as follows. Since $\tilde{v}$ is in $\tilde{V}$, there exists a sequence $\{\tilde{v}(n)\}_n$ in the convex hull of V such that $\tilde{v}(n) \to \tilde{v}$. By Carathéodory Theorem, for each n there exist $v^1(n), v^2(n), v^3(n)$ in V and $\lambda(n) \in \Re^3_+$ satisfying $(A4)$ such that $\tilde{v}(n) = \lambda_1(n)v^1(n) + \lambda_2(n)v^2(n) + \lambda_3(n)v^3(n)$. Finally, $(A3)$ and $A(5)$ are established by setting $\epsilon(n) = \tilde{v} - \tilde{v}(n)$ for all n.

where $\bar{v}^k \in V$ is the limit of $\{v^k(n)\}_n$ for $k = 1, 2, 3$. Suppose now $\bar{\lambda}_k = 0$ for some k but $\bar{\lambda}_{k'} > 0$ for $k' \neq k$. Without loss of generality, assume $\bar{\lambda}_1 = 0$. In this case,

$$\tilde{v} = \bar{w}^1 + \bar{\lambda}_2 \bar{v}^2 + \bar{\lambda}_3 \bar{v}^3,\ \bar{\lambda}_2, \bar{\lambda}_3 > 0,\ \bar{\lambda}_2 + \bar{\lambda}_3 = 1, \tag{A9}$$

where $\bar{w}^1$ is the limit of $\{\lambda_k(n)v^k(n)\}_n$ as in $(A7)$ and $\bar{v}^k \in V$ is the limit of $\{v^k(n)\}_n$ as in $(A8)$ for $k = 2, 3$. By $(A2)$ and $A(7)$,

$$a \cdot \bar{w}^1 = \lim_{n\to\infty} a \cdot \lambda_1(n)v^1(n) = \lim_{n\to\infty} \lambda_1(n)a \cdot v^1(n) \leq \lim_{n\to\infty} \bar{\lambda}_1 a \cdot \bar{v} = 0.$$

On the other hand, by $(A2)$ and $(A9)$,

$$a \cdot \bar{w}^1 = a \cdot \bar{v} - (\bar{\lambda}^2(a \cdot \bar{v}^2) + \bar{\lambda}^3(a \cdot \bar{v}^3)) \geq 0.$$

It follows that $a \cdot \bar{w}^1 = 0$. This shows that $\bar{w}^1$ lies on the line having normal vector normal vector $(\alpha, 1 - \alpha) \in \Re^2_+$ and passing though the origin. As the limit of $\{\lambda_1(n)v^1(n)\}$, $\bar{w}^1$ is in the closed cone generated by V. Thus, by $(A1)$, we must have $\bar{w}^1 = 0$. Suppose finally $\bar{\lambda}_k = 1$ for some k. Assume without loss of generality $\bar{\lambda}_3 \neq 0$. In this case, a similar proof as before shows $\bar{w}^1 = \bar{w}^2 = 0$.

By letting $\bar{v}^k$ be arbitrary element in V when $\bar{\lambda}_k = 0$, the preceding analysis establishes

$$\tilde{v} = \bar{\lambda}_1 \bar{v}^1 + \bar{\lambda}_2 \bar{v}^2 + \bar{\lambda}_3 \bar{v}^3. \tag{A10}$$

When $\bar{\lambda}_k > 0$, $(A2)$ and $(A10)$ together imply

$$a \cdot \bar{v}^k = \max_{v\in V} a \cdot v,$$

which implies $\bar{v}^k \in \partial V$. If $a = (\alpha, 1 - \alpha) \in \Re^2_{++}$, then the above equality and the assumption that (6) has a unique solution imply $\bar{v}^k = \tilde{v}$ whenever $\bar{\lambda}_k > 0$. Thus, in this case, $\tilde{v} \in \partial V \subset V$.

If $a = (1, 0)$, then

$$a \cdot \tilde{v} = \tilde{v}_1 = a \cdot \bar{v}^k = \bar{v}^k_1.$$

Assume without loss of generality $\bar{\lambda}_1 > 0$ and $\bar{v}^1_2 = \max\{\bar{v}^k_2 \mid \bar{\lambda}_k > 0\}$. Then, by $(A10)$,

$$\tilde{v}_2 = \bar{\lambda}_1 \bar{v}^1_2 + \bar{\lambda}_2 \bar{v}^2_2 + \bar{\lambda}_3 \bar{v}^3_2 \leq \bar{v}^1_2.$$

Consequently, in this case, $\tilde{v} \in \bar{v}^1 - \Re^2_+ \subset V$. If $a = (0, 1)$, then a similar proof establishes $\tilde{v} \in V$.

In summary, we have shown $\tilde{v} \in V$ for any $\tilde{v} \in \partial\tilde{V}$. Thus, by Lemma 3,

$$V \subseteq \tilde{V} = \partial\tilde{V} - \Re^2_+ \subseteq V - \Re^2_+ = V.$$

This concludes $V = \tilde{V}$; hence, V is closed convex. ■

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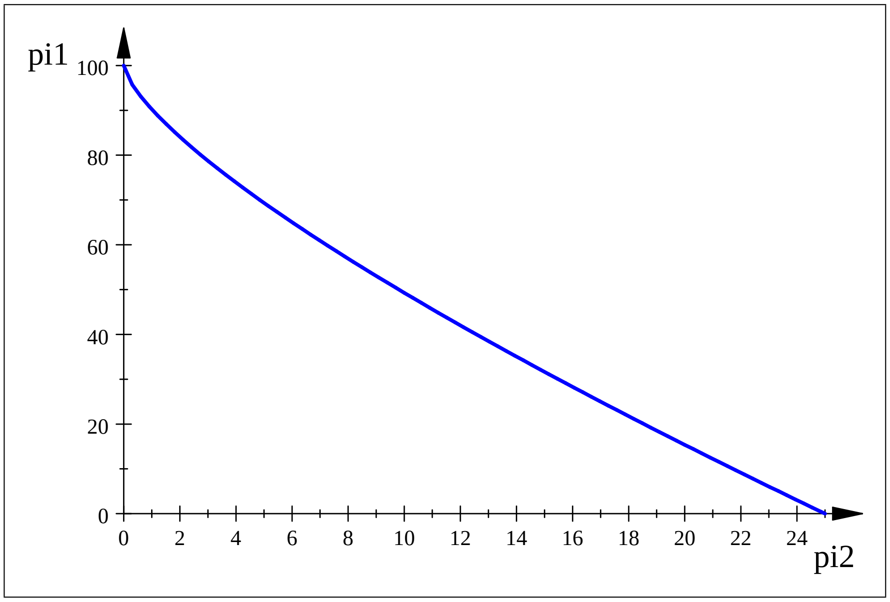

Figure 1 (a): The Pareto Profit Frontier for the Duopoly Problem
$(\pi_1 = 100 - \pi_2 - 3 \cdot (5\pi_2)^{\frac{2}{3}})$

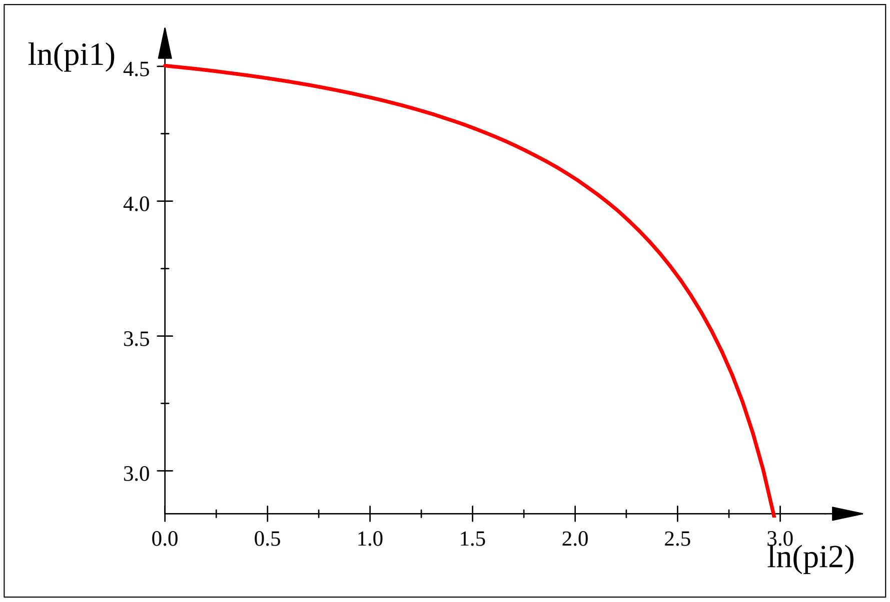

Figure 1 (b): Logarithmic Transformation of the Duopoly Profit Set

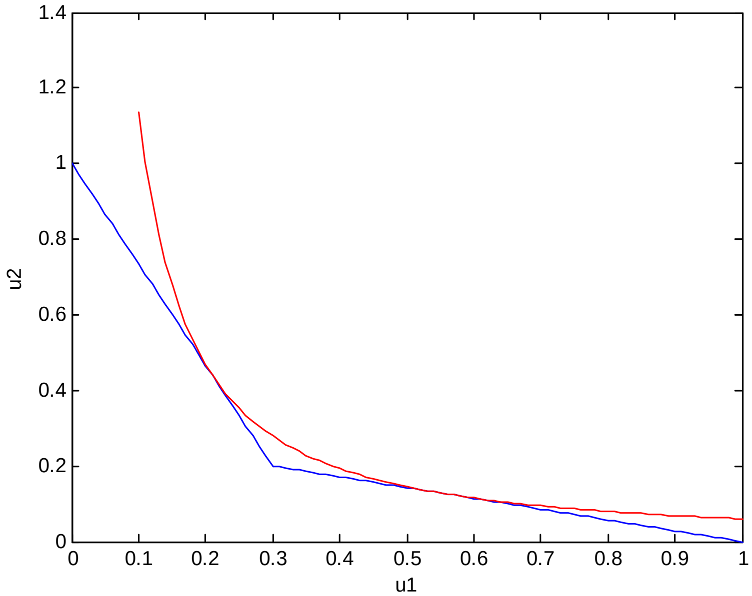

Figure 2: Multiple Solutions when the Bargaining Set is Non-Convex