

Partial Ordering Methods for Stationary Markovian Equilibrium in Overlapping Generations Models with Stochastic Production*

Jaime Erikson
SUNY-Oneonta

Olivier F. Morand
University of Connecticut

Kevin L. Reffett
Arizona State University

March 2009

Abstract

Using an order-theoretic approach, we derive sufficient conditions for the existence, characterization, and computation of recursive equilibrium (i.e., minimal state space Markovian equilibrium) and stationary Markov equilibrium (SME) for a large class of stochastic overlapping generations models. In contrast to previous work, our focus is exclusively on constructive methods. Our existence results are obtained for more general reduced-form stochastic production technologies, admissible public policy distortions, social security policies, monetary equilibrium, and production nonconvexities than in the present literature. We provide a collection of monotone iterative algorithms that converge uniformly to extremal recursive equilibrium (RE). We distinguish conditions under which there exist semicontinuous or Lipschitz continuous RE. We also provide the first results in the literature on the existence and computation of RE and SME for economies where capital income is not monotone. We conclude by showing how the methods can be extended to some models with elastic labor supply, general income processes, and we develop some equilibrium comparative statics results.

1 Introduction

Since its introduction in the seminal work of Samuelson [49] and Diamond [24], the overlapping generations (OLG) model has been workhorse in many areas

*We thank Elena Antoniadou, Robert Becker, Manjira Datta, Seppo Heikkilä, Manuel Santos, Jayaram Sethuraman, Yiannis Vailakis, Itzhak Zilcha, and especially John Stachurski for very helpful discussions on topics related to this paper. All mistakes are our own.

of applied dynamic general equilibrium theory.¹ For example, OLG models have been used extensively in macroeconomics and public finance to address theoretical and applied issues in intergenerational risk sharing and social security, human capital formation and public education, optimal taxation, economic growth and infrastructure or environmental degradation, macroeconomic fluctuations and equilibrium growth, public finance, and monetary economics. With a few notable exceptions, a majority of this applied work has focused on obtaining numerical characterizations of Markovian equilibrium. Some of this numerical work has been based on methods that seek to compute approximate solutions directly for recursive equilibrium decision rules within some class of functions, while others have focused on more indirect methods that first compute the set of sustainable equilibrium values via an operator on spaces of correspondences, and then they back out numerical characterizations of equilibrium that sustain those values. The latter methods are based on the work of Kydland and Prescott [?] and Abreu, Pearce, and Stacchetti [2], and are in the literature typically referred to as "APS" methods. This latter method is often used to prove existence of "generalized Markov equilibrium" in the sense of Duffie et.al. [26].² Finally, it is important to mention some negative results concerning the existence of recursive equilibrium (RE) that have been reported for some versions of the stochastic OLG model (e.g., see [35], as well as the related results in [12][13]).

Unfortunately, using existing methods, few sharp characterizations of RE via iterative procedures have been provided for even the the simplest case of stochastic OLG models with stochastic production. The important series of papers by Wang [56] [57] do obtain significant results on existence and uniqueness of continuous recursive equilibrium, but he requires rather strong monotonicity conditions on capital income associated with equilibrium production processes, and does not allow for equilibrium public policy.³ Morand and Reffett [42] do weaken Wang's conditions for existence of RE, but only for economies where the primitive data is consistent with RE that have investment processes that are monotone in endogeneous state variables. Finally, in most applied work using stochastic OLG models with production, researchers want to allow for elastic labor supply (e.g., studies of lifecycle labor supply decisions). It is therefore worth mentioning that none of the existing literature on conditions for existence of RE for stochastic OLG models are known to apply for models with elastic labor supply. Further, additional negative results associated with sta-

¹The OLG model was devised by Maurice Allais in 1947.

²By a recursive equilibrium (RE)), we mean a Markovian equilibrium decision process whose domain is exactly the domain of the initial states of the environment. A generalized Markovian equilibrium also can exist on such a partitioning of the state space for the sequential equilibrium, but only through compositions with additional "pseudo-state variables" such marginal utility of consumption, value functions, etc.

RE have been the focus of analysis in many dynamic macroeconomic models since the pioneering work of Lucas and Prescott [38], Prescott and Mehra [45], and Stokey, Lucas, and Prescott [52]. For work on stochastic OLG models using the generalized Markovian equilibrium approach via APS, see Miao and Santos [39].

³See also the conditions studied in the recent papers of Hausenchild.[31], Forni [27], and Matsuyama [37] which have generalized these results in some notable dimensions.

bility of Markovian equilibrium dynamics for OLG models have been recently reported in the work by Cazzavillan and Pintus [10] and Lloyd-Braga, Nourrey, and Venditt [36]. So what would be helpful is to discover conditions for models with elastic labor supply that have monotone investment processes.⁴

The situation is not much more encouraging using APS type methods. Although these methods can provide sufficient conditions under which equilibrium value correspondence are nonempty (and, therefore, for the existence of sequential and/or Markovian equilibrium), unfortunately they provide little rigorous characterization of the set of equilibrium decision rules that support the set of sustainable equilibrium values. For example, Miao and Santos [39] develop a APS approach to study the existence of Markov equilibrium for stochastic OLG models. The characterization of equilibrium they obtain is the existence of a measurable bounded selection from the equilibrium correspondence that sustains some element of the equilibrium value set. From a numerical and theoretical perspective, such statement questions are difficult to interpret. First, they do not allow one to characterize the long-run properties of the models (i.e., existence of Stationary Markov equilibrium). Second, other theoretical questions are difficult to address (e.g., equilibrium comparative statics). Finally, the mathematical foundation of constructing approximate solutions via a APS method has yet to be systematically addressed in the literature. For example, although convergence results for approximate solutions to strategic dynamic programming problems in pointwise convergence for value sets in Hausdorff topologies on powersets are known, how to translate such convergence statements into characterizations of the accuracy of approximate solutions for actual equilibrium that sustain such values is less clear. (See, for example, Judd, Yeltekin, and Conklin [33] and Sleet and Yeltekin [50] for the status of theoretical results on approximate solutions via APS).

This paper presents a new systematic approach for constructing RE for OLG models with stochastic production. Relative to the question of existence and computation of RE, we provide many new results. For example, we prove existence of RE for economies with very general technologies (in reduced-form) as well as policy distortions relative to the existing literature (e.g., Wang [56] and Morand and Reffett [42]). In particular, we allow for nonclassical production, as well as any policy distortion with commitment that can be written as an isotone decomposition. Using well-known arguments, our methods can be applied to economies with public infrastructure, valued fiat money, and monopolistic competition. We allow for cases of elastic labor supply, and very general income processes. Finally, in principle, our methods can be extended to studying sequential equilibria; which proves very helpful for optimal taxation problems

⁴Monotone investment processes in equilibrium would eliminate many of the indeterminacy problems that have been noted in the literature. As Wang shows in his work (e.g., Wang [57]), such indeterminacies can create difficulties obtaining sharp results on characterizing selections from the set of of generalized Markovian equilibrium constructed via the methods of Duffie et. al. [26]. Such problems leads one to seek a more general conditions under which one can guarantee the existence of generalized Markov equilibrium using KP-APS type methods for competitive economies (e.g., see [39]).

without commitment (e.g., as in Phelan and Stacchetti [46]).⁵ Given that equilibrium in OLG models are well-known to be suboptimal (even without such additional market frictions), our methods rely exclusively on an equilibrium Euler equation methods, and make no appeals to Negishi-like methods.⁶

As our methods are constructive, we are able to provide a direct connection between the characterization and the computation of RE, as well as new results on equilibrium comparative statics in the deep parameters of the economy. We are also able to study directly the question of computing (and characterizing) the structure probability measures that describe the Stationary Markov equilibrium (SME) that is induced by a particular RE. Therefore, our approach is quite different in methodological approach than that advocated in [56][57], [26], [31], [36], [12][13], and other important papers in this literature. In particular, our methods should be viewed as extensions of the recursive methods first put forth in Lucas and Prescott [38] and Prescott and Mehra [45], only applied to stochastic OLG models. As in this latter literature, we employ the “little k, big K” recursive equilibrium formulation standard in recursive competitive equilibrium theory (e.g., Lucas and Prescott [38]) so we can to isolate the continuation structure of a *particular* equilibrium from the APS equilibrium correspondence. This allows us to sharpen the characterization of this particular equilibrium (as opposed to what is obtained using APS). We then use an order-theoretic fixed point approach for computing RE as in the work of, for example, Coleman [14][15] Greenwood and Huffman [30], and Datta, Mirman, and Reffett [16], and Mirman, Morand, and Reffett [40], which allows us to obtain a unified approach to equilibrium selections (both theoretically and numerically) to studying this subset of sequential equilibrium computing via APS methods (e.g., Miao and Santos [39]).

In some cases, under very common assumptions used in the applied literature (e.g., capital income monotonicity and strictly concave utility), but under more general conditions on production and public policy, we prove existence of unique strictly positive locally Lipschitz continuous recursive equilibrium within a collection of equicontinuous decision rules. We show how to compute these decision rules via successive approximation. Equally as important, we are able to provide the first results in the literature on existence and computation of recursive equilibrium in the *absence* of capital monotonicity, for economies with very general income processes and production, as elastic labor supply. The case of elastic labor supply is shown to be a direct extension of our methods. For more general income processes, we develop a mixed-monotone expanded state space approach first discussed in Datta et. al. [18] for infinite horizon economies. This method allows one to prove existence and computation results for semi-continuous RE under very weak conditions on household income processes.

⁵ Although we do not pursue these questions in this paper, our methods prove quite powerful in studying subgame perfect equilibrium for competitive economies with public policy. We report such results in future work, as well as in Reffett [47].

⁶ In many recent papers on OLG models, researchers develop methods based on planning problems (e.g., Lloyd-Braga, Nourry, and Venditti [36]). Such methods seem inappropriate for the study of competitive equilibrium in economies where the welfare theorems fail.

⁷ In particular, we show for this case, there exists least and greatest upper-semi continuous and lower-semi continuous isotone recursive equilibrium that can be computed by successive approximation schemes converging in order (and in some cases topology) to the extremal RE (and their associated extremal SME).

There are a number of important results on the existence and characterization of Markovian equilibrium in OLG models with production that deserve mentioning. Galor and Ryder [29] obtained some of the first results concerning the existence, uniqueness, and stability of steady state for a deterministic setup with classical production. The work of Galor and Ryder [29] has been extended to a class of OLG models with stochastic “classical” production (i.e., a production setting identical to that used in the work of Brock and Mirman [9]) in a series of papers by Wang [56] [57] and Hauenschild [31]. For economies with identical and independently distributed (iid) production shocks and classical production, Wang [56] obtains sufficient conditions for the existence of a globally unique and stable non-trivial SME using a topological approach. Wang [57] notes that problems arise in economies without capital income monotonicity because of multiple equilibria and equilibrium indeterminacies. Hauenschild [31] extends some of Wang’s results to economies with pay-as-you-go financed social security. Finally Wang [57] provides some existence results along the lines of Duffie et. al. [26] for economies with stochastic classical production with Markov shocks. A key implication of Wang’s work (along with the storyline in the paper of Kubler and Polemarchakis [35]) is that when sunspots and multiplicities are considered, the state space that is used to construct SME can potentially become very large.⁸ As such solutions are difficult to compute, this provides one motivation for identifying conditions under which recursive equilibrium exist.

The paper is organized as follows. In section two, we provide some preliminary mathematical results useful in the sequel. In section three, we describe the set of economies we initially study. Section four studies the set of RE for these economies, and presents successive approximation algorithms that can construct extremal Markovian equilibrium investment decisions. In section five, we prove the existence and construction of extremal stationary Markov equilibrium (as invariant distributions). Section six discusses extensions of our result to the case

⁷We should note, the application of mixed-monotone decompositions is quite different in the present paper than in Datta et. al. [18]. In the present paper, this approach is applied to computing semi-continuous RE; not continuous.

⁸There are numerous pioneering papers in OLGs models which we have not mentioned directly in our remarks. Aside from the paper by Diamond, see Balasko and Shell [3][4], Okuno and Zilcha [44], Zilcha [58], Dechert and Yamamoto [23], Demange and Laroque [20][21], and Chattopadhyay and Gottardi [11], and Barbie, Hagedorn, and Kaul [5]). Many of these papers discuss the important question of how to define dynamic efficiency in an OLG environment, and often these papers have a much more complicated structure (e.g., many assets each period and incomplete markets, many goods each period, etc.)

We note that although some of these papers study economies with or without stochastic production, none of this related work addresses the questions addressed in this paper (i.e., the construction and usefulness of monotone methods in the study of MEDPs and SME in OLG models production subjected to i.i.d. shocks from both a theoretical and numerical point of view).

of elastic labor supply case, and general income processes. In this section, we also provide conditions for the uniqueness of RE, as well as providing conditions for monotone comparative statics for the set of RE.

2 Some Mathematical Prerequisites

We begin by providing a set of useful mathematical results that we shall use later in the paper. Throughout the paper, we consider the compact intervals $X = [0, k_{\max}] \subset \mathbb{R}$, where $k_{\max} > 0$, $Z = [z_{\min}, z_{\max}]$ with $0 < z_{\min} \leq z_{\max}$ (although many results can be generalized to compact subsets of \mathbb{R}_+^n). Deenote by $X^* = X \setminus \{0\}$, $S = X \times Z$, $S^* = X^* \times Z$, and let $\mathcal{B}(S)$ be the Borel algebra associated with S . All compact subsets are endowed with the relative pointwise partial order \leq and the usual topology on \mathbb{R}^n .

2.1 Definitions

A *partial order* \leq on a set X is a reflexive, transitive, and antisymmetric relation, in which case (X, \leq) is a *partially ordered set*, or *poset*. An *upper (resp. lower) bound* of $A \subset X$ is an element $u \in X$ (resp. $v \in X$) such that $\forall x \in A$, $u \geq x$ (resp. $v \leq x$). If x^u is an upper bound of A , and $x^u \leq y$ for any other upper bound y of A , then x^u is the least upper bound or supremum of A , denoted $\vee A$ (similarly one can define the greatest lower bound or infimum of A , denoted $\wedge A$). A *chain* C is a subset of X that can be linearly ordered, that is, $\forall (p, p') \in C$ $p \leq p'$ or $p' \leq p$. The poset (X, \leq) is a *lattice* if $\forall (x, x') \in X^2$, both $\wedge(x, x')$ and $\vee(x, x')$ exist and belong to X . If $B \subset X$ and (B, \leq) is a lattice, then we say that B is a *sublattice* of X . A lattice (X, \leq) is *complete* if $\vee B$ and $\wedge B$ exist for any $B \subset X$. If $X_1 \subset X$ is a sublattice, and X_1 is complete in its relative partial order to X , then X_1 is *subcomplete*. If every chain C in X is complete, then X is *chain complete*. If every chain C_c in X is countable and complete, then X is *countable chain complete*. Finally, a countable chain $\{x_n\}$ with $x_i \leq x_j$ order converges to x iff $\vee\{x_n\} = x$.

Let (X, \geq_X) and (Y, \geq_Y) be posets. A *mapping* is a relational statement between two spaces, say X and Y . We shall consider both “point-to-point” and “point-to-set” mappings. In the case of a “point-to-point” mapping, we refer to the mapping as a *function* (equivalently, as an *operator*). A function $f : X \rightarrow Y$ is *isotone* on X if it is “order-preserving”, i.e., $f(x') \geq_Y f(x)$, when $x' \geq_X x$, for $x, x' \in X$. If $f(x') \geq_Y f(x)$, $f(x') \neq f(x)$ when $x' \geq_X x$ for $x \neq x'$, for $x', x \in X$, we say the function f is *increasing*. If $f(x') >_Y f(x)$, when $x' >_X x$, for $x', x \in X$, we say the function f is *strictly increasing*. Dually, a mapping $f(x)$ is *antitone* (or, *order-reversing*) if $f(x) \geq_Y f(x')$ if $x' \geq_X x$. If $f(x) \geq_Y f(x')$, $f(x) \neq f(x')$, when $x' \geq_X x$, $x \neq x'$, $x, x' \in X$, we say the function f is *decreasing*. If $f(x) >_Y f(x')$ when $x' >_X x$, $x, x' \in X$, we say the function f is *strictly decreasing*. A function that is either isotone or antitone is *monotone*. The mapping $f(x)$ is a *self-mapping* or *transformation* of X if $f : X \rightarrow X$. In this case, we refer to the set X as the *transformation space* of $f(x)$.

Let X and Y be partially ordered spaces. Notions of monotonicity are available for multifunctions or correspondences. By a *correspondence or multifunction*, we always refer to a nonempty-valued “point-to-set” mapping $M : X \rightarrow 2^Y \setminus \emptyset$. We say a correspondence or multifunction is *ascending* in the set relation S (denoted by \geq_S) if $M(x') \geq_S M(x)$, when $x' \geq_X x$ where (X, \geq_X) is a partially ordered space. If this set relation \geq_S induces a partial order on the powerset 2^Y (or, perhaps, $2^Y \setminus \emptyset$), we refer to the ascending correspondence as an *isotone correspondence*. To make concrete the monotonicity properties of an ascending correspondence, let X and Y be lattices, $A, B \in 2^Y \setminus \emptyset$. Define the *Veinott-Strong Set Order* \geq_s on $2^Y \setminus \emptyset$ as follows: $A \geq_s B$, if for any $a \in A$, $b \in B$, $a \wedge b \in B$ and $a \vee b \in A$. Notice the set relation \geq_s is compatible with pointwise comparisons, and, therefore, it is useful when studying questions pertaining to isotone selections.

2.2 Useful Results

We first begin with discussing complete lattices. To prove a partially ordered set is a complete lattice sometimes requires much less work than the definition of completeness would have us believe, as demonstrated in the following well-known result we shall often apply (e.g., Davey and Priestley [19], Theorem 2.31).

Theorem 1 *The poset (P, \leq) is a complete lattice if and only if P has a top element and for any $P' \subset P$, $\bigwedge P'$ exists (in P).*

Given any bounded function $w : S \rightarrow \mathbb{R}^+$, the set $W = \{h : S \rightarrow \mathbb{R}^+, 0 \leq h \leq w\}$ is the set of “bounded functions”) endowed with the pointwise partial order \leq is clearly a complete lattice. If w is isotone (i.e., non-decreasing in its arguments), then the set $H = \{h \in W, h \text{ isotone}\}$ is subcomplete in W . If in addition, w is continuous in k for each $z \in Z$ (in the usual topology on \mathbb{R}), the set $H^u = \{h \in H, h \text{ upper semicontinuous in } k \in X \text{ for each } z \in Z\}$ is subcomplete in H , as established in the following Proposition.⁹

Proposition 2 *The poset (H^u, \leq) is a complete lattice. In addition any $h \in H^u$ is measurable.*

Proof. Given any $B \subset H^u$, denote $g(s) = \inf_{h \in B} h(s)$. Clearly $0 \leq g \leq w$, g is isotone, and $g(\cdot, z)$ is usc for any given z . Thus g is an upper bound of B , and it is easy to see that it is the least upper bound. Consequently, $g = \bigvee B$. Since w is the top element of H_k^u , the first result follows from the previous theorem. Next, since X is a compact interval of \mathbb{R} , denote by $\{x_0, x_1, \dots\}$ a

⁹Our results also extend to the set of bounded isotone lower semicontinuous functions, also a complete sublattice of H .

countable dense subset of X . Given any $\alpha \in \mathbb{R}$, we claim that:

$$\begin{aligned} & \{s \in S, h(s) \leq \alpha\} \\ & = \\ & \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty}]x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\}. \end{aligned}$$

This property implies that h is measurable: Indeed, since h is increasing in z for each k , it is $\mathcal{B}(Z)$ -measurable for each k which implies that $\{z \in Z, h(x_m, z) < \alpha + 1/n\} \in \mathcal{B}(Z)$, and that $\{s \in S, h(s) \leq \alpha\} \in \mathcal{B}(S)$. We prove now the stated claim. First, consider (k, z) such that $h(k, z) \leq \alpha$. Such h being usc and increasing in k (for each z), it is necessarily right continuous at k , and we have that:

$$\forall n \in \mathbb{N}, \exists m \text{ such that } x_m - 1/n < k < x_m \text{ and } h(x_m, z) < \alpha + 1/n.$$

Thus:

$$\forall n \in \mathbb{N}, \exists m \text{ such that } (k, z) \in]x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\},$$

which implies that:

$$\forall n \in \mathbb{N}, (k, z) \in \bigcup_{m=0}^{\infty}]x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\},$$

and therefore that:

$$(k, z) \in \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty}]x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\}.$$

Reciprocally, suppose that for all $n \in \mathbb{N}$, (k, z) belongs to $\bigcup_{m=0}^{\infty}]x_m - 1/n, x_m] \times \{z \in Z, h(x_m, z) < \alpha + 1/n\}$. This implies that for all n , there exists $m(n)$ such that $k \in]x_{m(n)} - 1/n, x_{m(n)}]$ and $h(x_{m(n)}, z) < \alpha + 1/n$. By construction the sequence $\{x_{m(1)}, x_{m(2)}, \dots\}$ converges to k and $x_{m(n)} \geq k$, so by continuity from the right at k of $h(\cdot, z)$, $h(x_{m(n)}, z)$ converges to $h(k, z)$ and necessarily $h(k, z) \leq \alpha$. Finally, we note that a similar result holds for the subset of H of lsc functions, since it can be shown that:

$$\begin{aligned} & \{(k, z) \in S, h(k, z) \geq \alpha\} \\ & = \\ & \bigcap_{n=1}^{\infty} \bigcup_{m=0}^{\infty} [x_m, x_m + 1/n[\times \{z \in Z, h(x_m, z) > \alpha - 1/n\}. \end{aligned}$$

■

Remark: It is important to note that for any $P \subset (H^u, \leq)$, $\wedge_{H^u} P$ coincide with $\wedge_H P$ (the pointwise inf) but that $\vee_{H^u} P$ and $\vee_H P$ (the pointwise sup) may differ.

A second class of sets of interest for this paper are spaces of probability measures. We use such spaces to study the existence of stationary Markov equilibria. Denote by $\Lambda(X, \mathcal{B}(X))$ the set of probability measures defined on the measurable space $(X, \mathcal{B}(X))$, and endow $\Lambda(X, \mathcal{B}(X))$ with the partial order of first stochastic dominance order \geq_s :

$$\mu \geq_s \mu' \text{ if } \int_X f(k)\mu(dk) \geq \int_X f(k)\mu'(dk),$$

for every isotone, and bounded function $f : X \rightarrow \mathbb{R}_+$, in which case we say that μ stochastically dominates μ' . We prove the following useful result:

Proposition 3 *The poset $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ is a complete lattice with minimal and maximal elements δ_0 and $\delta_{k_{\max}}$.*

Proof. It is easy to show that the set $\mathbb{D}(X)$ of functions $F : X \rightarrow [0, 1]$, that are increasing, upper semicontinuous, and satisfy $F(b) = 1$, is a complete lattice when endowed with the pointwise order. $\mathbb{D}(X)$ has maximal and minimal elements (respectively, the function $F(k) = 1$ for all $k \in X$, and the function $G(k) = 1$ if $k = b$ otherwise $G(k) = 0$), and is in fact the set of probability distributions over the compact set X . It is well-known that to any probability measure $\mu \in \Lambda(X, \mathcal{B}(X))$ corresponds a unique distribution function $F_\mu \in \mathbb{D}(X)$ and vice versa, and $\mu \geq_s \mu'$ is equivalent to $F_\mu \leq F_{\mu'}$ (see, for instance, Stokey & al. [52]).¹⁰ $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ is thus isomorphic to $(\mathbb{D}(X), \leq)$, and is therefore a complete lattice with minimal element the singular probability measure δ_0 , and maximal element the singular probability measure $\delta_{k_{\max}}$. ■

Notice, when the space $\Lambda(X, \mathcal{B}(X))$ is endowed with the weak topology, a sequence of probability measures $\{\mu_n\}$ in $\Lambda(X, \mathcal{B}(X))$ is said to weakly converge to $\mu \in \Lambda(X, \mathcal{B}(X))$ if for all continuous functions $f : X \rightarrow \mathbb{R}$:

$$\lim_{n \rightarrow \infty} \int_X f(k)\mu_n(dk) = \int_X f(k)\mu'(dk), \quad (\text{CV})$$

In this case, we write $\mu_n \Rightarrow \mu$, and we refer to μ as the weak limit of the sequence $\{\mu_n\}$. It is direct to see that in this case of support $X \subset \mathbb{R}$, a increasing sequence $\{\mu_n\}$ in $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ necessarily converges to its supremum, that is:

$$\mu_n \Rightarrow \mu = \vee \{\mu_n\}.$$

Similarly, for a decreasing sequence $\{\mu_n\}$:

$$\mu_n \Rightarrow \mu = \wedge \{\mu_n\}.$$

■

Next, when considering the set $\Lambda(S, \mathcal{B}(S))$ of probability measures defined on the measurable space $(S, \mathcal{B}(S))$ (also endowed with the stochastic order), $\Lambda(S, \mathcal{B}(S))$ is not a complete lattice (as $S = X \times Z \subset \mathbb{R}^2$). Still, $\Lambda(S, \mathcal{B}(S))$ is chain complete (see Hopenhayn and Prescott [32] for a proof).

¹⁰This is not true if $X \subset \mathbb{R}^l$ with $l \geq 2$, and this is one fundamental reason why the argument in this paper cannot be trivially generalized to economies with Markov shocks. See more on this at the end of Section 4 of the paper.

Proposition 4 $(\Lambda(S, \mathcal{B}(S)), \geq_s)$ is a chain complete lattice with minimal and maximal elements $\delta_{(0, z_{\min})}$ and $\delta_{(k_{\max}, z_{\max})}$.

The proofs of existence of recursive equilibrium and SME equilibria in this paper are based on extensions of Tarski's fixed point theorem in the case of order continuous operators. Recall, Tarski's theorem says that an isotone map F that transforms a complete lattice (P, \geq) has a nonempty, complete lattice of fixed points. Tarski's fixed point theorem is not constructive. Therefore, to obtain our successive approximation results, we study an stronger version of Tarski's result for order continuous operators (a result related to Theorem 4.2 in Dugundji and Granas [25]). Recalling that a countable chain $\{x_n\}$ with $x_i \leq x_j$ in the poset (P, \leq) order converges to $x \in P$ iff $\vee\{x_n\} = x$, order continuity is defined as follows:

Definition 5 Let (P, \geq) be a poset. A function $F : (P, \geq) \rightarrow (P, \geq)$ is order continuous if for any countable chain $C \subset P$ such that $\vee C$ and $\wedge C$ both exist,

$$\vee\{F(C)\} = F(\vee C) \text{ and } \wedge\{F(C)\} = F(\wedge C).$$

Order continuity implies isotonicity since $u \leq v$ implies $\vee\{F(u), F(v)\} = F(\vee\{u, v\}) = F(v)$; thus, $F(u) \leq F(v)$, successive iterations on F starting from extremal elements order converge monotonically to extremal fixed points, a property we exploit in the following theorem.

Theorem 6 Let (P, \geq) be a complete lattice with maximal element p_{\max} and minimal element p_{\min} . (a). If $F : (P, \geq) \rightarrow (P, \geq)$ is isotone, then the set of fixed points of F is a non-empty complete lattice with maximal and minimal elements. (b). If $F : (P, \geq) \rightarrow (P, \geq)$ is order continuous and there exists $a \in P$ such that $F(a) \geq a$, then $\vee\{F^n(a)\}_{n \in \mathbb{N}}$ is the minimal fixed point of F in the order interval $[a, p_{\max}]$. (c). If $F : (P, \geq) \rightarrow (P, \geq)$ is order continuous and there exists b such that $b \geq F(b)$, then $\wedge\{F^n(b)\}_{n \in \mathbb{N}}$ is the maximal fixed point in order interval $[p_{\min}, b]$.

Proof. (a). This is essentially Tarski's fixed point theorem (see Tarski [54]). Consider the set $Q = \{x \in P, x \leq F(x)\}$. Since $p_{\min} \in Q$, it is nonempty. Consider a chain C in Q , and $u = \vee C$. Then $c \leq u$ for all $c \in C$, so that by isotonicity of F , $c \leq F(c) \leq F(u)$ for all $c \in C$, which implies that $F(u) \geq u$. Thus $u \in Q$, and every chain in Q has an upper bound. By Zorn Lemma, Q has a maximal element, which we denote q . Since $q \leq F(q)$, $F(q) \leq F^2(q)$ so $F(q) \in Q$, which implies that $F(q) = q$, and q is clearly the maximal fixed point in P since any fixed point must belong to Q . Considering $Q = \{x \in P, F(x) \leq x\}$ and following a symmetric argument proves the existence of a minimal fixed point. (b). Suppose $F : (P, \geq) \rightarrow (P, \geq)$ is order continuous. Since $F(a) \geq a$ and F isotone, $\forall n \in \mathbb{N}$, $F^{n+1}(a) \geq F^n(a)$, and $\{F^n(a)\}_{n \in \mathbb{N}}$ is a countable chain. P is a complete lattice so $\vee\{F^n(a)\}_{n \in \mathbb{N}}$ exists, and if F is order continuous, $F(\vee\{F^n(a)\}_{n \in \mathbb{N}}) = \vee\{F(a)\} = \vee\{F^n(a)\}_{n \in \mathbb{N}}$ so that $\vee\{F^n(a)\}_{n \in \mathbb{N}}$ is a fixed point of F . Consider any $d \in P$ such that $F(d) = d$.

Since $d \geq a$ and F is isotone, it is easy to see that $\forall n \in \mathbb{N}$, $d \geq F^n(a)$ which implies that d is an upper bound of $\{F^n(a)\}_{n \in \mathbb{N}}$. Thus $d \geq \vee\{F^n(a)\}_{n \in \mathbb{N}}$ and $\vee\{F^n(a)\}_{n \in \mathbb{N}}$ is thus the least fixed point of F in $[a, p_{\max}]$, and thus the unique minimal fixed point. (c). Follows a similar argument to that in (b). ■

We remark, that In view of the above proof, the hypothesis of order continuity in (b) and (c) can often be weakened to that of order continuity along monotone recursive F -sequences. That is, sequences of the form $\{x, F(x), \dots, F^n(x), \dots\}$ where either $x \leq F(x)$ or $x \geq F(x)$ and as long as F is isotone.¹¹ We shall exploit this property, and state the following corollary:

Corollary 7 *The preceding theorem holds with $F : (P, \geq) \rightarrow (P, \geq)$ isotone and order continuous along monotone F -sequences.*

3 The Economic Environment

We now describe the class of economies we initially consider. There are a large number of agents born each period that live for two periods. In their first period of life, they are endowed with a unit of time which they supply inelastically, and they consume and save. Preferences are represented by a standard utility function for which we assumed to satisfy a standard intertemporal complementarity condition between consumption when young (denoted as c_1) and consumption when old (denoted as c_2). We allow for non-time separability in lifetime consumption, so the preferences in Wang [56] are a special case of those covered by Assumption 1:

Assumption 1. The utility function $u : X \times X \rightarrow \mathbb{R}$ is:

- I. twice continuously differentiable;
- II. strictly increasing in each of its arguments and jointly concave;
- III. satisfies $\forall c_2 > 0$, $\lim_{c_1 \rightarrow 0^+} u_1(c_1, c_2) = +\infty$ and $\forall c_1 > 0$, $\lim_{c_2 \rightarrow 0^+} u_2(c_1, c_2) = +\infty$;
- IV. has increasing differences in (c_1, c_2) (i.e., $u_{12} \geq 0$).

We next describe the shock processes in the model. As in Wang [56] and Hauenschild [31], we assume that the production shocks are *iid* with compact support.¹²

Assumption 2. The random variable z_t follows an iid process characterized by the probability measure denoted γ . The support of γ is the compact set $Z = [z_{\min}, z_{\max}] \subset \mathbb{R}$ with $z_{\max} > z_{\min} > 0$.

Following recent work in economies with public policy and nonclassical production (e.g., Greenwood and Huffman [30], Coleman [15], and Mirman, Morand, and Reffett [40]), we consider reduced-form production functions that admit a

¹¹Order continuity along monotone recursive F -sequences does not imply that F is increasing. Consider for instance $F : [0, 1] \rightarrow [0, 1]$ such that $F(x) = 1 - x$. F is clearly continuous along the monotone recursive F -sequence $\{1/2, 1/2, 1/2, \dots\}$, but F is not increasing.

¹²We remark that our existence of RE results do not require boundedness of the shock; the results on existence of SME do. Using methods in Stachurski [?], for example, one could generalize our results to the case of unbounded shocks. We save that case for future work.

nonclassical specification denoted $F(k, n, K, N, z)$. In all cases, we assume F is constant returns to scale in private inputs (k, n) for each level of aggregate inputs (K, N) . The following assumptions on F are adapted from the literature on nonoptimal stochastic growth are completely standard. Anticipating $n = 1 = N$ in any equilibrium with inelastic labor supply, we state some of our assumptions as follows:

Assumption 3. The production function $F(k, n, K, N, z) : X \times [0, 1] \times X \times [0, 1] \times Z \rightarrow \mathbb{R}_+$ is:

I. twice continuously differentiable in its first two arguments, and continuous in all arguments;

II. increasing in all arguments, strictly increasing and strictly concave in its first two arguments;

IIIa. $r(k, z) = F_1(k, 1, k, 1, z)$ is decreasing and continuous in k , and $\lim_{k \rightarrow 0} r(k, z) = +\infty$;

IIIb. $w(k, z) = F_2(k, 1, k, 1, z)$ is increasing and continuous in k , and $\lim_{k \rightarrow 0^+} w(k, z) = 0$;

IV. such that there exists a maximal sustainable capital stock k_{\max} (i.e., $\forall k \geq k_{\max}$ and $\forall z \in Z$, $F(k, 1, k, 1, z) \leq k_{\max}$, and $\forall k \leq k_{\max}$, $\exists z \in Z$, $F(k, 1, k, 1, z) \geq k_{\max}$), and with $F(0, 1, 0, 1, z) = 0$.

Also, IV implies that the set of feasible capital stock can be restricted to be in the compact interval $X = [0, k_{\max}]$ as long as we place the initial date zero capital stocks $k_0 = K_0$ in X . Notice, this condition, along with (IIIa and IIIb) also place restrictions on the amount of nonconvexity we can allow. Assumption 3 is only needed for seeking monotone RE (as we shall show in the last section of the paper when we weaken Assumption 3 considerably at the expense of losing the monotonicity properties of RE).

Finally, we shall state the following two additional assumptions as they shall often lead to sharper properties of the RE.¹³

Assumption 3'. Both $r(k, z)$ and $w(k, z)$ are continuous and increasing in z for all k .

Finally the last assumption is a sufficient condition to exclude economies in which 0 may be the only RE, and will lead to the construction of minimal RE by successive approximations.

Assumption 4 (i) $\lim_{k \rightarrow 0^+} r(k, z_{\max})k = 0$, and (ii) $\forall c_1 > 0$, $\lim_{c_2 \rightarrow 0} u_2(c_1, c_2) = \infty$.

¹³The monotonicity assumption is standard; it can be shown that the continuity assumption may be weakened to upper semicontinuity.

4 Existence and Computation of Recursive Equilibria

Our main existence results address the issues of existence, characterization and construction of extremal RE in spaces of functions with various properties, starting with bounded functions all the way to bounded isotone and usc functions. The proofs of these existence proposition through the use of the Euler equation method, a method traditionally reserved to the study of infinitely-lived agent models with non-classical production (see, for instance, Coleman[14], Greenwood and Huffman[30], Datta et. al [16], or Morand and Reffett [41]). The idea is to use the Euler equation to construct an isotone non-linear operator which maps a complete lattice of policies into itself.¹⁴ As a direct consequence of Tarski's fixed point theorem, the set of fixed points of this operator is a non-empty complete lattice, and by construction all fixed points but the trivial 0 are RE. The construction of extremal RE by successive iterations of the operator relies on the order continuity of the operator.

4.1 The Euler equation method

A young agent in any period earns the competitive wage w when supply her labor supply to the market, and must decide what amount y to consume today and save for period $t + 1$ consumption. To make this decision, the agent postulates a law of motion $h \in W$ for the capital stock, and uses that law of motion to compute the expected return on his capital investment. In addition, the competitive wage and return on capital are obtained from the firms optimization problem, who are assumed to simply profit maximize. That is, we have the equilibrium wage rate $w(k, z) = F_2(k, 1, k, 1, z)$ and return on capital $r(k, z) = F_1(k, 1, k, 1, z)$ taken as given. Thus, for $s \in S^*$, and $h \in W$, a young agent solves the following::

$$\max_{y \in [0, w(s)]} \int_{\mathcal{Z}} u(w(s) - y, r(h(s), z')y) \gamma(dz'),$$

Using the Euler equation associated with the agent's maximization problem we define a RE as follows:

Definition 8 *A Recursive Equilibrium (RE) is a bounded function h such that, for all $s \in S^*$, $0 < h(s)$ and:*

$$\begin{aligned} & \int_{\mathcal{Z}} u_1(w(s) - h(s), r(h(s), z')h(s)) \gamma(dz') \\ = & \int_{\mathcal{Z}} u_2(w(s) - h(s), r(h(s), z')h(s)) r(h(s), z') \gamma(dz'). \end{aligned} \tag{E}$$

¹⁴The non-linear operator in this paper differs from the one in the infinitely-lived agent models cited above.

Note that we restrict our search for RE depending only on the current states of the economy. To do so, we first define a nonlinear operator Ah for $h \in W$ as follows:

Definition 9 *Given any $h \in W$, define A as follows: If $h(s) > 0$, then $Ah(s)$ is the unique solution¹⁵ for y to:*

$$\begin{aligned} & \int_Z u_1(w(s) - y, r(h(s), z')y)\gamma(dz') \\ &= \int_Z u_2(w(s) - y, r(h(s), z')y)r(y, z')\gamma(dz'), \end{aligned} \quad (E')$$

and $Ah(s) = 0$ whenever $h(s) = 0$.

A function $h \in W$ is a RE if and only if it is a non-zero fixed point of the operator A . Hence, the issue of existence, characterization, and construction of extremal RE simply follow directly from the study of the set of *non-trivial* fixed points of A . Before proving our main Proposition, we need to prove three lemmas discussing useful properties of the operator A . In the first lemma, we show A is an isotone operator on several complete lattices. This implies by Tarski's theorem that the non-empty set of fixed points is a complete lattice. In the second lemma, we show A is order continuous along monotone sequences in some of these complete lattice; therefore, by the result of the previous section, we can construct of extremal fixed points of A by successive approximation. In the final lemma, under some minimum assumptions, we show Ah maps subintervals of these complete lattice up (with minimal elements of these subintervals strictly bigger than the trivial fixed point 0). This enables us to set aside the trivial fixed point 0, and construct minimal RE. We first prove the first step in this procedure.

Lemma 10 *Under Assumptions 1, 2, 3, the operator A is an isotone self map on (W, \leq) . Under Assumptions 1, 2, 3, 3', A is an isotone self map on (H, \leq) and on (H^u, \leq) .*

Proof. By construction A maps W into itself, and it is easy to verify that $Ah \geq Ah'$ whenever $h \geq h'$. Clearly Ah is isotone in k whenever h is, and Assumption 3' is sufficient for preservation of isotonicity in z , thus making A an isotone map on (H, \leq) . Consider $h \in H^u$, and therefore right continuous at every $k \in [0, k_{\max}[$ given any z . Since the unique solution to (E') can be expressed as a continuous function of h , w , and r , Assumption 1, 2, 3, 3' imply that Ah is right continuous in k as well, and therefore also usc in k since increasing. Thus A is an isotone self map on H^u .¹⁶ ■

We now provide sufficient conditions under which the operator A is order continuous; we, therefore, can use successive approximation to study extremal fixed points.

¹⁵It is easy to verify the existence of a unique solution under Assumption 1.

¹⁶Note that the same argument applies for semicontinuity in z given any k . As a result, under Assumptions 1, 2, 3, 3' A is an increasing self map on the set of isotone and (jointly) continuous functions.

Lemma 11 (i) Under Assumptions 1, 2, 3 the set of fixed points of A in (W, \leq) is a non-empty complete lattice and A is order continuous along any monotone sequence in (W, \leq) . (ii) Under Assumptions 1, 2, 3, 3' the set of fixed points of A in (H, \leq) (resp. (H^u, \leq)) is a non-empty complete lattice and A is order continuous along any monotone (resp. decreasing) sequence in (H, \leq) (resp. (H^u, \leq)).

Proof. The complete lattice structure of these sets of fixed points follows from Tarski's fixed point theorem. Next, we prove that order continuity along increasing sequences by showing that for an increasing sequence $\{g_n\}$ in (W, \leq) or in (H, \leq)

$$\sup(\{Ag_n(s)\}) = A(\sup\{g_n(s)\}).$$

For such a sequence and for all $s \in S$, the sequence of real numbers $\{g_n(s)\}$ is increasing and bounded above (by $w(s)$), thus $\lim_{n \rightarrow \infty} g_n(s) = \sup\{g_n(s)\}$. For the same reason $\lim_{n \rightarrow \infty} Ag_n(s) = \sup\{Ag_n(s)\}$. By definition, for all $n \in \mathbb{N}$, and all $s \in S^*$:

$$\begin{aligned} & \int_Z u_1(w(s) - Ag_n(s), r(g_n(s), z')Ag_n(s))\gamma(dz') \\ = & \\ & \int_Z u_2(w(s) - Ag_n(s), r(g_n(s), z')Ag_n(s))r(Ag_n(s), z')\gamma(dz') \end{aligned}$$

The functions u_1 and u_2 are continuous (Assumption 1), r is continuous in its first argument (Assumption 3), hence taking limits when n goes to infinity, we have:

$$\begin{aligned} & \int_Z u_1(w(s) - \sup\{Ag_n(s)\}, r(\sup\{g_n(s)\}, z')\sup\{Ag_n(s)\})\gamma(dz') \\ = & \\ & \int_Z u_2(w(s) - \sup\{Ag_n(s)\}, r(\sup\{g_n(s)\}, z')\sup\{Ag_n(s)\})r(\sup\{Ag_n(s)\}, z')\gamma(dz'), \end{aligned}$$

which implies that $A(\sup\{g_n(s)\}) = \sup\{Ag_n(s)\}$. A symmetric argument can easily be made for any decreasing sequence $\{g_n\}$ in (W, \leq) or in (H, \leq) noting that in this case, the sequences of real numbers $\{g_n(s)\}$ is decreasing and bounded below by 0, therefore $\lim_{n \rightarrow \infty} g_n(s) = \inf\{g_n(s)\}$. This establishes (i) and (ii). Note that the proof also holds for a decreasing sequence in (H^u, \leq) (since, as we have noted before, \wedge_H and \wedge_{H^u} coincide). ¹⁷ ■

Finally, we prove a result useful for construction appropriate subintervals under which we can study our existence questions without the presence of trivial fixed points.

Lemma 12 Under assumption 1, 2, 3, 4 there exists a function $h_0 \in (H, \leq)$ lower semicontinuous in k and continuous in z such that (i) $\forall s \in S^*$, $Ah_0(s) > h_0(s) > 0$, and (ii) $\forall h \in]0, h_0]$, $Ah > h$ on S^* .

¹⁷Similarly, A is order continuous along any increasing sequences in the set of bounded isotone lsc functions.

Proof. See Appendix A. Note that we construct h_0 to be lsc so that $\{A^n h_0\}$ will be an increasing sequence of lsc functions, which therefore converges pointwise to the lsc function $\vee\{A^n h_0\}$. ■

We are now prepared to prove the the first key proposition of this section on the existence of RE in the class of functions W :

Proposition 13 *Under Assumptions 1, 2, 3, 3', 4 (i) the set of Recursive Equilibrium (RE) in (W, \leq) (in (H, \leq)) is a non-empty complete lattice and (ii) there exists a countable set $\{h^n\}_{n \in \mathbb{N}}$ of bounded measurable functions such that any bounded RE satisfies $h(s) \in cl\{h^1(s), h^2(s), \dots\}$ for each s . (iii) The minimal RE in (W, \leq) (in (H, \leq)) is an isotone lsc and measurable function h_{\min} ; the maximal RE in (W, \leq) (in (H, \leq)) is an isotone usc and measurable function h_{\max} , and both can be constructed by successive approximations.*

Proof. (i) It is precisely the set of fixed points of A in (W, \leq) (which we prove is non-empty in Lemma 13 above) minus 0. To show that there is fixed point strictly greater than 0, note that as a consequence of lemma 14 above, the isotone operator A maps the complete lattice $[h_0, w] \subset W$ into itself, and therefore A must have a fixed point greater than h_0 . Also by lemma 14 there cannot be any other strictly positive fixed point in W smaller than h_0 . By Theorem 4 in Section 2, the minimal fixed point of A in $W \cap [h_0, w]$ (and thus the minimal RE in W) must therefore be $h_{\min} = \vee\{A^n h_0\}$. It is also the minimal RE in H with the addition of Assumption 3'. Note that:

$$h_{\min}(s) = \vee\{A^n h_0\}(s) = \lim_{n \rightarrow \infty} A^n h_0(s) = \sup\{A^n h_0(s)\}.$$

Further, h_{\min} is lsc since it is the upper envelope of a family of elements of lsc functions, and is therefore the minimal bounded isotone and lsc RE. Similarly, the maximal RE in (W, \leq) is obtained as the inf (pointwise limit) of a decreasing sequence beginning at w . That is, is:

$$h_{\max}(s) = \wedge\{A^n w\}(s) = \lim_{n \rightarrow \infty} A^n w(s) = \inf\{A^n w(s)\},$$

which implies that $h_{\max} \in H^u$ since it is the lower envelope of a family of elements of (H^u, \leq) . This proves (iii). Note that the same arguments applies for Proposition 2.

(ii). The function $p : S^* \times X \rightarrow R$ defined as:

$$p(s, y) = - \left| \int_Z [u_1(w(s) - y, r(y, z')y) - u_2(w(s) - y, r(y, z')y)r(y, z')] \gamma(dz') \right|$$

is continuous, and the correspondence $\Psi : S \rightarrow \mathbb{R}$ defined as $\Psi(s) = [0, w(s)]$ is non-empty, compact valued and measurable. Note that the RE are constructed has the non-zero maximizers of p . As a consequence of the measurable maximum theorem (see for instance, Aliprantis and Border[1], corollary 17.8), the correspondence Φ defined as $\Phi(s) = \arg \max_{y \in \Psi(s)} p(s, y)$ is measurable,

nonempty and compact valued. By Castaing's theorem (see Aliprantis and Border[1]), this implies that there exists a countable sequence $\{h^n\}_{n \in \mathbb{N}}$ of measurable selectors from Φ satisfying:

$$\forall s \in S, \Phi(s) = cl\{h^1(s), h^2(s), \dots\}.$$

■

We now prove a second key existence theorem relating to the existence and computation of RE with the subclass of function H^u :

Proposition 14 *Under Assumptions 1, 2, 3, 3', 4 the set of RE in (H^u, \leq) is a non-empty complete lattice with minimal g_{\min} and maximal elements h_{\max} , and both can be constructed by successive approximations. All the RE in (H^u, \leq) are measurable.*

Proof. We follow the same argument as that of the proof of Proposition 13 (i). Then it is then only a matter of correcting h_{\min} at most at a countable number of points to obtain the minimal bounded isotone and usc RE, as shown in the next lemma. Specifically, we can show that the minimal RE in (H^u, \leq) is the function $g_{\min} : S \rightarrow X$ defined as:

$$g_{\min}(s) = \inf_{k' > k} \{\sup\{A^n h_0(k', z)\}\} = \inf_{k' > k} \{\vee\{A^n h_0\}(k', z)\} \text{ for all } s = (k, z) \in [0, k_{\max}[\times Z$$

and $g_{\min}(k_{\max}, z) = \vee\{A^n h_0\}(k_{\max}, z)$. Indeed, by construction $g_{\min} \in H^u$, $g_{\min}(\cdot, z)$ and $q(\cdot, z) = \vee\{A^n h_0\}(\cdot, z)$ differ at most at the discontinuity points of $\vee\{A^n h_0\}(\cdot, z)$, and $g_{\min}(\cdot, z)$ is the smallest usc function greater than $\vee\{A^n h_0\}(\cdot, z)$. In addition, since $\vee\{A^n h_0\}$ is lsc, for any $s \in S$, $g_{\min}(s) = \lim_{k' \rightarrow k^+} \vee\{A^n h_0\}(k', z)$. For any $s = (k, z) \in [0, k_{\max}[\times Z$, and for all $k' > k$, by definition of $q(\cdot, z)$:

$$\begin{aligned} & \int_Z u_1(w(k', z) - q(k', z), r(q(k', z), z')q(k', z))\gamma(dz')) \\ = & \int_Z u_2(w(k', z) - q(k', z), r(q(k', z), z')q(k', z))r(q(k', z), z')\gamma(dz')). \end{aligned}$$

Both functions u_1 and u_2 are continuous and r is continuous in its first argument so taking limits when $k' \rightarrow k^+$ on both sides of the previous equality implies that:

$$\begin{aligned} & \int_Z u_1(w(s) - g_{\min}(s), r(g_{\min}(s), z')g_{\min}(s))\gamma(dz') \\ = & \int_Z u_2(w(s) - g_{\min}(s), r(g_{\min}(s), z')g_{\min}(s))r(g_{\min}(s), z')\gamma(dz'), \end{aligned}$$

which proves that, $Ag_{\min}(s) = g_{\min}(s)$. The set of RE in (H^u, \leq) is then simply the set of fixed point of A that are bounded, isotone, and usc (a complete lattice by Proposition 2 in Section2). ■

Finally, note that it is also easy to modify the usc function h_{\max} at most at a countable number of points to construct the maximal bounded isotone and lsc RE.

4.2 Uniqueness under capital income monotonicity

Under the additional assumption of capital income monotonicity, the only case discussed in Wang[56], we prove the existence of a unique bounded isotone h^* . Since it is both a minimal and maximal RE, by our previous findings it is both usc and lsc, and therefore continuous. In addition, we prove that the Markovian equilibrium consumption decision policy corresponding to this RE is also isotone, and thus also continuous (if both $w - h^*$ and h^* are isotone when w is continuous, they both must also be continuous.¹⁸

Proposition 15 *Under Assumption 1, 2, 3, 3', and 4, if $r(y, z)y$ is increasing in y for all $z \in Z$ (an hypothesis we call "capital income monotonicity") there exists a unique bounded isotone RE h^* in H . The corresponding (Markovian) equilibrium consumption policy, $w - h^*$ is also isotone, which implies that both h^* and $w - h^*$ are continuous.*

Proof: Under capital income monotonicity, for all $s \in S^*$, it is easy to see that the following equation in y :

$$\begin{aligned} & \int_Z u_1(w(s) - y, r(y, z')y) \gamma(dz') \\ = & \int_Z u_2(w(s) - y, r(y, z')y) r(y, z') \gamma(dz'). \end{aligned}$$

has a unique solution. Note that the function h^* is therefore the maximal and minimal RE, and thus usc and lsc in k , i.e., continuous in k . By definition, for all $s \in S^*$:

$$\begin{aligned} & \int_Z u_1(w(s) - h^*(s), r(h^*(s), z')h^*(s)) \gamma(dz') \\ = & \int_Z u_2(w(s) - h^*(s), r(h^*(s), z')h^*(s)) r(h^*(s), z') \gamma(dz'). \end{aligned} \tag{E''}$$

Suppose there exists $s = (k, z) \in X^* \times Z$ such that $w(k, z) - h^*(k, z)$ decreases with an increase in k . Then, for all $z' \in Z$, the expression:

$$u_1(w(k, z) - h^*(k, z), r(h^*(k, z), z')h^*(k, z))$$

increases with k under the assumption of capital income monotonicity, and given that $h^*(k, z)$ is increasing in k , $u_{12} \geq 0$ and $u_{11} \leq 0$. However, for all $z' \in Z$, the expression:

$$u_2(w(k, z) - h^*(k, z), r(h^*(k, z), z')h^*(k, z)) r(h^*(k, z), z')$$

¹⁸In fact, both are Lipschitz continuous functions, an important property when considering numerical implementations of our methods since Lipschitz continuous functions can be approximated with greater accuracy and convergence rates than merely continuous functions.

necessarily decreases with an increase in k . Thus LHS and RHS in equation (E'') above move in opposite direction when k increases, which is impossible. As a result, $w(k, z) - h^*(k, z)$ must be increasing in k . The same argument works to show that $w(k, z) - h^*(k, z)$ must be increasing in z . Finally, under the assumption that w is continuous, if both the equilibrium investment and the equilibrium consumption policies are increasing, they both necessarily must be continuous. ■

We emphasize that both w and $w - h^*$ are in fact Lipschitz continuous functions since their gradient fields are all bounded by the variation in the wage rate in equilibrium, an important property when considering numerical implementations of our methods since Lipschitz continuous functions can be approximated with greater accuracy and convergence rates than merely continuous functions.

We finally remark that capital income isotonicity is not necessary for uniqueness of RE. We can see this with the following example.:

Example 16 Consider, for instance, preferences represented by the utility function:

$$\ln(c_t) + \ln(c_{t+1}),$$

in which case the maximization problem of an agent is:

$$\max_{y \in [0, w(s)]} \left\{ \ln(w(s) - y) + \int_Z \ln(r(h(s), z')y) \gamma(dz') \right\},$$

and the associated first order condition is:

$$(w(s) - y) = y.$$

Thus, for example, for any concave production function that generates r , there exists a unique Markovian equilibrium decision policy (the function $h = .5w$).

5 Existence and construction of stationary Markov equilibria

We define a Stationary Markov Equilibrium (SME) as a “non-trivial” (i.e., not all mass is concentrated at the origin) invariant distribution, in line with the work of Hopenhayn and Prescott [32] and Futia[28], and in contrast to Wang[56][57] who follows the path of Duffie & al.[26] and focuses on ergodic distributions. Our main contribution is to establish algorithms converging to extremal invariant probability measures corresponding to any isotone and measurable RE h . The invariants measures are precisely the fixed points of a stochastic operator mapping the chain complete lattice $\Lambda(S, B(S))$ into itself, and most everything follows from the order continuity of this operator. When the RE is also continuous (which is the case under capital income monotonicity), the order continuous operator maps the complete lattice $\Lambda(X, B(X))$ into itself, and the set of SME is a non-empty complete lattice. We treat the case of h continuous first before proceeding with the general case.

5.1 SME associated with a continuous isotone RE

Any measurable bounded RE h induces a Markov process for the capital stock represented by the transition function P_h defined as:

$$\begin{aligned} \forall A \in \mathcal{B}(X), P_h(k, A) &= \Pr\{h(k, z) \in A\} = \gamma(\{z \in Z, h(k, z) \in A\}) \\ &= \int_Z \chi_A(h(k, z)) \gamma(dz). \end{aligned}$$

That is, $P_h(k, A)$ is the probability that the capital stock is in the set A one period after being equal to k .¹⁹ If we denote by μ_t the probability measure associated with the random variable k_t , then μ_{t+1} is defined by applying the operator $T_h^* : (\Lambda(X, \mathcal{B}(X)), \geq_s) \rightarrow (\Lambda(X, \mathcal{B}(X)), \geq_s)$ to μ_t so that:

$$\forall B \in \mathcal{B}(X), \mu_{t+1}(B) = T_h^* \mu_t(B) = \int P_h(k, B) \mu_t(dk). \quad (\text{M1})$$

That is, $\mu_{t+1}(B)$ is the probability that the k_{t+1} lies in the set B if k_t is drawn according to the probability measure μ_t , and we define a stationary Markov equilibria (SME) as a non-trivial fixed point of T_h^* .

Definition 17 *Given a measurable RE h , a Stationary Markov Equilibrium (SME) is a probability measure $\mu \in \Lambda(X, \mathcal{B}(X))$ distinct from δ_0 such that:*

$$\forall B \in \mathcal{B}(X), \mu(B) = T_h^* \mu(B) = \int P_h(k, B) \mu(dk).$$

It is easy to verify that if h is isotone and continuous, the Markov operator T_h^* is an isotone self map on $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ and P_h has the Feller property or, equivalently, that T_h^* is a weakly continuous and isotone operator (see, for instance, Exercises 8.10 and 12.7 in Stokey & al.[52]). These two properties imply that T_h^* is order continuous along any monotone sequence. Indeed, if the sequence $\{\mu_n\}$ is increasing, then $\mu_n \Rightarrow \mu = \vee\{\mu_n\}$, so that $T_h^*(\mu_n) \Rightarrow T_h^*(\vee\{\mu_n\})$ by weak continuity. Since T_h^* is an isotone operator, the sequence $\{T_h^*(\mu_n)\}$ is also increasing and therefore $T_h^*(\mu_n) \Rightarrow \vee\{T_h^*(\mu_n)\}$. By uniqueness of the limit, $T_h^*(\vee\{\mu_n\}) = \vee\{T_h^*(\mu_n)\}$, which proves continuity along any increasing sequence. The existence and computational results below follow directly from Theorem 6 of Section 2.

Proposition 18 *For any continuous and increasing RE h , the set of fixed points of T_h^* is a non-empty complete lattice with maximal and minimal elements, respectively $\wedge\{T_h^{*n} \delta_{k_{\max}}\}$ and $\vee\{T_h^{*n} \delta_0\}$.*

Since our definition of SME excludes δ_0 , the previous result does not necessarily imply the existence of a SME. Indeed, suppose for instance that:

$$\forall(k, z) \in S^*, 0 < h^*(k, z) < k.$$

¹⁹The $\mathcal{B}(S)$ -measurability of h implies that P_h is indeed a transition function since for each $k \in X$, $P_h(k, \cdot)$ is a probability measure, and for each A , $P_h(\cdot, A)$ is a measurable function.

It is then easy to see that given any initial distribution of capital stock, in the long run the capital stock will be 0. The only fixed point of $T_{h^*}^*$ is δ_0 , and the set of SME is therefore empty, a case taking place for instance when $w(k, z) < k$ for all (k, z) in $S^* : \forall (k, z) \in S^*, w(k, z) < k$. Thus one needs to think of sufficient conditions under which the set of SME is non-empty (see, for instance Wang[56]) but it is most useful to express any such condition in terms of restrictions on the primitives of the problem (unlike in Wang[56]).

While condition (I) in Assumption 5 below is necessary, condition (II) is sufficient for the existence of a specific element h_0 of H to be mapped up strictly by A . It implies that the isotone operator A maps the order interval $[h_0, w] \subset H$ (a complete lattice when endowed with the pointwise order) into itself, so that A must have a fixed point in this interval. Since under the assumption of capital income monotonicity, the fixed point h^* of A in H is unique it must be that:

$$\forall k \in [0, k_0] \text{ and } \forall z \in Z, h^*(k, z) > h_0(k, z) (> k).$$

Given this property of h^* , we show that there exist a fixed point of $T_{h^*}^*$ that is distinct from δ_0 . The argument is the following: Consider any measure μ_0 with support in $[0, k_0]$ and distinct from δ_0 (we write $\mu_0 >_s \delta_0$). Since h^* maps up strictly every point in $[0, k_0]$, μ_0 is mapped up strictly by the operator $T_{h^*}^*$. By isotonicity of $T_{h^*}^*$ the sequence $\{T_{h^*}^{*n} \mu_0\}$ is increasing, and by order continuity along monotone sequences of $T_{h^*}^*$ it weakly converges to a fixed point of $T_{h^*}^*$. Clearly by construction this fixed point is strictly greater than δ_0 . The rest of this section formalizes this argument.

Assumption 5: Assume that:

(I). There exists a right neighborhood Δ of 0 such that for all $k \in \Delta$ and all $z \in Z$, $w(k, z) \geq k$.

(II). The following inequality holds:

$$\begin{aligned} & \lim_{k \rightarrow 0^+} u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) \\ < & \lim_{k \rightarrow 0^+} u_2(w(k, z_{\min}) - k, r(k, z_{\max})k)r(k, z_{\min}). \end{aligned}$$

Note that for log separable utility, condition.(II) is equivalent to:

$$\lim_{k \rightarrow 0^+} (w(k, z_{\min})/k) > 2,$$

and under a Cobb-Douglass production function with multiplicative shocks, it is trivially satisfied (and so is condition (I)). For a polynomial utility of the form $u(c_1, c_2) = (c_1)^{\eta_1} (c_2)^{\eta_2}$ the condition is equivalent to:

$$\lim_{k \rightarrow 0^+} (w(k, z_{\min})/k) > \left[1 + \frac{\eta_1 r(k, z_{\max})}{\eta_2 r(k, z_{\min})} \right],$$

also trivially satisfied with Cobb-Douglass production and multiplicative shocks.

We can now state a key proposition that extends the uniqueness result in Coleman [15] and Morand and Reffett [41] obtained for infinite horizon economies to the present class of OLG models under assumption 5.

Proposition 19 *Under Assumption 5, the set of SME associated with an isotone continuous RE h is a non-empty complete lattice. The maximal SME is $\wedge\{T_{h^*}^{*n}\delta_{k_{\max}}\}$, and there exists $k_0 \in X$ such that the minimal SME is $\vee\{T_{h^*}^{*n}\delta_{k'}\}$ for any $0 < k' \leq k_0$.*

Proof: The proof is in two parts. Part 1 establishes the existence of h_0 that is mapped up strictly by the operator A , and Part 2 shows the existence of a probability measure μ_0 that is mapped up $T_{h^*}^*$, where h^* is the unique RE.

Part 1. By continuity of all functions in k , the inequality in Assumption 5 must be satisfied in a right neighborhood of 0. That is, there exists of $\Theta =]0, k_0] \subset \Delta$ such that, $\forall k \in \Theta$:

$$\begin{aligned} & u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) \\ & < \\ & u_2(w(k, z_{\min}) - k, r(k, z_{\max})k)r(k, z_{\min}). \end{aligned}$$

Consequently, $\forall k \in \Theta =]0, k_0]$:

$$\begin{aligned} & \int_Z u_1(w(k, z) - k, r(k, z')k)G(dz') \\ & \leq \\ & u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) \\ & < \\ & u_2(w(k, z_{\min}) - k, r(k, z_{\max})k)r(k, z_{\min}) \\ & \leq \\ & \int_Z u_2(w(k, z) - k, r(k, z')k)r(k, z')G(dz'). \end{aligned}$$

Next, consider the function $h_0 : X \times Z \rightarrow X$ defined as:

$$h_0(k, z) = \begin{cases} 0 & \text{if } k = 0, z \in Z \\ k & \text{if } 0 < k \leq k_0, z \in Z \\ k_0 & \text{if } k \geq k_0, z \in Z \end{cases} .$$

We prove now that $Ah_0 > h_0$. First, consider $0 < k \leq k_0, z \in Z$, and suppose that $Ah_0(k, z) \leq h_0(k, z) = k$. Then:

$$\begin{aligned} & \int_Z u_1(w(k, z) - k, r(k, z')k)G(dz') \\ & < \\ & \int_Z u_2(w(k, z) - k, r(k, z')k)r(k, z')G(dz') \\ & \leq \\ & \int_Z u_2(w(k, z) - Ah_0(k, z), r(k, z')Ah_0(k, z))r(Ah_0(k, z), z')G(dz'), \end{aligned}$$

where the first inequality stems from the result just above, and the second from $u_{22} \leq 0$, $u_{12} \geq 0$ and r decreasing in its first argument. By definition of Ah_0 , this last expression is equal to:

$$\int_Z u_1(w(k, z) - Ah_0(k, z), r(k, z')Ah_0(k, z))G(dz').$$

Thus, we have $Ah_0(k, z) \leq k$ and:

$$\begin{aligned} & \int_Z u_1(w(k, z) - k, r(k, z')k)G(dz') \\ < & \int_Z u_1(w(k, z) - Ah_0(k, z), r(k, z')Ah_0(k, z))G(dz'). \end{aligned}$$

which contradicts the hypothesis that $u_{11} \leq 0$ and $u_{12} \geq 0$. It must therefore be that for all $k \in]0, k_0]$ and all $z \in Z$, $Ah_0(k, z) > h_0(k, z) = k$, that is A maps h_0 strictly up at least in the interval $]0, k_0]$. Finally, for $k > k_0$, since Ah_0 is increasing in its first argument:

$$Ah_0(k, z) \geq Ah_0(k_0, z) > h_0(k_0, z) = k_0 = h_0(k, z).$$

We have thus established that A maps h_0 up (strictly). Since the order interval $[h_0, w]$ in (H, \leq) is a complete lattice when endowed with the pointwise order, then by isotonicity of A there must exist a fixed point of A in that interval. Under capital income isotonicity, $h^* \in [h_0, w]$.

Part 2. Consider any probability measure in $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ with support in the compact interval $[0, k_0]$ and distinct from δ_0 . We show that $T_{h^*}^* \mu_0 \geq_s \mu_0$. Consider any $f : X \rightarrow \mathbb{R}_+$ measurable, increasing and bounded, we have:

$$\begin{aligned} & \int [\int f(k')P_{h^*}(k, dk')] \mu_0(dk) = \int [\int f(h^*(k, z))\lambda(dz)] \mu_0(dk) \\ = & \int_{[0, k_0]} [\int_Z f(h^*(k, z))\lambda(dz)] \mu_0(dk) + \int_{[k_0, k_{\max}]} [\int f(h^*(k, z))\lambda(dz)] \mu_0(dk) \\ \geq & \int_{[0, k_0]} f(k) \mu_0(dk) \end{aligned}$$

since $h^*(k, z) > k$ on $[0, k_0]$. Note that if f is strictly positive on $[0, k_0]$ then the last inequality is strict.

We have just demonstrated that $T_{h^*}^* \mu_0 \geq_s \mu_0$ and that $T_{h^*}^* \mu_0$ is distinct from μ_0 , so we write $T_{h^*}^* \mu_0 >_s \mu_0 (>_s \delta_0)$. By order continuity along any monotone sequence of $T_{h^*}^*$, necessarily the increasing sequence $\{T_{h^*}^{*n} \mu_0\}$ converges weakly to a fixed point of $T_{h^*}^*$ strictly greater than δ_0 . In addition, it is easy to see that there cannot be any fixed point of $T_{h^*}^*$ with support in $[0, k_0]$ other than δ_0 so that the minimal non-trivial (i.e., distinct from δ_0) fixed point of $T_{h^*}^*$, which is by definition the minimal SME, can be constructed as the limit of the sequence

$\{T_{h^*}^{*n} \mu_0\}$, where $\mu_0 = \delta_{k'}$ for any $0 < k' \leq k_0$. This completes the proof that the set of SME is the non-empty complete lattice of fixed points of $T_{h^*}^*$ minus δ_0 , and that the maximal SME and minimal SME can be obtained as claimed. ■

5.2 Existence and construction of extremal SME for Semi-continuous RE

Continuity of h , however, is not necessary for T_h^* to be order continuous along recursive monotone T_h^* -sequences. Indeed, recall the result in Morand and Reffett[?] that in an OLG model with Markov shocks associated with the transition function Q , if Q is increasing and satisfies Doeblin's condition, then the $\mathcal{B}(S)$ -measurability of any isotone RE h is sufficient for T_h^* is order continuous along recursive monotone T_h^* -sequences, a property leading to the construction of extremal SME by successive approximation.

In order to apply this result, consider iid shocks as a special case of Markov shocks for which the transition function is defined as $Q(z, B) = \gamma(B) = \Pr(z' \in B \text{ given } z)$, and recall that a Markov transition function Q satisfies Doeblin's condition if there exists $\delta \in \Lambda(Z, \mathcal{B}(Z))$ and $\theta < 1$ and $\eta > 0$ such that:

$$\forall B \in \mathcal{B}(Z), \delta(B) \geq \theta \text{ implies that } \forall z \in Z, Q(z, B) \geq \eta.$$

In particular, since $Q(z, B) = \gamma(B)$, then any $\theta = \eta < 1$ and $\delta = \gamma$ show that iid shock trivially satisfy Doeblin's condition. It is easy to see that if Q satisfies Doeblin's condition (D), then the transition function P_h corresponding to any $\mathcal{B}(S)$ -measurable RE h and defined by:

$$\forall A \times B \in \mathcal{B}(S), P_h(x, z; A, B) = \begin{cases} Q(z, B) & \text{if } h(x, z) \in A \\ 0 & \text{otherwise.} \end{cases}$$

also satisfies Doeblin's condition. Consequently, by Theorem 11.9 in Stokey & al.[52], the n-average of any recursive T_h^* -sequence converges in the total variation norm, and therefore weakly converges, to a fixed point of the isotone T_h^* . This implies that any monotone recursive T_h^* -sequence weakly converges, and that the limit is a fixed point of T_h^* . This precisely proves that T_h^* is order continuous along recursive monotone T_h^* -sequences, and an application of Theorem 6 of Section 2 gives the following important result.

Proposition 20 *Under Assumptions 1, 2, 3, 3', 4, 5 and if shocks satisfy Doeblin's condition, then for any $\mathcal{B}(S)$ -measurable RE h in H , there exists a non-empty set of SME with maximal and minimal elements respectively given by $\gamma_{\max}(h) = \wedge \{T_h^{*n} \delta_{(k_{\max}, z_{\max})}\}$ and $\gamma_{\min}(h) = \vee \{T_h^{*n} \mu_0\}$, where $\mu_0 = \delta_{(k', z_{\min})}$ for any $0 < k' \leq k_0, k_0$ constructed from Assumption 5.*

Finally, for economies satisfying Assumption 4, by our results in the previous section of the paper, there exist minimal and maximal RE h_{\min} and h_{\max} in H , and both are $\mathcal{B}(S)$ -measurable. Necessarily, any other RE h in H satisfies $h_{\min} \leq h \leq h_{\max}$, and therefore:

$$T_{h_{\min}}^* \mu_0 \leq T_h^* \mu_0,$$

and recursively,

$$\gamma_{\min}(h_{\min}) = \vee \{T_{h_{\min}}^{*n} \mu_0\}_{n \in \mathbb{N}} \leq \vee \{T_h^{*n} \mu_0\}_{n \in \mathbb{N}} = \gamma_{\min}(h).$$

By a similar argument:

$$\gamma_{\max}(h_{\max}) = \wedge \{T_{h_{\max}}^{*n} \delta_{(k_{\max}, z_{\max})}\}_{n \in \mathbb{N}} \geq \wedge \{T_h^{*n} \delta_{(k_{\max}, z_{\max})}\}_{n \in \mathbb{N}} = \gamma_{\max}(h),$$

and this proves that $\gamma_{\max}(h_{\max})$ and $\gamma_{\min}(h_{\min})$ are the greatest and least SME, respectively. We state this very general result in the last proposition of the paper.

Proposition 21 *Under Assumptions 4 and 5, and when shocks satisfy Doeblin's condition, the set of SME is nonempty and there exist a greatest and a least SME, respectively $\gamma_{\max}(h_{\max}) = \wedge \{T_{h_{\max}}^{*n} \delta_{(k_{\max}, z_{\max})}\}_{n \in \mathbb{N}}$ and $\gamma_{\min}(h_{\min}) = \vee \{T_{h_{\min}}^{*n} \mu_0\}_{n \in \mathbb{N}}$ where $\mu_0 = \delta_{(k', z_{\min})}$ for any $0 < k' \leq k_0$, k_0 constructed from Assumption 5.*

6 Extensions

6.1 Models with Elastic labor supply

In this section, we show how our methods can be extended to economies where households have preferences over consumption and leisure. To do this, we consider a simple setting where preferences are additively separable in consumption and leisure²⁰, so that the utility function is $u(c) + v(l)$ where u satisfies Assumption 1, and v satisfies Assumption 1' below.

Assumption 1'. The function $v : [0, 1] \rightarrow \mathbb{R}$ is:

- I. twice continuously differentiable on $(0, 1]$,
- II. strictly increasing and strictly concave,
- III. such that $\lim_{l \rightarrow 0} v'(l) = +\infty$.

For the sake of exposition, we present our analysis in the context of a standard classical production function $F(k, n)$, although as will be clear, our analysis can be easily extended to a non-classical production functions of the type discussed in the main part of the paper. Specifically, we make the following assumption on F :

Assumption 3". The production function $F(k, n, z) : X \times [0, 1] \times Z \rightarrow \mathbb{R}_+$ is:

- I. twice continuously differentiable in its first two arguments, and continuous in all arguments;
- II. increasing in all arguments, strictly increasing and strictly concave in its first two arguments;

²⁰Most applied work in macroeconomics using OLG models with elastic labor supply assume additive separability between consumption and leisure. Our methods work for more general preferences but at the expense continuity of the RE. We discuss such cases in future work.

IIIa. such that $F_1(k, n, z)$ is decreasing and continuous in k , for all $n > 0$, and $\lim_{k \rightarrow 0} F_1(k, n, z) = +\infty$;

IIIb. such that $F_2(k, n, z)$ is increasing and continuous in k , for all $n > 0$, and $\lim_{k \rightarrow 0^+} F_2(k, n, z) = 0$.

Definition 22 A Recursive Equilibrium (RE) is a bounded function h and a bounded function n such that, for all $s = (k, z) \in S^*$, $0 < n(s) < 1$, $0 < h(s) < w(s)n(s)$ and

$$v'(1 - n(s)) = F_2(k, n(s), z) \quad (1)$$

and also:

$$\begin{aligned} & \int_Z u_1[w(s)n(s) - h(s), F_1(h(s), n(h(s), z'))h(s)]\gamma(dz') \\ = & \\ & \int_Z u_2[w(s)n(s) - h(s), F_1(h(s), n(h(s), z'))h(s)]F_1(h(s), n(h(s), z'))\gamma(dz'). \end{aligned} \quad (2)$$

where $w(s) = F_2(k, n(s), z)$.

Consider the first equation $v'(1 - n) = F_2(k, n, z)$. By assumptions 1 and 3", its unique solution $n^*(s)$ is increasing in s , which implies that the equilibrium wage $w(k, z) = F_2(k, n^*(k, z), z)$ is also increasing in $s = (k, z)$ since it is equal to $v'(1 - n^*(s))$ and v is strictly concave (assumption 1). We then apply the Euler equation method and define a nonlinear operator A as follows:

Definition 23 Given any $h \in W$, if $h(s) > 0$, then $Ah(s)$ is the unique solution for y to:

$$\begin{aligned} & \int_Z u_1(w(s)n^*(s) - y, F_1(h(s), n^*(y, z'), z')y)\gamma(dz') \\ = & \\ & \int_Z u_2(w(s)n^*(s) - y, F_1(h(s), n^*(y, z'), z')y)F_1(y, n^*(h(s), z'))\gamma(dz') \end{aligned} \quad (E)$$

and $Ah(s) = 0$ whenever $h(s) = 0$.

Defining the quantity,

$$\begin{aligned} Z(s, h, y) = & \\ & \int_Z u_1(w(s)n^*(s) - y, F_1(h(s), n^*(y, z'), z')y)\gamma(dz') \\ & - \\ & \int_Z u_2(w(s)n^*(s) - y, F_1(h(s), n^*(y, z'), z')y)F_1(y, n^*(h(s), z'))\gamma(dz') \end{aligned}$$

it is easy to see that Z is (strictly) increasing in y , being negative for y close to 0 and positive for y close to $w(s)n(s)$, which implies that $Ah(s)$ is the unique

solution to $Z(s, h, y) = 0$. Also, if h is an increasing function, then Z is decreasing in s , which implies that Ah is also an increasing function. Finally, Z is decreasing in h , which implies that A is an isotone operator. Thus A is an isotone self map on (W, \leq) , (H, \leq) and (H^u, \leq) , and it is straightforward to apply our results concerning the existence of recursive equilibria of Section 3 of the paper. That is, under Assumptions 1', 2, 3' and 3'', there exist complete lattices of RE. Note that capital income monotonicity is in this case not a sufficient condition for uniqueness of RE.

■

6.2 Models with General Income Processes

We now use our methods to construct RE for economies with general income processes. In such economies, although one loses the monotonicity properties of equilibrium investment and/or consumption dynamics, conditions can still be provided under which RE exist and can be computed by successive approximations. We focus on economies where household income processes admit a mixed-monotone decomposition on an expanded aggregate state space.²¹ Recall $X \subset \mathbb{R}$ is the space for the endogeneous aggregate capital stock, and is endowed with the standard pointwise partial order. Let \hat{X} be X endowed with the dual partial order on X , $k_e = (k, \hat{k}) \in X_e = X \times \hat{X}$. If f a real-valued function on X , we say $g : X_e \rightarrow \mathbb{R}$ is a *mixed-monotone decomposition of f* if g satisfies the following two properties: (i) $f(k) = g(k, \hat{k})$ for all $k_e \in X_e$ when $k = \hat{k}$, and (ii) $g(k_e)$ is isotone on X_e .²² We denote $s_e = (k_e, z) \in S_e = X \times \hat{X} \times Z$.

Assuming the bounded function for wages admits a mixed-monotone decomposition $w^e : S_e \rightarrow \mathbb{R}_+$, we define mixed-monotone versions of the sets W , and H discussed in Section 2. That is, we have the sets $W^e = \{h^e | h^e : S_e \rightarrow \mathbb{R}_+, 0 \leq h^e \leq w^e\}$, $H^e = \{h^e | h^e \text{ isotone on } S_e\} \subset W^e$. Clearly the set W^e is a complete lattice when endowed the pointwise partial order, and H^e is subcomplete in W^e . We now modify the specification of the reduced-form production function in assumption 3 in section 2 as follows:

Assumption 3'''. The function $F(k, n, K, n, x, n, z) : X \times [0, 1] \times X \times [0, 1] \times \hat{X} \times [0, 1] \times Z \rightarrow \mathbb{R}_+$ is:

- I. twice continuously differentiable in its first two arguments, with all partials continuous in all arguments;
- II. increasing in all arguments, strictly increasing and strictly concave in its first two arguments when $x = k$;

²¹See Datta, Mirman, Morand, and Reffett ([18], section 5) for examples of infinite horizon economies with income processes that admit mixed-monotone decompositions. Most economies with concave constant returns to scale production processes as well as nonclassical production, for example, admit mixed-monotone production processes on an expanded aggregate state space. When adding public policy to such models, one does need to require that policy preserve this mixed monotone structure.

²²Note that if $g(k_e)$ is isotone on X_e , then on the space $X \times X$, $g(\cdot, \hat{k})$ is (i) isotone for any given $\hat{k} \in X$; and (ii) antitone for any given $k \in X$; hence, the reference to a "mixed-monotone" decomposition.

IIIa. $r(k, x, z) = F_1(k, 1, k, 1, x, 1, z)$ decreasing and continuous in k , increasing and continuous in x , increasing and continuous in z , with $\lim_{k \rightarrow 0} r(k, x, z) = +\infty$ when $k = x$;

IIIb. $w(k, x, z) = F_2(k, 1, k, 1, x, 1, z)$ increasing and continuous in k , decreasing and continuous in x , increasing and continuous in z , with $\lim_{k \rightarrow 0^+} w(k, x, z) = 0$ when $k = x$.

Notice that under Assumption 3''', wage income w admits a mixed-monotone decomposition w^e on S_e , and first period is given by $y_1(s_e) = w(s_e)$, while second period income is $y_2(s_e) = r(s_e)k$.²³ We can now use the methods of section 2, only now studying the problem in the space of mixed-monotone equilibria. For economies under Assumptions 1, 2, 3''', 4, assume households use the law of motion $h \in H^e$ to calculate the continuation of the aggregate economy in their first period of life in a candidate RE, and solve a standard optimization problem. In particular, letting $S_e^* = X^* \times \hat{X}^* \times Z$, for $s_e \in S_e^*$, $h(s_e) \in H^e$, a young agent solves:

$$\max_{y \in [0, w]} \int_Z u(w(s_e) - y, r(h(s_e), h(s_e), z')y) \gamma(dz'),$$

We define a RE as in Section 3, but noting we now require additionally $k = \hat{k}$.

To construct such a RE, we modify our Euler equation method of section 3 to accommodate our more general framework for modeling the aggregate state space for minimal state RE by defining a nonlinear operator Ah as follows:

Definition 24 For $k_e \gg 0$, any $h \in H^e$, define A as follows: if $h(s_e) > 0$, then $Ah(s_e)$ is the unique solution for y to:

$$\begin{aligned} Z(y, s_e, h) = & \int_Z u_1(w(s_e) - y, r(y, h(s_e), z')y) \gamma(dz') - \\ & \int_Z u_2(w(s_e) - y, r(y, h(s_e), z')y) r(y, h(s_e), z') \gamma(dz') = 0 \quad (E') \end{aligned}$$

and $Ah(s_e) = 0$ whenever $h(s_e) = 0$.

We have the following properties for $Z(y, s_e, h)$ on $\mathbb{R}_+ \times S_e^* \times W^e$ when $h(s_e) > 0$: (i) Z is strictly increasing in y , each (s_e, h) ; (ii) Z is strictly decreasing in (s_e, h) , for each $y \in \mathbb{R}_+$. Therefore, for each (s_e, h) where $Ah(s_e)$ is

²³Consider, for instance, that $f(k, n, z) = zg(k, n)$ where $g(k, n)$ is CES, and assume that the government places an income tax on households each period given by $\theta(s_e) = (1 - \tau^e(s_e))$ that is a mixed-monotone decomposition of an arbitrary continuous tax policy $\tau(s)$. Let $J_1(s_e)$ and $J_2(s_e)$ be transfers functions for agents in their first and second period of life, respectively. The income processes on the expanded state space s_e will then be:

$$\begin{aligned} y_1(s_e) &= \theta(s_e)w(k, 1, z) \\ y_2(s_e) &= \theta(s_e)r(\hat{k}, 1, z) \\ J_1(s_e) + J_2(s_e) &= \tau(s_e)(y_1(s_e) + y_2(s_e)) \end{aligned}$$

and will satisfy all of our conditions.

defined, define the mapping $y^*(s_e, h) = \{y | Z(y, s_e, h) = 0\}$. We can now state the following extensions of the main theorems in Section 3 to this setting of general income processes:

Lemma 25 *Under Assumptions 1, 2, 3''', the operator A is an isotone self map on (W^e, \leq) , (H^e, \leq) .*

Proof. For (s_e, h) where Z is defined, by the fact that Z is continuous and strictly increasing in y , $Ah(s_e)$ is well-defined (i.e., nonempty and single valued), and the Ah is also clearly bounded. It is easy to see A is an isotone operator on W^e . In addition, if h is isotone in s_e , then so is Ah , which implies that A is an isotone self map on (H^e, \leq) . ■ ■

Finally, it is important to note that if h is a measurable function, then the objective in the young agent's seeks maximization problem is in fact a Caratheodory function (continuous in y and measurable in s). Given that the domain $[0, w]$ is compact valued, by the measurable maximum theorem Ah is then measurable. Such an important property (that A maps measurable functions into measurable functions) has important consequences for the constructions of extremal fixed points of A , as detailed in the following result.

Lemma 26 *(i) Under Assumptions 1, 2, 3''' the set of fixed points of A in (W^e, \leq) is a non-empty complete lattice and A is order continuous along any monotone sequence in (W^e, \leq) ; (ii) Under Assumptions 1, 2, 3''' the set of fixed points of A in (H, \leq) is a non-empty complete lattice and A is order continuous along any monotone sequence in (H^e, \leq) (resp. (H^{eu}, \leq)); (iii) All extremal fixed points of A are measurable; (iv) Under Assumptions 1, 2, 3''', if f is such that $r(K, K, z)K$ is increasing in K , then A has a unique fixed point that is strictly positive when $K > 0$, each $z \in Z$.*

Proof. The proof of (i), (ii) and (iv) are similar to that of Lemma 11. For (iii), simply notice that the greatest fixed point of A can be constructed as the pointwise limit of the decreasing sequence $\{A^n w\}$ of measurable functions, and is therefore measurable. A similar argument holds for the smallest fixed point of A . ■

6.3 Equilibrium Comparative statics

In this application of our methods, we show that the results of Hauenschild [31] incorporating a social security system in the overlapping generation model of Wang [56] can easily be derived from our setup. This example thus illustrates the power of monotone methods to generate (weak) comparative statics results. Recall that in Hauenschild [31], a Markovian equilibrium investment policy is a

function h satisfying:

$$\begin{aligned}
& \int_Z u_1((1-\tau)w(k, z) - h(k, z), r(h(k, z), z')h(k, z) + \tau w(h(k, z), z'))\gamma(dz') \\
= & \int_Z u_2((1-\tau)w(k, z) - h(k, z), r(h(k, z), z')h(k, z) + \tau w(h(k, z), z')) \\
& r(h(k, z), z')\gamma(dz').
\end{aligned} \tag{B1}$$

Consider the following equation in y :

$$\begin{aligned}
& \int_Z u_1((1-\tau)w(k, z) - y, r(h(k, z), z')y + \tau w(y, z'))\gamma(dz') \\
= & \int_Z u_2((1-\tau)w(k, z) - y, r(h(k, z), z')y + \tau w(y, z'))r(y, z')\gamma(dz').
\end{aligned}$$

For any $(k, z) \in X \times Z$ and $h \in E$, denote $Ah(k, z)$ the unique solution to this equation. It is easy to see that, in addition to being an order continuous isotone operator mapping E into itself, A is also isotone in $-\tau$. Consequently, an increase in τ generates a decrease (in the pointwise order) of the extremal Markovian equilibrium investment policies $h_{\tau, \max}$ and $h_{\tau, \min}$.

Next, recall that any equilibrium investment policy h induces a Markov process for the capital stock defined by the following transition function P_h :

$$\text{For all } A \in \mathcal{B}(X), P_h(k, A) = \Pr\{h(k, z) \in A\} = \lambda(\{z \in Z, h(k, z) \in A\}).$$

Consider two Markovian equilibrium policies $h' \geq h$ and their respective transition functions $P_{h'}$ and P_h . For any $k \in X$ and any function $f : X \rightarrow \mathbb{R}_+$ bounded, measurable and increasing:

$$\int f(k')P_{h'}(k, dk') = \int f(h'(k, z))\lambda(dz) \geq \int f(h(k, z))\lambda(dz) = \int f(k')P_h(k, dk').$$

Thus, for any $\mu \in \Lambda(X, B(X))$:

$$\begin{aligned}
\int f(k')T_{h'}^*\mu(dk') &= \int [\int f(k')P_{h'}(k, dk')] \mu(dk) \\
&\geq \\
\int [\int f(k')P_h(k, dk')] \mu(dk) &= \int f(k')T_h^*\mu(dk'),
\end{aligned}$$

which establishes that $T_{h'}^*\mu \geq T_h^*\mu$. Thus the natural ordering on the set of taxes τ induces an ordering by stochastic dominance of the corresponding extremal stationary Markov equilibria in the following way:

$$\tau' \geq \tau \text{ implies } h_{\tau, \max} \geq h_{\tau', \max} \text{ implies } \lim_{n \rightarrow \infty} T_{\tau'}^{*n} \delta_{k \max} \geq_s \lim_{n \rightarrow \infty} T_{\tau}^{*n} \delta_{k \max}.$$

In particular, we obtain proposition 2 in Hauenschild[31] by taking $\tau = 0$ (a "pure economy") and $\tau' > 0$.

7 Appendix A.

Lemma 27 *Under Assumption 4, for all $k \in X^*$, there exists a right neighborhood $\Omega =]0, \bar{k}]$ with $0 < \bar{k} \leq w(k, z_{\min})$ and $M > 0$ such that, for all $x \in \Omega$,*

$$u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) > M.$$

Proof. If $\lim_{x \rightarrow 0^+} r(x, z_{\max})x = 0$ then for all $k \in X^*$:

$$\lim_{x \rightarrow 0^+} u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) = u_2(w(k, z_{\min}), \lim_{x \rightarrow 0^+} r(x, z_{\max})x) = \infty.$$

The expression $u_2(w(k, z_{\min}) - x, r(x, z_{\max})x)$ can therefore be made arbitrarily large in a right neighborhood of 0, and the existence of Ω thus follows. ■

Lemma 28 *For all $s = (k, z) \in S^*$, there exists $h_0(s) \in]0, w(s)[$ such that:*

$$\begin{aligned} & \int_Z u_1(w(s) - h_0(s), r(h_0(s), z')h_0(s))G(dz') & (E0) \\ < & \\ & \int_Z u_2(w(s) - h_0(s), r(h_0(s), z')h_0(s))r(h_0(s), z')G(dz'). \end{aligned}$$

In addition, h_0 can be chosen increasing in k for each z , constant in z (and therefore continuous and increasing in z) for each k .

Proof. Fix $k \in X^*$. For all $z \in Z$:

$$\begin{aligned} & \lim_{x \rightarrow 0^+} \int_Z u_1(w(k, z) - x, r(x, z')x)G(dz') \\ & = \int_Z u_1(w(k, z), 0)G(dz') \\ & \leq \\ & u_1(w(k, z_{\min}), 0). \end{aligned}$$

Thus there exists a right neighborhood of 0, denoted $\Psi =]0, \bar{x}]$, such that, for all $x \in \Psi$:

$$\begin{aligned} & \int_Z u_1(w(k, z) - x, r(x, z')x)G(dz') \\ < & \\ & .5u_1(w(k, z_{\min}), 0). \end{aligned}$$

Next, for $x \in \Omega$:

$$\begin{aligned} & \int_Z u_2(w(k, z) - x, r(x, z')x)r(x, z')G(dz') \\ & \geq \\ & \int_Z u_2(w(k, z_{\min}) - x, r(x, z_{\max})x)r(x, z')G(dz') \\ & \geq \\ & \int_Z Mr(x, z')G(dz'), \end{aligned}$$

where the first inequality stems from $u_{12} \geq 0$ and u_2 decreasing, and the second from the Lemma above. This last expression can be made arbitrarily large, independently of z , by choosing x in Ω sufficiently close to 0. That is, it is always possible to choose x^* sufficiently small in $\Omega \cap \Psi$ so that:

$$\int_Z Mr(x^*, z')F(dz') \geq .5u_1(w(k, z_{\min}), 0). \quad (\text{E1})$$

Pick such an x^* and set $\delta_0(k, z) = x^*$ for all $z \in Z$. By construction, any $x \in]0, \delta_0(s)]$ satisfies:

$$\begin{aligned} & \int_Z u_1(w(s) - x, r(x, z')x)G(dz') \\ & < \\ & \quad .5u_1(w(k, z_{\min}), 0) \\ & \leq \\ & \int_Z Mr(x, z')G(dz') \\ & \leq \\ & \int_Z u_2(w(s) - x, r(x, z')x)r(x, z')G(dz'). \end{aligned}$$

That is, for all $x \in]0, \delta_0(s)]$:

$$\begin{aligned} & \int_Z u_1(w(s) - x, r(x, z')x)G(dz') \\ & < \\ & \int_Z u_2(w(s) - x, r(x, z')x)r(x, z')G(dz'). \end{aligned} \quad (\text{E2})$$

We repeat the same operation for each k in X^* , thus constructing a function $\delta_0 : S \rightarrow X$, setting $\delta_0(0, z) = 0$. By construction, for each $k \in X$, $\delta_0(k, z)$ is constant in z , and therefore increasing in z . In addition, any function smaller (pointwise) than δ_0 also satisfies (E2). In particular, the function $p_0 : X \times Z \rightarrow X$ defined as:

$$p_0(k, z) = \min_{k' \geq k} \{\delta_0(k', z)\}.$$

satisfies (E2), is increasing in k for all z , and constant in z for all k (and thus continuous in z for all k). Finally, the function h_0 defined as follows:

$$h_0(k, z) = \left\{ \begin{array}{l} \sup_{0 < k' < k} p_0(k', z) \text{ for } (k, z) \in X^* \times Z \\ 0 \text{ for } k = 0, z \in Z \end{array} \right\}$$

is smaller than p_0 (and therefore than δ_0 , hence it satisfies (E2)), increasing in k for all z , constant in z for all k , and lower semicontinuous in k for all z . ■

Proposition 29 $\forall s \in S^*, h_0(s) \geq h(s) > 0$ implies that $Ah(s) > h(s) > 0$.

Proof. Suppose that there exists $s \in S^*$ such that $Ah(s) \leq h(s)$. Then:

$$\begin{aligned}
& \int_Z u_1(w(s) - h(s), r(h(s), z')h(s))G(dz') \\
& < \int_Z u_2(w(s) - h(s), r(h(s), z')h(s))r(h(s), z')G(dz') \\
& \leq \int_Z u_2(w(s) - Ah(s), r(h(s), z')Ah(s))r(Ah(s), z')G(dz'),
\end{aligned}$$

where the first inequality stems from (E2) (since $0 < x = h(s) \leq h_0(s) \leq \delta_0(s)$) and the second from $u_{22} \leq 0$, $u_{12} \geq 0$ and r decreasing in its first argument. By definition of Ah , this last expression is equal to:

$$\int_Z u_1(w(s) - Ah(s), r(h(s), z')Ah(s))G(dz').$$

Summarizing, we have:

$$\begin{aligned}
& \int_Z u_1(w(s) - h(s), r(h(s), z')h(s))G(dz') \\
& < \int_Z u_1(w(s) - Ah(s), r(h(s), z')Ah(s))G(dz').
\end{aligned}$$

which is contradicted by the hypothesis that $u_{11} \leq 0$ and $u_{12} \geq 0$. Thus, necessarily, $Ah(s) > h(s)$. In particular, A maps h_0 strictly up. ■

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