Partial Ordering Methods for Constructing Symmetric Markovian Equilibrium in a Class of Dynamic Games*

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April 2009

Abstract

This paper uses lattice programming methods along with the extension of Tarski's fixed point theorem due to Veinott[22] and Zhou[24] to establish sufficient conditions for existence of sequential symmetric Markov equilibrium in a large class of dynamic games. Dynamic game frameworks are important for studying strategic, dynamic interaction between economic agents and the class of Markov equilibrium solutions are fundamental ones to analyze. Our method is constructive and we provide specific algorithms for computing symmetric Markov equilibrium. These results are applied to a dynamic, joint-access, renewable resouce extraction problem commonly known as "fishwar" game, in the context of a finite horizon.

Keywords: Dynamic games, Nash equilibrium, existence, computation.

1 INTRODUCTION

This paper develops an application of lattice methods to analyze existence, characterize, and compute symmetric sequential Markov equilibria in an important class of dynamic games: the classic "Great Fish War."[1] This particular class of games provides a framework for studying strategic interaction in dynamic economies and has extensive application in economics. Often, in economic models, the sets of feasible strategies available to the agents' and their payoffs are

*This draft is preliminary and incomplete. We thank Rabah Amir, Federico Echenique, Jean-Marie Viaene as well as seminar participants at Erasmus University, Jadavpur University, Wesleyan University, and the University of Connecticut for helpful discussions.

[1]See Levhari and Mirman[12], Amir[1], and Sundaram[18] for an introduction to this dynamic game.

interdependent and, in addition, the agents' choices in the current period affect their payoffs and their choice sets in the future. This framework commonly appears in the study of macroeconomics, public finance, industrial organization, international trade, environmental economics etc. See, for example, Datta[6] for an analysis of market-clearing prices over time in a dynamic model with trade, Datta and Mirman[7][8] for discussions on international policy coordination in models of trade and resource extraction. Analytical results are hard to get in the literature on dynamic games and results are often summarized in terms of specific examples or numerical simulations. We prove the existence of equilibrium and provide a theoretical framework for computing and characterizing sequential symmetric Markov equilibrium in dynamic games usimg lattice theoritic techniques.

A prototype model that we consider in this paper can be described as follows: some amount of resource is available at the beginning of the period for current consumption and investment, current consumption generates payoff in the current period while investment generates resources in the future and supports future consumption. The agents' care about own payoffs only, both in current and future periods, but future stock of the resource depends on aggregate investment made in the economy in the current period. Suppose that $f(k)$ units of the resource are available for consumption, and 2 identical players each pick their consumption, denoted c_i for player i[2]. The leftover resource, $k' = f(k) - \sum_{i=1}^{n} c_i$, is investment in the current period. In the next period, $f(k')$ units of the good are available, where f can be considered as the production function or, in general, the law of motion for the state variable; and the same sequence of actions takes place in the following period. The game ends after T periods, and each agent seeks to maximize the discounted sum of utility from their own consumption, $\sum_{t=0}^{T} \beta^t u(c_{i,t})$ over T-periods. Levhari and Mirman[12] solve for a sequential Markov equilibrium (and also the stationary Markov equilibrium) for a version of this game in which closed form solutions can be computed. Unfortunately, the method they use does not work in more general settings even for situations where agents are identical (See Mirman[15] for a discussion). For the general framework, Sundaram[18] proves an existence result for stationary symmetric Markov equilibrium in an infinite horizon version of this game, but does not consider the issue of sequential Markov equilibrium for the finite horizon. His argument is topological and based on the continuity of a mapping from the interval $[0, f]$ of the space of bounded, uppersemicontinuous (usc) and non-decreasing functions into itself. The stock belongs to a compact interval of the real line (called Θ) and that interval is shown to be compact in the weak topology, the appropriate mapping has the fixed point property by an application of Schauder's theorem. However, Schauder's theorem is an existential result, and, therefore, proofs along those line do not provide guidance toward a constructive procedure for computing stationary symmetric Markov equilibria.

In this paper, we take a different approach to characterize Markovian equi-

[2] It can be easily generalized to a finite number of players.

librium in dynamic games. First, we restrict our attention in this paper to *Sequential Symmetric Markov Equilibrium* (SSME) A SSME is a sequence of pure strategy actions that each player finds optimal to follow, given that the other players follow the very same sequence of actions. The concept of SSME is a special case of a Markov perfect equilibrium (see, for instance, Maskin and Tirole[13]). We look for a SSME in the set of sequences of functions of the set Θ for the Great Fish War as introduced in Levhari and Mirman[12]. Second, we present a powerful set of constructive methods for computing SSME, using lattice programming methods and solving the game recursively by backward induction. Specifically, in computing the set of symmetric Markovian equilibrium n periods from the terminal period, we parametrize the continuation value function for each player using the history of Markovian equilibrium $n-1$ periods from the terminal period. We use lattice programming to construct a monotone operator mapping Θ into itself. Finally, exploiting the property that Θ, endowed with the pointwise partial order, is a complete lattice, we apply a generalization of Tarski's famous fixed point theorem to prove existence of a sequential symmetric Markov equilibrium.

The essential tool (aside from some important lattice programming results from Topkis[21]) is a powerful class of order theoretic fixed point theorems for ascending correspondences mapping complete lattices into their powersets. This result has been derived independently by Veinott[22] and Zhou[24], and is essentially an extension of Tarski's result for monotone functions on complete lattices that are self maps. Our paper makes an important contribution to the dynamic game literature by providing an application of a lattice theoretic fixed point construction to characterize equilibrium. Another important contribution of the paper is methodological: we present constructive methods and algorithms to compute the extremal sequential symmetric Markov equilibrium, thereby, relating the analytical results to numerical solutions. This paper is part of a growing literature on application of lattice theoretic arguments to games. Sleet[17] studies a large anonymous game, and the equilibrium concept is "open-loop," where agents only interact via a probability measure on states and types. In a finite agent version (where agents are not atomless), this methods do not apply. Amir[1] uses lattice methods to set up an application of Schauder's theorem. Curtot[5] studies stochastic games with lattice methods, but his methods do not cover the class of game we consider. In particular, closed-loop Markov games, in general, do not have double increasing differences a la Granot and Veinott[11].

The paper is organized as follows. Section 2 introduces the tools used in the paper. Sections 3 and 4 present the main results of the paper within the context of the so-called "fishwar game" of Mirman [15] and Levhari and Mirman [12]. Section 5 discusses future work and concludes.

2 MATHEMATICAL PRELIMINARIES

This paper uses tools from lattice theory and from fixed point theory and, in order to be self-contained, we provide a brief introduction. We present definitions and properties of partially ordered sets (or, posets) and lattices that are useful. We also provide a set of sufficient conditions on the objective and domain of a maximization problem for the set of maximizers to have important topological and order properties, and for the set of maximizers to be monotone in specific parameters.

Order and Lattice: A partially ordered set $(X, \leq)$ is a *lattice* if the greatest lower bound and lowest upper bound of any two elements x and x' in X exist (in X); these are denoted as $x \wedge x'$ and $x \vee x'$, respectively.

Example 1. The set $[0, 1] \times [0, 1]$ endowed with the pointwise partial order is a lattice; the $\wedge$ and $\vee$ operations are the componentwise inf and sup, respectively. This is the Euclidean lattice.

Example 2. Let $g_{\min}$ and $g_{\max} : K \to R^+$ be two increasing (i.e., non-decreasing) and continuous function on a compact subspace of R, such that $g_{\min}(k) \leq g_{\max}(k)$ for all $k \in K$. Consider the set X of functions $c : K \to R^+$, such that:
(i) $g_{\min}(k) \leq c(k) \leq g_{\max}(k)$,
(ii) $c(k)$ increasing in k, and
(iii) $c(k)$ continuous k, for all $k \in K$
The set X, endowed with the pointwise partial order, is a lattice. The $\wedge$ and $\vee$ operations are the pointwise inf and sup, respectively.

A partially ordered set $(X, \leq)$ is a *complete lattice* if the greatest lower bound and lowest upper bound of any subset X' of X exist (in X); and, these are denoted by $\wedge X'$and $\vee X'$, respectively.

Fortunately, to show that a partially ordered set is a complete lattice requires less work than the definition of completeness would have us believe. And, for this purpose, it is important to define maximal elements.

A partially ordered set $(X, \leq)$ has a *maximal element* $x'_{\max}$ in $X' \subset X$ if $x'_{\max} \leq x \in X'$implies $x'_{\max} = x$. The element $x'_{\max}$ in $X'$is the *greatest (or maximum) element* of X' if $x \leq x'_{\max}$ for every $x \in X'$. The greatest element of X', if it exists, is called the *top element* of X'.

Theorem 0 - Characterization of a Complete Lattice (Davey and Priestley [9]): *Let $(X, \leq)$ be a non-empty partially ordered set. Then X is a complete lattice if and only if X has a top element and for any $X' \subset X$, $\wedge X'$ exists in X.*

Example 3. The set X of example 2 is not a complete lattice (since the pointwise limit of a sequence of continuous functions is not necessarily continuous). However, for the same $g_{\min}$ and $g_{\max}$ as in example 2, consider the set Θ of functions $h : K \to K$, such that:
(i) $g_{\min}(k) \leq h(k) \leq g_{\max}(k)$,and

(ii) $h(k)$ increasing (i.e., non-decreasing) in k and
(iii) $h(k)$ is upper semicontinuous (usc) in k, for all $k \in K$.
The set Θ, endowed with the pointwise partial order, is a complete lattice.

Proof. The pointwise inf of any family of elements in Θ is bounded below by $g_{\min}$ and the pointwise sup of any family of elements in Θ is bounded above by $g_{\max}$, and is non-decreasing. The pointwise inf of a family of usc functions is usc (See Bourbaki [4] or Aliprantis and Border[2]), and $g_{\max}$ is the top element of Θ. Thus, by Theorem 0, Θ is a complete lattice. ∎

The complete lattice $(\Theta, \leq)$ of Example 3 is used extensively in this paper. And, there are two points worth mentioning:
(a) For all $\Theta' \subset \Theta$, $\forall k \in K$, $\wedge\Theta'(k) = \inf\{h(k),\ h \in \Theta'\}$ and $\vee\Theta'(k) = \sup\{h(x),\ h \in \Theta'\}$.
(b) If $\Theta' = \{h_n\}_{n=0}^{\infty}$ with $h_i \geq h_{i+1}$ then the sequence $\{h_n\}_{n=0}^{\infty}$ converges pointwise to $\wedge\Theta'$.

Consider $a < b$, $a, b \in X$ and the interval $[a, b] \subset X$ is said to be an *order interval* if $[a, b]$ is closed in X. Tht is, an *order interval* is a set $I[a, b] = \{x | a \leq x \leq b; a, b \in X\}$.

Isotone (or Order Preserving) Mappings on a Poset: Let $(X, \geq_X)$ and $(Y, \geq_Y)$ be partially ordered sets. A function $f : X \to Y$ is *isotone* (or *monotone)* on X if $f(x') \geq_Y f(x)$, when $x' \geq_X x$, for $x, x' \in X$. If $f(x') >_Y f(x)$ when $x' >_X x$ for $x,\ x' \in X$, then we say f is *increasing.* If $f(x') >_Y f(x)$ when $x' \geq_X x$, $x' \neq x$, then we say f is *strictly increasing.* We say $f(x)$ is *antitone* if $f(x) \geq_Y f(x')$ if $x' \geq_x x$. A real-valued function $f(x)$ is *upper semi-continuous (usc)* if for each real a, the set $f^{-1}(-\infty, a)$ is open and *lower semi-continuous (lsc)* if $-f$ is usc. We use capital letters to denote correspondences (e.g., multifunctions). A correspondence (or multifunction) $F : X \to 2^Y \backslash\varnothing$ is *ascending* in the set relation S , denoted by $\geq_S$, if $F(x') \geq_S F(x)$, when $x' \geq_X x$.If $\geq_S$ defines a partial order on $2^Y\backslash\varnothing$, then we say F is isotone. We say a (non-empty valued) correspondence $F : X \to 2^Y$ is *upper hemi-continuous (uhc)* if for each closed $A \in 2^Y\backslash\varnothing$, $F^{-1}(A)$ is closed, and *lower hemi-continuous (lhc)* if for each open $B \in 2^Y\backslash\varnothing, F^{-1}(B)$ is open. A (non-empty valued) correspondence is *continuous* if it is both uhc and lhc.

Fixed points: Let $\mu : X \to 2^X$ be a non-empty valued correspondence for each $x \in X$. The correspondence μ is said to have a *fixed point* if there exists an x such that $x \in \mu(x)$. And, if μ is a function, then a fixed point is an x^* such that $x^* = \mu(x^*)$.

Supermodular Functions and Increasing Difference: Suppose that X is a lattice. A function $f : X \to \bar{R}$ is *supermodular (strictly supermodular)* in x if for all x and y in X, $f(x \vee y) + f(x \wedge y) \geq (>) f(x) + f(y)$. As is clear in the sequel, a supermodular function is a super* function from a lattice to a chain. Here * is a binary "+" operation on a chain. Thus, perhaps more appropriately, a supermodular (strictly supermodular) function is referred to as a *superadditive (strictly superadditive)* function.

Consider a partially ordered set $\Psi = X_{d1} \times P$ (with order $\geq$), and $B \subset \Psi$. The function $f : B \longrightarrow R$ has *increasing difference* in (x_1, p) if for all $p_1, p_2 \in P$, $p_1 \leq p_2 \Longrightarrow f(x, p_2) - f(x, p_1)$ is non-decreasing in $x \in B_p$, where B_p is the p section of B. If this difference is strictly increasing in x then f has *strictly increasing difference* on B.

An important property of the class of supermodular functions is that they are closed under pointwise limits. (Topkis[**?**], Lemma 2.6.1).

There are various ways to define monotonicity of a set E in a parameter t, depending on the order one endows to the powerset of $E(t)$. We mainly use two definitions of monotonicity:

Ascending sets: A lattice $(E(t), \leq_E)$, where $t \in (T, \leq_T)$ is *ascending* in t if $t_1 \leq_T t_2$ implies that $E(t_1) \leq_E E(t_2)$. Depending on the order of $E(t)$, ascending has different implications. Given $t_1 \leq_T t_2$ for the *induced or strong set order* on $E(t)$, ascending requires that for $y_i \in E(t_i), i = 1, 2, y_1 \wedge y_2 \in E(t_1)$ and $y_1 \vee y_2 \in E(t_2)$. For the *set inclusion order* on $E(t)$, ascending means that $E(t_1) \subseteq E(t_2)$. Clearly, these imply different orderings, as the following examples show.

Example 4. Consider $E(t = 0) = [1, 2]$ and $E(t = 1) = [0, 3]$; E is ascending in t in the set inclusion but not in the induced or strong set order.

Example 5. Consider $f : X \to R$ and $g : X \to R$, both increasing functions such that $g \leq f$ pointwise then the set $E(t) = [g(t), f(t)]$ is ascending in t in the induced set order, but not necessarily in the set inclusion order.f

Example 6. For any increasing function $f : X \to R^+$, $E(t) = [0, f(t)]$ is ascending in t in both orders.

Unless otherwise specified, ascending is taken in the sense of induced or strong set order, in this paper.

2.1 Characterization of the Optimal Correspondence

We study the structure of optimal solutions of a collection of parametrized problems emerging from a study of pure strategy equilibrium in dynamic games. Here, the decision maker controls a set of variables x while facing an environment that can be characterized by two sets of interesting parameters, a finite dimensional parameter vector $t \in T$, and an infinite dimensional parameter $g \in \Theta$, where Θ is a lattice of isotone functions. Specifically, we consider the following parametrized decision problem:

$$\max_{x \in S(t;g)} H(x, t; g). \tag{M}$$

Here, the decision vector $x \in S(t; g) \subset X$ and $\gamma(t; g)$ is a solution to the maximization problem. Our focus is on comparative static results.

In particular, we study monotonicity of the optimal policy correspondence in the strong set order in t, for each g, that allows construction of selections in Θ. Also, monotonicity in the strong set order in g, for each t, helps define a monotone operator on Θ,a which is very useful. The existence and comparative statics for the problem is particularly difficult from a lattice programming perspective (aside from the recursive nature of the dynamic programming problem and the issue of the preservation of supermodularity under the sup operation). The presence of the infinite dimensional parameter g not only in the objective, but also in the domain (as an aspect of the feasible correspondence) makes the analysis difficult.

For clarity of exposition we distinguish between the state variable $t \in T$, a real vector, and the state variable g, a function belonging to a set Θ. Throughout the paper we assume that $S(t;g)$, X, and T are compact subsets of R (therefore, compact sublattices)[3]. Also, assume that Θ is the set defined in Example 3 (in section 2.1). In other words, Θ is a complete lattice and an order interval of the space of bounded, usc, increasing functions $g : T \to X$, endowed with the pointwise partial order. We also assume that for all $t \in T$,

$$g(t) \in S(t;g) \subset [g_{\min}(t), g_{\max}(t)], \tag{M'}$$

and that $S(t;g)$ is an upper hemicontinuous correspondence in t. In the next section, we interpret $H(x,t;g)$ as the objective of a player in a given period, assuming the other player plays the strategy g in that same period, and $S(t;g)$ is the set of feasible actions for given values g and t. Our first result stems from lattice theory and the work of Topkis on the maximization of a supermodular objective.

Theorem 1 (Topkis, [**?**] Lemma 2.8.1 and Theorem 2.8.3): *Suppose that $S(t;g)$ is ascending in t for all $g \in \Theta$ and is ascending in g for all $t \in T$. Assume that $H(x,t;g)$ is supermodular in $x \in S(t;g)$ and is upper-semicontinuous on $S(t;g)$; and for all $t^2 \le t^1$ and for all $x^2 \in S(t^2;g)$ and $x^1 \in S(t^1;g)$,*

$$H(x^1,t^1;g) + H(x^2,t^2;g) \le H(x^1 \wedge x^2, t^2;g) + H(x^1 \vee x^2, t^1;g). \tag{1}$$

And, suppose for any $g_1 \le g_2$, $t \in T$,and $x_i \in S(t;g_i)$,for $i = 1,2$:

$$H(x_1,t;g_1) + H(x_2,t;g_2) \le H(x_1 \wedge x_2, t;g_1) + H(x_1 \vee x_2, t;g_2).$$

Then, for each t in T and g in Θ, $\gamma(t;g) = \arg\max_{x \in S(t;g)} H(x,t;g)$ is a non-empty compact and subcomplete sublattice of R^n with a greatest and a least element, and the greatest and least elements are both increasing in t and in g, and $\gamma(t;g)$ is ascending in t and in g.

We exploit the supermodularity properties of the objective to generate monotonicity properties of the optimal correspondence. We also like to have sufficient topological properties to insure compactness of the set of maximizers, namely the upper semicontinuity of the objective. Furthermore, we are working on a recursive dynamic setup; given the objective H in a particular period, we need to make an upper semicontinuous selection from the optimal correspondence so

[3] Generalizations to X and $S(t;g)$ compact subsets of R^n with a sublattice structure and T compact sublattice of R^m are possible.

as to, recursively, construct the objective for the next period. The next proposition shows that such a selection exists, under additional assumptions on the objective function and the domain of the decision problem. In particular, the monotonicity of the objective in the state parameter t is critical: not only is the set of increasing and upper semicontinuous optimal selections, non-empty, but it is also a complete lattice. Furthermore, we can construct the smallest increasing upper semicontinuos optimal selection, as shown in Proposition 2.

Proposition 1: *In addition to the hypothesis of Theorem 1, suppose that (i) the domain $S(t;g)$ is upper hemicontinuous in t for each g in Θ, (ii) the objective $H(x,t;g)$ is increasing in t, for each g in Θ, and (iii) for all $t_1 \geq t_2$* and all g in Θ, $S(t_2;g) \subset S(t_1;g)$. *Then* $\max \gamma(t;g)$ *is an upper-semicontinuous function of t.*

Proof. In this proof, we suppress the dependency on g for clarity of exposition. Recall from Theorem 1, that the optimal selection $\max \gamma(t)$ is an increasing function, defined on a compact space, and therefore it is continuous almost everywhere. To show that it is usc, it is sufficient to show that it is continuous to the right,, at every point of discontinuity.

Consider $\bar{t}$ as a point of discontinuity of $\max \gamma(t)$ and $\{t_n\}_{n=0}^{\infty}$ a decreasing sequence converging to $\bar{t}$ from the right. By monotonicity of $\max \gamma(t)$, the sequence $\{\max \gamma(t_n)\}_{n=0}^{\infty}$ is decreasing and is bounded below (by $\max \gamma(\bar{t})$), and therefore converges to $\alpha \geq \max \gamma(\bar{t})$. If we prove that $\alpha = \max \gamma(\bar{t})$, then $\max \gamma(.)$ is continuous from the right at $\bar{t}$, and is, therefore, an usc selection. We prove it by contradiction: we postulate that $\alpha > \max \gamma(\bar{t})$ then arrive at a contradiction.

Since $S(t)$ is upper hemicontinuous in t by hypothesis (i), we have $\alpha \in S(\bar{t})$. By the monotonicity of $H(x,t)$ in t (hypothesis (ii)), for all $x \in S(\bar{t})$:

$$H(x,t_n) \geq H(x,t_{n+1}) \geq ... \geq H(x,\bar{t}),$$

Hypothesis (iii) implies $S(t_{n+1}) \subset S(t_n) \subset S(\bar{t})$, so that,

$$\max_{S(t_n)} H(x,t_n) \geq \max_{S(t_{n+1})} H(x,t_{n+1}) \geq ... \geq \max_{S(\bar{t})} H(x,\bar{t}).$$

Consequently,

$$H(\max \gamma(t_n),t_n) \geq H(\max \gamma(t_{n+1}),t_{n+1}) \geq ... \geq H(\alpha,\bar{t}),$$

and the objective H is monotonic in $(\max \gamma(t_n),t_n)$.

Recall that the decreasing sequence $\{(\max \gamma(t_n),t_n)\}_{n=0}^{\infty}$ converges (monotonically from the right) to $(\alpha,\bar{t})$,and H is usc in t and x by hypothesis (of Theorem 1). Together with the monotonicity of H in $(\max \gamma(t_n),t_n)$, this imply that,

$$H(\max \gamma(t_n),t_n) \text{ converges to } H(\alpha,\bar{t}) < H(\max \gamma(\bar{t}),\bar{t}).$$

Here the last inequality stems from the hypothesis that $\alpha > \max \gamma(\bar{t})$ (i.e., α cannot be a maximizer of $H(x,\bar{t})$ in the domain $S(\bar{t})$).

Consequently, for n sufficiently large,

$$H(\max\gamma(t_n), t_n) < H(\max\gamma(\bar{t}), \bar{t}). \tag{L1}$$

Finally, since $t_n \geq \bar{t}$, $\max\gamma(\bar{t}) \in S(t_n)$ for all n, and therefore, by the monotonicity of $H(x,t)$ in t,

$$H(\max\gamma(\bar{t}), t_n) \geq H(\max\gamma(\bar{t}), \bar{t}). \tag{L2}$$

Results (L1) and (L2) imply that:

$$H(\max\gamma(\bar{t}), t_n) > H(\max\gamma(t_n), t_n),$$

which contradicts the definition of $\max\gamma(t_n)$. ■

Theorem 2: *In addition to the hypothesis of Theorem 1, suppose that (i) the domain is $S(t;g)$ is upper hemicontinuous in t, (ii) the objective $H(x,t;g)$ is increasing in t on T, and (iii) for all $t_1 \geq t_2$ and all g in Θ, $S(t_2;g) \subset S(t_1;g)$. Then for each g in Θ, the set Φ of upper semicontinuous and increasing selections in t belonging to the optimal correspondence $\gamma(t;g)$ is a nonempty complete lattice with the pointwise partial order.*

Proof. By Theorem 1, for any t in T, $\gamma(t;g)$ is nonempty and compact. By Proposition 1, $h_{\max}(t;g) = \max\gamma(t;g)$, is upper semicontinuous, in addition to be increasing in t (by Theorem 1). Thus, Φ is nonempty and has a maximal element. Second, consider any subset V of Φ, that is, for any $v \in \Phi$, and any t in T, $v(t) \in \gamma(t;g)$. The lower envelope of V, $\inf_{v\in V}\{v(t)\}$ is increasing and usc (See Bourbaki [4]or Aliprantis and Border [2]). For all t, $\gamma(t;g)$ is a compact set, and if $v(t) \in \gamma(t;g)$, then $\inf_{v\in V}\{v(t)\} \in \gamma(t;g)$. This proves that the lower envelope is in Φ. Therefore, by Theorem 0, Φ is a complete lattice. ■

While the maximal element of Φ is $\max\gamma(t;g)$, the minimal element is generally not $\min\gamma(t;g)$ (since $\min\gamma$ may not be an upper semicontinuous function of t). However,

Proposition 2: *In addition to the hypothesis of Theorem 1, suppose that, for each g in Θ (i) the domain is $S(t;g)$ is upper hemicontinuous in t, (ii) the objective $H(x,t;g)$ is increasing in t on T, and (iii) for all $t_1 \geq t_2$, $S(t_2;g) \subset S(t_1;g)$. The minimal element of Φ is the function:*

$$h_{\min}(t;g) = \inf_{x>t}\{\min\gamma(x;g)\}$$

Proof. The proof is similar to that of Proposition 1. By Theorem 1, $\min\gamma(t)$ is increasing in t, and so is $h_{\min}(t)$. Note that,

$$h_{\min}(t) = \lim_{y\to t^+}\min\gamma(y).m$$

By construction $h_{\min}(t)$ is usc in t, and the two functions, $h_{\min}$ and $\min\gamma$, coincide almost everywhere since they differ only at the discontinuity points of the increasing function $\min\gamma$. Recall that $\min\gamma(t)$ is an increasing function defined on a compact interval of the real line and is therefore almost everywhere continuous. To show that $h_{\min}(t) \in \gamma(t)$, it is sufficient to demonstrate it only at the points where $\min \gamma(t)$ is discontinuous. Let $\widetilde{t}$ be a point of discontinuity of $\min \gamma(t)$, and $\{t_n\}_{n=0}^{\infty}$ be a decreasing sequence $t_n > \widetilde{t}$, converging to $\widetilde{t}$. All we need to prove is that $h_{\min}(\widetilde{t})$, which is exactly $\lim_{n\to\infty} \min \gamma(t_n)$, belongs to $\gamma(\widetilde{t})$. First, $h_{\min}(\widetilde{t}) \in S(\widetilde{t})$ since $h_{\min}(\widetilde{t}) \leq \max \gamma(\widetilde{t})$.The monotonicity of $\min \gamma$ (from Theorem 1), the sequence $\min \gamma(t_n)$ is decreasing, and converges to $h_{\min}(\widetilde{t})$. Our strategy is to assume that $h_{\min}(\widetilde{t})$ does not belong to $\gamma(\widetilde{t})$ and then arrive at a contradiction. Recall that $(t_n, \min \gamma(t_n)) \geq (t_{n+1}, \min \gamma(t_{n+1})) \geq ... \geq (\widetilde{t}, h_{\min}(\widetilde{t}))$. By monotonicity of the objective in t, for all $x \in S(\widetilde{t})$:

$$H(x, t_n) \geq H(x, t_{n+1}) \geq ... \geq H(x, \widetilde{t}).$$

Since $S(\widetilde{t}) \subset ... \subset S(t_{n+1}) \subset S(t_n)$,

$$H(\min \gamma(t_n), t_n) \geq H(\min \gamma(t_{n+1}), t_{n+1}) \geq ... \geq H(h_{\min}(\widetilde{t}), \widetilde{t}).$$

As $(t_n, \min \gamma(t_n))$ converges monotonically to $(\widetilde{t}, h_{\min}(\widetilde{t}))$, H is usc in x and in t, and H is monotone in $(t_n, \min \gamma(t_n))$,

$$H(\min \gamma(t_n), t_n) \text{ converges to } H(h_{\min}(\widetilde{t}), \widetilde{t}) < H(\min \gamma(\widetilde{t}), \widetilde{t}).$$

Here the last inequality follows from the assumption that $h_{\min}(\widetilde{t})$ does not belong to $\gamma(\widetilde{t})$.

Consequently, it must be that, for n sufficiently large,

$$H(\min \gamma(t_n), t_n) < H(\min \gamma(\widetilde{t}), \widetilde{t}).$$

Since $t_n \geq \widetilde{t}$, $\min \gamma(\widetilde{t}) \in S(t_n)$, by monotonicity of H,

$$H(\min \gamma(\widetilde{t}), \widetilde{t}) \leq H(\min \gamma(\widetilde{t}), t_n).$$

Combining the last two inequalities implies that, for n large,

$$H(\min \gamma(t_n), t_n) < H(\min \gamma(\widetilde{t}), t_n),$$

which contradicts the definition of $\min \gamma(t_n)$. ■

It is worth noting that if the objective H is strictly supermodular in (x, t) (strict inequality in condition (1) of Theorem 1), then all selections from the optimal policy correspondence are increasing in t. In this case, the optimal policy correspondence is a continuous single-valued function, except on a countable number of discontinuity points. As a result, the maximal usc increasing and minimal usc increasing selections coincide (i.e., $\max \gamma(t; g) = h_{\min}(t; g)$).

3 EQUILIBRIUM: EXISTENCE AND CHARACTERIZATION

In this section we analyze the classic dynamic game known as "Great Fish War," studied by Levhari and Mirman[12] with specific functional forms. This game is also analyzed subsequently by others, in the general case, earliest examples are Amir[1] and Sundaram [18]. In this paper, we prove the existence of a sequential symmetric Markov equilibrium by combining the results on existence of static equilibrium with the results on existence and optimality from the recursive structure of the game. Our methodology is based on proving that the appropriate recursively defined objective has the properties hypothetized in Theorems 1 and 2.

3.1 Assumptions and Definitions

For simplicity of exposition, we consider a two-agent version of the game. Each agent or player has preferences represented by the utility function u (identical for both players) defined over the single consumption good in each period. Consumption generates current period utility while investment is required to generate future stock of the resource which in turn can support future consumption. The payoff is given by sum of discounted utility over a finite horizon. Player i aims,

$$\max_{\{c_{i,t}\}} \sum_{t=0}^{T} \beta^t u(c_{i,t}) \quad (\Pi)$$

$$\begin{aligned} \text{subject to the feasibility condition, } c_{1,t} + c_{2,t} + k_{t+1} &\leq f(k_t), & (1) \\ \text{and nonnegativity conditions, } (c_{1,t}, c_{2,t}, k_{t+1}) &\geq 0 \text{ for } t = 0, ..., T, & (2) \\ \text{given the initial capital } k_0 &> 0. & (3) \end{aligned}$$

We assume that the law of motion, $f(.)$, and the utility function, $u(.)$, satisfy the following,

(A*f*) $f : R^+c \to R^+$ is a continuous, strictly increasing function, and there exists $\overline{k} > 0$ such that $f(\overline{k}) = \overline{k}$ and $f(k) < k$ for all $k > \overline{k}$;

(A*u*) $u : R^+ \to R$ is strictly increasing, continuous, concave, and $u'(0) = \infty$.

Note that there is no concavity condition on f, and that the state space can be restricted to the compact set $K = [0, \overline{K}] \subset R$, where $\overline{K} = \max(k_0, \overline{k})$. The continuity of u and the compactness of K imply that u is bounded on K.

A symmetric Markov equilibrium in period t is a consumption (as a function of current capital stock) that maximizes the utility for a player when the other player chooses that same consumption function, $c_{it} = c_t$ for $i = 1, 2$. A sequential symmetric Markov equilibrium is a particular sequence of symmetric Markov equilibrium for each period. We formalize these in the following definitions:

Sequential Symmetric Markov Equilibrium in Consumption: *A sequential symmetric Markov equilibrium (SSME) in consumption is a sequence of consumption functions* $\{\mu_t(k)\}_{t=0}^T$*,* $\mu_t : K \to K$*, such that the sequence* $\{c_t\}_{t=0}^T = \{\mu_t(k_t)\}_{t=0}^T$ *where* $k_t = f(k_{t-1}) - 2\mu_{t-1}(k_{t-1})$ *for* $t = 1, ..., T+1$ *maximizes* (Π) *for one player, given* k_0*, and given that the other player chooses the same sequence of actions* $\{c_t\}_{t=0}^T = \{\mu_t(k_t)\}_{t=0}^T$*.*

Clearly, if μ_t is a symmetric Markov equilibrium, then necessarily $0 \leq \mu_t(k) \leq f(k)/2$ for all $k \in K$, and $f - 2\mu_t : K \to K$ is the aggregate investment function in period t. As a consequence, a sequential symmetric Markov equilibrium can also be defined in terms of investment functions.

Sequential Symmetric Markov Equilibrium in Investment: *A sequential symmetric Markov equilibrium (SSME) in investment is a sequence of investment functions* $\{g_t(k)\}_{t=0}^T$, $g_t : K \to K$, *such that the sequence* $\{\mu_t\}_{t=0}^T = \{[f(k_t) - g_t(k_t)]/2\}_{t=0}^T$, *where* $k_t = g_{t-1}(k_{t-1})$, *maximizes* (Π) *for one player, given* k_0, *and given that the other player chooses the sequence of actions* $\{\mu_t\}_{t=0}^T = \{[f(k_t) - g_t(k_t)]/2\}_{t=0}^T$.

That is, in period t, g_t is the investment function that results if μ_t is the best response consumption function to itself (i.e., the best consumption response of a player whenever the other player chooses the consumption function μ_t). A symmetric Markov equilibrium is said to be stationary if $g_t \equiv g_{t-1}$, for $t = 0, ..., T-1$.

Rather than working with consumption functions, we follow Sundaram [18] and build our arguments using investment functions. As noted before, any choice of consumption function $\mu_{1,t}$ of player 1 that is a candidate symmetric Markov equilibrium must satisfy $0 \leq \mu_{1,t} \leq f/2$ and corresponds to a unique investment strategy $g_{1,t} = f - 2\mu_{1,t}$ ($g_{1,t}$ is the investment strategy that would result if player 2 chooses the consumption function $\mu_{1,t}$). Equivalently, to any investment strategy $0 \leq g_{1,t} \leq f$ corresponds a unique symmetric consumption function $\mu_{1,t} = (f - g_{1,t})/2$. Denote player 2's consumption best response to $\mu_{1,t}$ by $\mu_{2,t}$.To the consumption function, $\mu_{2,t}$, corresponds the unique investment function $g_{2,t} = f - 2\mu_{2,t}$. Since $\mu_{2,t}$ is the best (consumption) response to $\mu_{1,t}$ if and only if $g_{2,t}$ is a best (investment) response to $g_{1,t}$, a symmetric Markov equilibrium in period t arises if and only if $g_{1,t}$ and $g_{2,t}$ coincide. We construct an operator from the set of investment functions into itself, and, as in Sundaram [18], the proof of existence of a symmetric Markov equilibrium in period t relies on a fixed point theorem applied to this operator.

3.2 Objective and Best Response Correspondence

Denote by v^*_{T-i+1} the value function of player 1 in period $T-i+1$, and assume that $v^*_{T-i+1} : K \to R$ belongs to the set V of real-valued usc, increasing, and bounded functions.[4] In period $T-i$, consider player 1 who takes as given the action $g \in \Theta$ of player 2. To any action g corresponds a unique symmetric consumption strategy via the rule $\mu(k) = [f(k) - g(k)]/2$, which leaves the maximum quantity,

$$f(k) - [f(k) - g(k)]/2 = [f(k) + g(k)]/2,$$

for player 1's consumption. The amount that player 1 does not consume is investment, denoted k', and is the input for period $T-i+1$ production.

[4] We prove these properties recursively in the next section.

The problem that player 1 faces can therefore be written as follows,

$$\max_{k' \in S(k;g)} \{u([f(k)+g(k)]/2 - k') + \beta v^*_{T-i+1}(k')\}, \tag{4}$$

where the choice domain is $S(k;g) = \{k' \in R, 0 \le k' \le [f(k)+g(k)]/2\}$. To use the notation developed in Section 2, we identify the objective by the function H, that is,

$$H(k', k; g, v^*_{T-i+1}) = u([f(k)+g(k)]/2 - k') + \beta v^*_{T-i+1}(k').$$

Since $g \in \Theta$ and v^*_{T-i+1} is usc, the objective of the maximization problem is usc in k', and, therefore, a maximum exists. However, the maximum is generally not unique, and the set of maximizers of the objective in (3), also called the "optimal (investment) correspondence" or "best response (investment) correspondence" are,

$$\gamma(k; g, v^*_{T-i+1}) = \arg \max_{k' \in S(k;g)} H(k', k; g, v^*_{T-i+1}).$$

Given player 2's action $g(k)$, if player 1's indirect utility function in period $T-i$ is,

$$v_{T-i}(k; g) = \max_{k' \in S(k;g)} H(k', k; g, v^*_{T-i+1}).$$

Note that, since u is strictly increasing in k, the indirect utility function v_{T-i} is increasing in k.[5]

Note that the assumptions of theorems 1 and 2 are satisfied. Indeed:

(i). $S(k;g) = [0, (f(k)+g(k))/2]$ is a compact subset of K ascending in k in both the set inclusion and the induced set order and ascending in g in the set inclusion order,

(ii). $S(k;g)$ is an upper hemicontinuous correspondence in k whenever $g \in \Theta$ (since f is continuous), and $H(k', k; g, v^*_{T-i+1})$ is upper semicontinuous on $S(t;g)$ whenever v^*_{T-i+1} is upper semicontinuous (Theorem 2).

(ii). $H(k', k; g, v^*_{T-i+1})$ is trivially supermodular in $k' \in S(t;g)$ (Theorem 1).

(iii). $H(k', k; g, v^*_{T-i+1})$ has increasing differences in (k, k') (condition (1) in Theorem 1), and increasing differences in (k, g) (condition (2) in Theorem 1).

Thus, problem (M) can be interpreted as the maximization problem of an agent whose objective depends on the action g of another agent, in a one period game. By definition γ is the best response correspondence of the agent, and a fixed point of γ in Θ is a function $\mu \in \Theta$ such that, for all t in T $\mu(t) \in \gamma(t;\mu)$. When identical agents play the game, the fixed points of γ are precisely the set of symmetric Markov equilibrium in Θ. Existence, characterization, and computation of the fixed points of γ are analysed for the game (M).

[5] We show below that when g is a fixed point of γ (i.e., a Nash equilibrium) in Θ, v_{T-i} is also usc in k.

3.3 Existence of a Symmetric Markov Equilibrium

Instead of using a topological fixed point argument to establish the existence of a fixed point of γ, we exploit the monotonicity of the optimal correspondence and apply Veinott[22]/Zhou[24] extension of Tarski's famous fixed point theorem.

Theorem 3: *(Veinott[22], Zhou[24]) S a complete lattice and Γ an ascending correspondence from S to S, and for every s in S, $\Gamma(s)$ is a non-empty closed sublattice. Then the set of fixed points of Γ is a non-empty complete lattice.*

Note that, by Theorem 2, for any $g \in \Theta$, the set of increasing and usc selections of γ is a non-empty complete lattice, and therefore a non-empty closed sublattice. Thus, γ is ascending in g by Theorem 1. The Veinott/Zhou theorem, therefore, applies and we have the following existence result.

Proposition 3: *The set of fixed points of γ in Θ is a non-empty complete lattice.*

For any increasing, usc and bounded v^*_{T-i+1}, consider the correspondence Γ defined as follows:

$$\forall g \in \Theta,\ \Gamma g(k) = \gamma(k; g, v^*_{T-i+1}) = \arg \max_{k' \in S(k;g)} H(k', k; g, v^*_{T-i+1}).$$

By results in Section 2, for any k, $\Gamma g(k)$ is nonempty and contains a greatest element. Let $Tg = \Gamma g \cap \Theta$, for any g in Θ. That is, Tg is the set of optimal selections that are increasing and usc. By Theorem 4, the greatest element of Γg is an increasing function of k that is also usc, which implies that Tg is nonempty.

A function $g^*_{T-i} \in \Theta$ is a symmetric Markov equilibrium in period $T-i$ if it is best response to itself, that is, if it is a fixed point of the operator T, i.e. if:

$$\text{for all } k,\ g^*_{T-i}(k) \in \Gamma g^*_{T-i}(k).$$

The following result is a direct application of Proposition 3.

Proposition 4 - Existence of a Symmetric Markov Equilibrium. *The set of symmetric Markov equilibrium in period $T-i$ is a nonempty complete lattice, with the greatest element and the least element.*

Given the period $T-i+1$ value function v^*_{T-i+1}, to each symmetric Markov equilibrium g^* in period $T-i$ corresponds a unique period $T-i$ value function v^*_{T-i} that obtains when both players play g^*, i.e.:

$$\begin{aligned} v^*_{T-i}(k) &= \max_{k' \in S(k;g^*)} H(k', k; g^*, v^*_{T-i+1})\} \\ &= \max_{k' \in S(k;g^*)} \{u([f(k) + g^*(k)]/2 - k') + \beta v^*_{T-i+1}(k')\}. \end{aligned} \tag{5}$$

Equation (2), therefore, implicitly defines a mapping from the complete lattice of Markov equilibrium g^* into the partially ordered set V of value functions in period $T-i$ given by (2) (and contingent on v^*_{T-i+1}). The monotonicity of this mapping generates some important structure on the set V.

3.4 Computation of a Symmetric Markov Equilibrium

Order-based fixed point theorems generally are more than existential in the sense that they can, in many cases, provide an insight for a procedure to compute some the extremal fixed points. Here we show a method for constructing the maximal fixed point and finding a lower bound for the minimal fixed point.

To compute the maximal fixed point we construct a monotone mapping A from Θ into itself (which has a nonempty set of fixed points by Tarski's theorem). Since Θ has a top element, the monotonicity of A generates a decreasing sequence of elements of Θ, starting from the top element. We show, in the next proposition, that this sequence converges uniformly to the maximal fixed point of γ in Θ. Computing the minimal fixed point is similar, except for one complication. We construct a monotone mapping B from Θ into itself, and generate an increasing sequence of elements of Θ starting from the bottom element of Θ. We show that this sequence converges pointwise, almost everywhere, to the minimal fixed point of γ in Θ.

Proposition 5 - Computation of the Maximal Symmetric Markov Equilibrium: *Let us define an operator A such that $Ag(t) = \max \gamma(t; g)$ for any $g \in \Theta$ and for all $t \in T$. The operator A has the following properties:*

(i) It maps Θ into itself and is monotone[6].

(ii) The sequence $\{A^n g_{\max}\}_{n=0}^{\infty}$ converges pointwise to the maximal fixed point of A in Θ.

(iii) The maximal fixed point of A in Θ is the maximal fixed point of γ in Θ.

Proof. (i) For all $t \in T$, $Ag(t) \in S(t; g)$ by construction, and thus, $g_{\min}(t) \leq Ag(t) \leq g_{\max}(t)$ by condition (M'). $Ag(t)$ is increasing in t (Theorem 1), and usc (Proposition 1). The monotonicity of the operator A is a consequence of condition (2) in Theorem 1; indeed, consider $g_1 \leq g_2$ pointwise and suppose that A is not monotone, i.e. that $Ag_1(t) > Ag_2(t)$ for some t in T. Since A is ascending in g, $Ag_2(t) \in S(t; g_1)$ (and $Ag_1(t) \in S(t; g_2)$), therefore,

$$H(t, Ag_1(t); g_1) - H(t, Ag_2(t); g_1) \geq 0.$$

Using condition (2),

$$H(t, Ag_1(t); g_2) - H(t, Ag_2(t); g_2) \geq H(t, Ag_1(t); g_1) - H(t, Ag_2(t); g_1) \geq 0,$$

and thus:

$$H(t, Ag_1(t); g_2) - H(t, Ag_2(t); g_2) \geq 0.$$

By the optimality of $Ag_2(t)$, this last inequality implies that $Ag_1(t) \in \gamma(t; g_2)$, and therefore requires that $Ag_1(t) \leq Ag_2(t) = \max \gamma(t; g_2)$, thus contradicting the hypothesis that $Ag_1(t) > Ag_2(t)$.

(ii) Note that

$$Ag_{\max}(t) \in S(t; g_{\max}) \leq g_{\max}(t) \text{ and } Ag_{\min}(t) \in S(t; g_{\min}) \geq g_{\min}(t).$$

[6] A is monotone if $g_1 \leq g_2$ implies that $Ag_1 \leq Ag_2$ in the pointwise partial order for all $g_1, g_2 \in \Theta$.

By Tarski, the set of fixed points of A is a nonempty complete lattice. There is, thus, a maximal fixed point for A in Θ, which we denote $h_{\max}$. Since $Ag_{\max}(t) \leq g_{\max}(t)$ and A is monotone, the sequence $\{A^n g_{\max}\}_{n=0}^{\infty}$ is a decreasing sequence of increasing usc functions, and therefore converges (pointwise) to its pointwise inf, which we denote $\overline{h}$ (See example 3). In addition,

$$\lim_{n\to\infty} A^n g_{\max} = \overline{h} = A(\overline{h}),$$

and $\overline{h}$ is a fixed point of A (See Appendix A for a proof). Suppose that $\overline{h}$ is not the maximal fixed point of A, i.e, suppose there exists h in Θ and t in T such that,

$$\overline{h}(t) < h(t) \leq g_{\max}(t) \text{ and } Ah(t) = h(t).$$

By the monotonicity of A, for all n,

$$A^n g_{\max}(t) \geq h(t) > \overline{h}(t),$$

therefore,

$$\lim_{n\to\infty} A^n g_{\max}(t) = \overline{h}(t) \geq h(t) > \overline{h}(t),$$

which is impossible. Thus $\overline{h} = h_{\max}$.

(iii) Clearly, any fixed point of A is a fixed point of γ. Suppose that the maximal fixed point of A is not the maximal fixed point of γ, i.e, there exists a fixed point μ of γ and a $t \in T$ such that:

$$h_{\max}(t) \leq \mu(t) \leq g_{\max}(t).$$

Since μ is a fixed point of γ, $\mu(t) \in \gamma(t;\mu)$ and either (a) $\mu(t) = \max \gamma(t;\mu)$, in which case μ is a fixed point of A, and necessarily $\mu(t) \leq h_{\max}(t)$, or (b) $\mu(t) < \max \gamma(t;\mu) = A\mu(t)$ and by monotonicity of A,

$$h_{\max}(t) = Ah_{\max}(t) \leq \mu(t) < A\mu(t) \leq Ag_{\max}(t).$$

Recursively,

$$h_{\max}(t) \leq \mu(t) \leq \lim_{n\to\infty} A^n g_{\max}(t),$$

implying that $\mu(t) = h_{\max}(t)$, which contradicts the assumption that $\mu(t) < A\mu(t)$. ■

The fixed point constructed in this recursive procedure is the maximum symmetric Markov equilibrium for problem (M).

To compute a lower bound for the minimal fixed point, we construct a monotone mapping B from Θ into itself. Since Θ has a bottom element, the monotonicity of B generates an increasing sequence of elements of Θ, starting from the bottom element. We show, in the next proposition, that this sequence converges pointwise almost everywhere to the minimal fixed point of γ in Θ.

Proposition 6 - Computation of the Minimal Symmetric Markov Equilibrium. *Define an operator B such that $Bg(t) = \inf_{x>t}\{\min \gamma(x;g)\}$ for any $g \in \Theta$ and all $t \in T$. The operator B has the following properties:*

(i) It maps Θ into itself and is monotone.

(ii) The sequence $\{B^n g_{\min}\}_{n=0}^{\infty}$ converges pointwise to a function l and $\underline{h}(t) = \inf_{x>t} l(x)$ *is the minimal fixed point of B in Θ.*

(iii) The minimal fixed point of B in Θ coincides with the minimal fixed point of γ in Θ.

Proof. (i) By Proposition 2 $Bg(t)$ is in Θ. The monotonicity of B follows from the monotonicity of $\min \gamma(t; g)$ in g, which can be shown by an argument similar to that of Proposition 5.

(ii) Since $Bg_{\min}(t) \geq g_{\min}(t)$ and B is monotone, for each $t \in T$, the sequence $B^n g_{\min}(t)$ is an increasing sequence of R^n which is bounded above by $\max_{t \in T} g_{\max}(t)$, and, therefore, converges. Let l be the pointwise limit of the sequence $\{B^n g_{\min}\}_{n=0}^{\infty}$, and $\underline{h}(t) = \inf_{x>t} l(x)$. We prove in Appendix B that $\underline{h}$ is a fixed point of B. Suppose that there exists a fixed point h of B in Θ such that, for some $t \in T$,

$$g_{\min}(t) \leq h(t) \leq \underline{h}(t),$$

therefore, by the monotonicity of B, for all n,

$$B^n g_{\min}(t) \leq h(t) \leq \underline{h}(t).$$

If t is a continuity point of $\underline{h}$, then $\underline{h}(t) = \lim_{n \to \infty} B^n g_{\min}(t)$, and the previous inequality implies that $h(t) = \underline{h}(t)$. Thus, h and $\underline{h}$ coincide when $\underline{h}$ is continuous. Since both functions, h and $\underline{h}$ are usc, they must be identical.

The proof of (iii) can easily be adapted from the proof of Proposition 5. ∎

4 SEQUENTIAL SYMMETRIC MARKOV EQUILIBRIUM

The structure of the set of value functions has important consequences. We use these properties to prove existence of a sequential symmetric Markov equilibrium. Also, we outline a method for computing this equilibrium.

Proposition 7: *The set of value functions V is a nonempty lattice with minimal and maximal elements.*

Proof. For a given v^*_{T-i+1}, consider the set of two period symmetric Markov equilibrium in Θ for period $T-i$, denoted g^* and h^*, and suppose that $g^* \leq h^*$. Because u is increasing, for any $y \in S(k; g^*)$,

$$\begin{aligned} H(y, k; g^*, v^*_{T-i+1}) &= u([f(k) + g^*(k)]/2 - y) + \beta v^*_{T-i+1}(y) \\ &\leq \\ u([f(k) + h^*(k)]/2 - y) + \beta v^*_{T-i+1}(y) &= H(y, k; h^*, v^*_{T-i+1}). \end{aligned}$$

Taking the maximum of both expressions over the set $S(k; g^*)$, and since $S(k; g^*) \subset S(k; h^*)$ the previous inequality implies that:

$$\max_{y \in S(k, g^*)} H(y, k; g^*, v^*_{T-i+1}) \leq \max_{y \in S(k, h^*)} H(y, k; h^*, v^*_{T-i+1}).$$

Thus, the one to one correspondence between the set of symmetric Markov equilibrium in period $T-i$ and the set V of value functions in period $T-i$ preserves order. Also, the greatest and least symmetric Markov equilibrium points generate, respectively, the minimal and maximal elements of V. ∎

This result implies the partial pointwise ordering of the set of symmetric Markov equilibrium for any period and that the maximum symmetric Markov equilibrium is also Pareto optimal.

The proof of the existence of sequential symmetric Markov equilibrium relies on repeated applications of the previous theorem, starting from the last period of the game and proceeding by backward induction.

Proposition 8. Existence of Sequential Symmetric Markov Equilibrium: *There exists a sequential symmetric Markov equilibrium in Θ for the finite horizon game.*

Proof. First, consider the final period T of the game. The symmetric Markov equilibrium in period T is unique, and corresponds to the equal sharing among the players of the available output. In terms of investment functions, $g_T^* = 0$ is thus the unique period T symmetric Markov equilibrium. The resulting value function of each player in period T is then $v_T^*(.) = u(f(.)/2)$, and is clearly usc, increasing and bounded. Next, consider the game in period $T-1$. Proposition 4 guarantees the existence of a symmetric Markov equilibrium g_{T-1}^* in period $T-1$. By construction, $g_{T-1}^*(k)$ solves the maximization problem,

$$g_{T-1}^*(k) \in \arg \max_{k' \in S(k; g_{T-1}^*)} \{u([f(k) + g_{T-1}^*(k)]/2 - k') + \beta v_T^*(k')\}.$$

Denote $v_{T-i}^*(k; g_{T-1}^*)$, the indirect utility function,

$$v_{T-i}^*(k; g_{T-1}^*) = \max_{k' \in S(k; g_{T-1}^*)} \{u([f(k) + g_{T-1}^*(k)]/2 - k') + \beta v_T^*(k')\}. \qquad (6)$$

The right-hand side of (3) is usc in k and in k', and the correspondence $S(k; g_{T-1}^*)$ is upper hemicontinuous in k. Consequently by Berge ([3] Theorem 2 page 116), the function $v_{T-i}^*(k; g_{T-1}^*)$ is upper semi continuous in k. Since the right-hand side in (3) is increasing in k, $v_{T-i}^*(k; g_{T-1}^*)$ is also increasing in k. Using the value function $v_{T-i}^*(k; g_{T-1}^*)$, we can construct the game in period $T-2$; proceeding recursively the sequence $\{g_0^*, g_1^*, ..., g_T^* = u(f/2)\}$ is a sequential symmetric Markov equilibrium. ∎

It is important to note that the set of sequential symmetric Markov equilibrium is generally not a lattice, if $T \geq 2$, despite the fact that the set of symmetric Markov equilibria in any given period (and given future choices of players) is a complete lattice. For $T = 3$, consider two sequential symmetric markov equilibrium (g_0^*, g_1^*, g_2^*) and (h_0^*, h_1^*, g_2^*). While $g_1^* \vee h_1^*$ is a symmetric Markov equilibrium in the second period of the game[7] (given that both players

[7] The set of sequential symmetric markov equilibrium for the two-period game is therefore a (complete) lattice.

play g_2^* in the terminal period), does not imply that $g_0^* \vee h_0^*$ is a symmetric equilibrium in period 0.

A consequence of Proposition 7 is that, in each period, the maximum symmetric Markov equilibrium is better (in a Pareto sense) than all other symmetric Markov equilibria. This gives a particular importance to the sequence of maximum symmetric Markov equilibrium points, which can be computed period by period using the result in Proposition 5: setting $v_T^*(k) = u(f(k)/2)$, the value function in the terminal period corresponding to the unique symmetric Markov equilibrium $h_{T,\max}(k) = 0$ (i.e, the two players share equally the entire output and leave nothing), we note that v_T^* is usc, increasing and bounded. For period $T-1$, consider the sequence $\{A^n f\}_{n=0}^{\infty}$ (where $A^0 f = f$), defined recursively as,

$$A^{j+1}f(k) = \max(\arg \max_{k' \in S(k;A^j f)} \{u([f(k) + A^j f(k)]/2 - k') + \beta v_T^*(k')\}),$$

with $S(k; A^j f) = [0, (f(k) + A^j f(k))/2]$.

By Proposition 5, the sequence $\{A^n f\}_{n=0}^{\infty}$ converges uniformly to the maximal symmetric Markov equilibrium in period $T-1$, which we denote $h_{T-1,\max}$ (contingent on both players following the symmetric Markov equilibrium, $h_{T,\max} \equiv 0$ in period T). Note that $h_{T-1,\max}(k)$ is the unique function that solves,

$$h_{T-i,\max}(k) = \max(\arg \max_{k' \in S(k;h_{T-i,\max})} \{u([f(k) + h_{T-i,\max}(k)]/2 - k') + \beta v_T^*(k')\}),$$

where $S(k; h_{T-1,\max}) = [0, (f(k) + h_{T-1,\max}(k))/2]$. Then,

$$v_{T-1}^*(k) = \max_{0 \le k' \le [f(k) + h_{T-1,\max}(k)]/2} \{u([f(k) + h_{T-1,\max}(k)]/2 - k') + \beta v_T^*(k')\},$$

or, equivalently,

$$v_{T-1}^*(k) = u([f(k) - h_{T-1,\max}(k)]/2) + \beta v_T^*(h_{T-i,\max}(k)).$$

As demonstrated in the proof of Proposition 8, v_{T-1}^* is usc and increasing, and also bounded. Thus, we can recursively generate the sequence of functions $\{h_{0,\max}(k), h_{1,\max}(k), ..., h_{T,\max}(k)\}$, which by construction is the sequence of maximal symmetric Markov equilibria for the finite horizon game. Each $h_{i,\max}(k)$ is an increasing, upper semicontinuous, and bounded investment function in period i, and the associated consumption function that both player follow in period i is $[f(k) - h_i(k)]/2$, and is lower semicontinuous, bounded, but not necessarily increasing, although it is of bounded variations. That sequence, however, is not necessarily monotone in the pointwise partial order (note that if it were, it would be decreasing since $h_{T,\max}(k) = 0$). Finally, it is important to note that the sequence of equilibrium output in each period, $\{f(k_0), f(k_1), ..., f(k_T)\}$ is not necessarily monotone.

5 CONCLUSION

This paper suggests additional work using lattice methods to study both nonstationary and stationary symmetric Markov equilibria. The most important extension for future work concerns the possible development of monotone comparative statics on the space of games. Such an extension is not only interesting from a theoretical perspective, but also for computation. However, that will require additional tools over those presented in the present paper. Recall that in the case of finite horizon games, the set of sequential symmetric Markov equilibria is generally not a lattice whenever the game exceeds two periods. This is quite remarkable, as for any given "history", the continuation set of Markov equilibria is a complete lattice. Because of this problem, strong set order comparative statics in the sense of Veinott[22] is not be available. This does not, however, preclude the possibility of obtaining so-called "weak set order" comparative statics. Smithson[**?**] introduces such set orders, and they do not require a lattice structure of the underlying space(also see Topkis[**?**] who refers to similar set relations and develops and characterizes the "weak induced set order"). These weak set order could prove useful to conduct monotone comparative statics on the operator A and B.

6 Appendix A

Proof of the claim that $A(\overline{h}) = \overline{h}$ in Proposition 5. Recall that,

$$H(x, t; g) = u([f(t) + g(t)]/2 - x) + \beta v^*(x).$$

For all t, $\overline{h}(t) = \inf\{A^n g_{\max}(t)\}_{n=0}^{\infty}$ and, for all n, $\overline{h} \leq A^n g_{\max}$, thus by the monotonicity of A, $A\overline{h} \leq A^{n+1} g_{\max}$ and $A\overline{h} \leq \overline{h}$. We prove that $A\overline{h} = \overline{h}$ by contradiction.

Suppose that there exists t in T such that $A\overline{h}(t) < \overline{h}(t)$. Then $\overline{h}(t)$ does not belong to $\gamma(t; \overline{h})$, so that there exists y in $S(t; \overline{h})$ and $\varepsilon > 0$ such that,

$$H(y, t; \overline{h}) > H(\overline{h}(t), t; \overline{h}) + 4\varepsilon. \tag{A1}$$

Since u is continuous on the compact domain K (and therefore uniformly continuous), and the sequence $\{A^n g_{\max}(t)\}_{n=0}^{\infty}$ converges to $\overline{h}(t)$, for $\varepsilon > 0$, there exists N_0, such that, for all $n \geq N_0$,

$$u([f(t) + A^n g_{\max}(t)]/2 - y) \geq u([f(t) + \overline{h}(t)]/2 - y) - \varepsilon. \tag{A2}$$

Recall that v^* is usc and strictly increasing, and, therefore, continuous from the right. As the sequence $\{A^n g_{\max}(t)\}_{n=0}^{\infty}$ is decreasing, it converges to $\overline{h}(t)$ from the right, and the sequence $\{v^*(A^n g_{\max}(t))\}_{n=0}^{\infty}$, thus, converges to $v^*(\overline{h}(t))$. Therefore, there exists, N_1 such that, for all $n \geq N_1$,

$$v^*((A^n g_{\max}(t)) \leq v^*(\overline{h}(t)) + \varepsilon. \tag{A3}$$

Finally, since the sequence $\{A^n g_{\max}(t)\}_{n=0}^{\infty}$ converges to $\overline{h}(t)$, there exists N_2 such that, for all $n \geq N_2$,

$$u([f(t) + \overline{h}(t)]/2 - \overline{h}(t)) + \varepsilon \geq u([f(t) + A^n g_{\max}(t)]/2 - A^n g_{\max}(t)) \tag{A4}$$

Combining (A3) and (A4) implies that, for all $n \geq \max(N_1, N_2)$,

$$\begin{aligned} & u([f(t) + A^n g_{\max}(t)]/2 - A^{n+1} g_{\max}(t)) + v^*(A^{n+1} g_{\max}(t)) \\ \leq & \\ & u([f(t) + \overline{h}(t)]/2 - \overline{h}(t)) + v^*(\overline{h}(t)) + 2\varepsilon, \end{aligned}$$

that is,

$$H(A^{n+1} g_{\max}(t), t; A^n g_{\max}) \leq H(\overline{h}(t), t; \overline{h}) + 2\varepsilon. \quad \text{(A5)}$$

Similarly, combining (A1) and (A2) implies that, for all $n \geq N_0$,

$$H(y, t; A^n g_{\max}) \geq H(y, t; \overline{h}) - \varepsilon > H(\overline{h}(t), t; \overline{h}) + 3\varepsilon. \quad \text{(A6)}$$

Inequalities (A5) and (A6) imply that there exists y and $\varepsilon > 0$, such that, for all $n \geq \max(N_0, N_1, N_2)$,

$$H(y, t; A^n g_{\max}) \geq H(A^{n+1} g_{\max}(t), t; A^n g_{\max}) + \varepsilon.$$

This is impossible because $A^{n+1} g_{\max}(t) = \max \gamma(t; A^n g_{\max})$, and thus contradicts the initial hypothesis of the existence of a t, such that $A\overline{h}(t) < \overline{h}(t)$. QED.

7 Appendix B

Proof of the claim that $B(\underline{h}) = \underline{h}$ in Proposition 6. Since (i) $\underline{h}(t) = \vee_n \{B^n g_{\min}(t)\}$ and (ii) $B\underline{h}(t) \geq B^n g_{\min}(t)$ for all n, and (iii) $B\underline{h}(t) \in \Theta$, therefore, $B\underline{h}(t) \geq \underline{h}(t)$.We prove that $B\underline{h}(t) = \underline{h}(t)$ by contradiction.

Suppose that $B\underline{h}(t) > \underline{h}(t)$ at a point t where $\underline{h}$ is continuous. Recall that both functions are increasing and usc, and, therefore, they are continuous functions with atmost a countable number of discontinuity points. Note that $B\underline{h}(t) = \inf_{x>t}\{\min \gamma(x, \underline{h}) \in \gamma(t, \underline{h})\}$, by Proposition 2.

If $B\underline{h}(t) > \underline{h}(t)$, then there exists a $y \in \gamma(t; \underline{h})$, $y > \underline{h}(t)$ and $\underline{h}(t) \notin \gamma(t; \underline{h})$. Consider any y in $\gamma(t, ; \underline{h}) \subset S(t; \underline{h}) = [0, (f(t) + \underline{h}(t))/2]$. Then, $y < (f(t) + \underline{h}(t))/2$, since $u'(0) = \infty$. By construction, $\underline{h}(t)$ is the pointwise limit of the increasing sequence $\{B^n g_{\min}(t)\}$, as t is not a discontinuity point of $\underline{h}(t)$. Hence, there exists an N_0 such that, for all $n \geq N_0$, $y \in S(t; B^n g_{\min}) = [0, (f(t) + B^n g_{\min}(t))/2]$.

By the choice of y,

$$u((f(t) + \underline{h}(t))/2 - y) + \beta v^*(y) > u((f(t) + \underline{h}(t))/2 - \underline{h}(t)) + \beta v^*(\underline{h}(t)). \quad \text{(B1)}$$

The sequence $\{B^n g_{\min}(t)\}$ converges to $\underline{h}(t)$. Hence, for all $\varepsilon > 0$, there exists an N_1 such that, for all $n \geq N_1$,

$$u((f(t) + B^n g_{\min}(t))/2 - y) + \beta v^*(y) > u((f(t) + \underline{h}(t))/2 - y) + \beta v^*(y) + \varepsilon, \quad \text{(B2)}$$

by the continuity of u. For the same reason, there exists N_2 such that, for all $n \geq N_2$,

$$u((f(t) + \underline{h}(t))/2 - \underline{h}(t)) + \varepsilon > u((f(t) + B^n g_{\min}(t))/2 - B^{n+1} g_{\min}(t)).$$

Recall that v^* is increasing, and, therefore, $v^*(\underline{h}(t)) \geq v^*(B^{n+1}g_{\min}(t))$. Combining this result with inequality (B2) implies that,

$$
\begin{aligned}
& u((f(t)+\underline{h}(t))/2-\underline{h}(t))+\beta v^*(\underline{h}(t))+\varepsilon \\
> & \\
& u((f(t)+B^n g_{\min}(t))/2-B^{n+1}g_{\min}(t))+\beta v^*(B^{n+1}g_{\min}(t)).
\end{aligned}
\tag{B3}
$$

Finally, from (B1), (B2), and (B3), for all $n \geq \max(N_0, N_1, N_2)$,

$$u((f(t)+B^n g_{\min}(t))/2-y)+\beta v^*(y) > u((f(t)+B^n g_{\min}(t))/2-B^{n+1}g_{\min}(t))+\beta v^*(B^{n+1}g_{\min}(t)).$$

This inequality contradicts the property that $B^{n+1}g_{\min}(t) = \inf_{x>t}\{\min \gamma(t; B^n g_{\max})\} \in \gamma(t; B^n g_{\min})$, since there exists a y in $S(t; B^n g_{\min})$ that is preferred to $B^{n+1}g_{\min}(t)$. This contradiction implies that the initial hypothesis that $B\underline{h}(t) > \underline{h}(t)$ cannot be correct. It, therefore, must be the case that $B\underline{h}(t) = \underline{h}(t)$, at every point t where $\underline{h}$ is continuous. Since two usc functions coincide on their continuity points they must be identical, we have $B\underline{h} = \underline{h}$. QED.

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