BAYESIAN AND DOMINANT STRATEGY IMPLEMENTATION IN THE INDEPENDENT PRIVATE VALUES MODEL

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Abstract. We prove—in the standard independent private-values model—that the outcome, in terms of expected probabilities of trade and expected transfers, of *any* Bayesian mechanism, can also be obtained with a dominant strategy mechanism.

Key words: Independent private values, incentive compatibility, Bayesian implementations, dominant strategy implementation, adverse selection, bilateral trade, mechanism design.

1. Introduction

We prove, in the independent private-values model, that the outcome—in terms of expected probabilities of trade and expected transfers—of any Bayesian incentive compatible mechanism, can also be obtained with a dominant strategy mechanism. In other words, a mechanism is Bayesian incentive compatible if and only if there is a dominant-strategy incentive compatible mechanism that generates the same expected probability of trade.

The model has a single indivisible object and finitely many agents. Every agent has private information represented by a type, customarily interpreted as the agent's valuation for the object. Payoffs are linear on valuation and transfers. From each agent's viewpoint, other agents' types are random variables. Agents' types are independently distributed according to known distribution functions. The setup is sufficiently flexible to include a privately informed seller and heterogeneous buyers.

A direct mechanism consists of two maps per agent, a probability function and a transfer function. Every agent, after observing his or her type, sends a report to the mechanism

Date: July 16, 2008.

JEL Classification. D42, D44, D82, D86.

We are very grateful to Martín Besfamille, Hector Chade, Estelle Cantillon, Giuseppe Lopomo, Jeroen Swinkels, and Lin Zhou for comments on an earlier draft.

designer. Given the reported profile of types, the probability function specifies the probability with which the agent receives the object, and the transfer function determines the amounts that the agent must pay. Thus, a direct mechanism defines a game where an agent's strategy is her report given her private information, and an agent's payoff is determined by the functions mentioned.

A mechanism is *Bayesian incentive compatible* if reporting truthfully constitutes a Bayesian-Nash equilibrium. A mechanism is *dominant strategy incentive compatible* if reporting truthfully is a weakly dominant strategy for every agent.

In a Bayesian incentive compatible mechanism, agents do not observe their opponents' reports, but presume, by way of equilibrium analysis, that opponents report their types truthfully. The distribution of opponents' reports is therefore the distribution of opponent's types. When choosing a report, an agent evaluates her expected payoff given her type. This payoff is determined by the *expected* probability of trade and *expected* transfer, both computed using the distribution of opponent's types. (The actual probability of trade and transfer depend on the realization of opponents' reports.)

With respect to the notion of outcome described—expected probability of trade and expected transfer—we prove that an outcome is attainable with a Bayesian incentive compatible mechanism if and only if it is attainable with a dominant strategy incentive compatible mechanism. Thus, there is no gain in moving from dominant strategy to Bayesian implementation in terms of implementable outcomes. Henceforth, we use the term outcome in the sense described.

In the independent private-values model with ex ante identical bidders, the first-price sealed-bid auction (a Bayesian mechanism) and the second-price sealed-bid auction (a dominant-strategy mechanism) generate the same *actual* probabilities of trade and the same expected revenue (see, for instance, Krishna (2002) for the necessary assumptions). Our results go beyond the implication of revenue equivalence. We prove that any Bayesian incentive compatible outcome, can be obtained with a dominant strategy mechanism. This equivalence is in terms of *expected* probabilities of trade and *expected* transfers and holds in environments with heterogeneous agents and nonsymmetric mechanisms.

Our equivalence result is valuable. Dominant-strategy mechanisms have advantages over Bayesian mechanisms. For instance, one may be more confident that a rational agent will play a dominant strategy (if one is available) than that the same agent will play a Nash equilibrium strategy.[1]

Finally, as an illustration of potential applications, consider the question of efficient allocations in a bilateral trade setting. The question can be traced to Vickrey (1961) who shows that there is no dominant strategy incentive compatible mechanism that achieves ex post Pareto efficiency without outside subsidies. After Vickrey's observation, considerable effort was spent in an attempt to restore efficiency by relaxing the equilibrium notion from dominant strategy to Bayesian incentive compatibility.[2] Our equivalence result shows that, in some sense, that effort was misguided: Solely moving from dominant strategy to Bayesian mechanisms yields no gain in terms of implementable outcomes; any such gain must come from variations in other constraints such as ex ante or interim budget balance or individual rationality.

Our equivalence result is specific to the independent private values model with linear payoffs. (It is not relevant in other environments; see for instance Barberà and Jackson (1995) and Crémer and McLean (1988)).

We borrow several ideas from Border (1991) and Matthews (1984). Border characterizes the functions that are the expected probability of trade for some mechanism; he proves, elegantly and in great generality, a conjecture of Matthews' (1984), first established for the real line by Chen (1986). The relationship to the cited papers will be indicated throughout the paper.

The formal results are presented in three sections. Section 4 deals with ex ante identical bidders and symmetric mechanisms. This case allows us to present the main ideas in the the proofs. Sections 5 and 6 use similar arguments to those introduced in Section 4 and treat heterogeneous agents and nonsymmetric mechanisms.

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[1] See Mas-Colell, Whinston, and Green (1995), page 870, for a brief discussion of this point.

[2] Myerson and Satterthwaite (1983) prove that there is no Bayesian incentive compatible mechanism that achieves ex post efficiency, is interim individually rational, and consumes no outside resources.

2. Notation

Vectors are represented in bold face. If $\boldsymbol{b}$ is a vector in $\mathbb{R}^K$, b_k is its k^{th} coordinate, $\boldsymbol{b}_k$ denotes the vector $(0, 0, \ldots, 0, b_k, b_{k+1}, \ldots, b_K) \in \mathbb{R}^K$, $\boldsymbol{b}_{-k} = (b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_K) \in \mathbb{R}^{K-1}$, and $(a, \boldsymbol{b}_{-k}) = (b_1, \ldots, b_{k-1}, a, b_{k+1}, \ldots, b_K) \in \mathbb{R}^K$. The vector whose k^{th}'s coordinate is 1 and all others are 0 is denoted by $\boldsymbol{e}_k$.

A sum with no terms is defined to be zero; for instance, $\sum_{j=3}^{2} b_j = 0$. The complement of a set B is denoted by B^c, and the number of elements in B by $|B|$.

Let I be a positive integer and $\mathcal{I} = \{1, 2, \ldots, I\}$. For $i \in \mathcal{I}$, let $X_i \subset \mathbb{R}$, and λ_i be a probability distribution on X_i. Then, $\lambda = \prod_{i \in \mathcal{I}} \lambda_i$ is the product distribution and $\lambda_{-j} = \prod_{i \in \mathcal{I} \setminus \{j\}} \lambda_i$. All functions are assumed to be measurable with respect to the corresponding Borel σ-algebras; product spaces are endowed with the product σ-algebras. We use the following conventions for expectations. Given a function $q : \prod_{i=1}^{I} X_i \to \mathbb{R}$,

$$Eq(x_j) = \int_{\prod_{i \in \mathcal{I} \setminus \{j\}}} q(x_1, \ldots, x_I)\, d\lambda_1 \ldots\, d\lambda_{j-1}\, d\lambda_{j+1} \ldots\, d\lambda_I.$$

Thus the expectation is taken over $\boldsymbol{x}_{-j}$. When confusion about the variable of integration is unlikely, we will write simply Eq.

Analogous notation is applied to other objects.

Step functions are prominently used in our analysis. Any increasing step function can be represented by a collection of pairs $\{(b_k, \beta_k)\}_{k=1}^{K}$ where k represents the k^{th} step, β_k is the value of Q on that step, and b_k is the size of the step according to the underlying probability distribution. The following definition makes this observation precise.

Definition 1. *Let $X \subset \mathbb{R}$, let λ be a probability distribution on X, and let $Q : X \to [0, 1]$. We say that Q is a step function with K steps if $Q(X)$ has K elements and $[\beta \in Q(X) \implies \lambda(Q^{-1}(\beta)) > 0]$.*

Any nondecreasing step function Q can be represented by K pairs $\{(b_k, \beta_k)\}_{k=1}^{K}$ where

(1) $Q(X) = \{\beta_k\}_{k=1}^{K}$, $1 \geq \beta_K > \beta_{K-1} > \ldots > \beta_k > \beta_{k-1} > \ldots > \beta_1 \geq 0$ and
(2) for $k = 1, \ldots, K$, $b_k = \lambda\,(Q^{-1}(\beta_k))$.

We denote such Q by $\{b_k, \beta_k\}_{k=1}^K$ or by $(\boldsymbol{b}, \boldsymbol{\beta})$ where $\boldsymbol{b} = (b_1, \dots, b_K)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_K)$. Abusing notation, we may write $Q = \{b_k, \beta_k\}_{k=1}^K = (\boldsymbol{b}, \boldsymbol{\beta})$ when the same function is intended.

Note that if the function Q has K steps and is nondecreasing, then $\beta_k > \beta_{k-1}$ for all k with $1 < k \leq K$, otherwise it would have fewer than K steps.

3. Model

We use a standard independent private-values model. There is a single indivisible object and a finite set $\mathcal{I} = \{1, 2, \dots, I\}$ of agents. Agent i's type is an element $x_i \in X_i = [\underline{x}_i, \overline{x}_i] \subset \mathbb{R}$, distributed according to a probability distribution λ_i. Agents are risk neutral. Preferences are linear in type and money: If t_i is the amount paid by agent i, and q_i is the probability that i obtains the object, i's utility is $x_i q_i - t_i$; hence the interpretation of an agent's type as her valuation for the object.

A direct mechanism consists of two functions per agent, $q_i(\boldsymbol{x})$ and $t_i(\boldsymbol{x})$, where $q_i(\boldsymbol{x})$ is the probability that i is assigned the object and $t_i(\boldsymbol{x})$ is the amount i pays when the profile of reports is $\boldsymbol{x}$. The sum over i of the probabilities $q_i(\boldsymbol{x})$ must be less than or equal to one. (Strict inequality is allowed.)

Fix a direct mechanism $\{q_i, t_i\}_{i \in \mathcal{I}}$. If i reports her type truthfully (and other players report $\boldsymbol{x}_{-i}$), then i's payoff is $u_i(x_i, \boldsymbol{x}_{-i}) = q_i(x_i, \boldsymbol{x}_{-i})x_i - t_i(x_i, \boldsymbol{x}_{-i})$. Assuming other players also report truthfully, i's expected payoff is $Eu_i(x_i) = Eq_i(x_i)x_i - Et_i(x_i)$.

A direct mechanism is incentive compatible if truthful reporting is an equilibrium. For different equilibrium concepts, i.e. dominant-strategy or Bayesian-Nash equilibrium, there are different characterizations of incentive compatibility in terms of probabilities of trade. The following well-known results follow from Myerson (1981). A mechanism is

(1) dominant-strategy incentive compatible if and only if for all i and $\boldsymbol{x}_{-i}$, $q_i(x_i, \boldsymbol{x}_{-i})$ is nondecreasing on x_i and $t_i(x_i, \boldsymbol{x}_{-i}) = q_i(x_i, \boldsymbol{x}_{-i})x_i - \int_{\underline{x}_i}^{x_i} q(z, \boldsymbol{x}_{-i})dz - u(\underline{x}_i, \boldsymbol{x}_{-i})$. See also Mookherjee and Reichelstein (1991).

(2) Bayesian incentive compatible if and only if for all i, $Eq_i(x_i)$ is nondecreasing on x_i and $Et_i(x_i) = Eq_i(x_i)x_i - \int_{\underline{x}_i}^{x_i} Eq_i(z)dz - Eu_i(\underline{x}_i)$.[3]

This characterization justifies the usage summarized in the following definition.

Definition 2. *Let $\{q_i\}_{i\in\mathcal{I}}$ be a collection of I functions $q_i : \prod_{i=1}^{I} X_i \to [0,1]$ such that for every $x \in \prod_{i=1}^{I} X_i$, $\sum_{i\in\mathcal{I}} q_i(\boldsymbol{x}) \leq 1$.*

If for every i and $\boldsymbol{x}_{-i}$, $q_i(x_i, \boldsymbol{x}_{-i})$ is nondecreasing in x_i, then $\{q_i\}_{i\in\mathcal{I}}$ is a dominant-strategy incentive compatible mechanism.

If for every i, $Eq_i(x_i)$ is nondecreasing in x_i, then $\{q_i\}_{i\in\mathcal{I}}$ is a Bayesian incentive compatible mechanism.

The omitted transfer functions are recovered, up to a constant, using the corresponding incentive compatibility characterizations.

The framework presented is sufficiently flexible to include, among other things, a seller with private information. These and other features are discussed farther in the following sections.

4. Ex ante identical bidders

In this section we assume that the I agents or bidders are ex ante identical: types are identically and independently distributed according to the probability distribution λ_b in $X = [\underline{x}, \overline{x}]$, assumed for simplicity to be nonatomic.

We require that mechanisms be symmetric, i.e. that ex ante identical bidders be treated ex ante identically. (We introduce a privately informed seller, heterogeneous bidders, and nonsymmetric mechanisms in the following sections.)

Symmetric mechanisms are interesting on their own right. An alleged advantage of competitive bidding over direct adjudication is that competitive bidding tends to reduce agency problems. Favoring a particular bidder when they are all ex ante identical may diminish this advantage. For instance, in several countries, government agencies must use competitive bidding for their purchases and are often not permitted to favor a particular

[3]Myerson (1981) assumes that distributions have densities, and that the densities are strictly positive on their supports. Monteiro and Svaiter (2007) extend the characterization to arbitrary measures.

bidder when all bidders are ex ante identical. A symmetric mechanism can be defined using a single probability-of-trade function as follows.

Definition 3. *Let* $q : X^I \to [0,1]$ *be such that for every* $\boldsymbol{x} \in X^I$, $\sum_{i=1}^{I} q(\sigma_i(\boldsymbol{x})) \leq 1$, *where* $\sigma_i(x_1, \dots, x_I) = (x_i, x_2, \dots, x_{i-1}, x_1, x_{i+1}, \dots, x_I)$, *i.e* $\sigma_i(\boldsymbol{x})$ *interchanges the first and ith coordinate of the vector* $\boldsymbol{x}$.

(1) q *is a symmetric, dominant-strategy incentive compatible, direct mechanism with* I *bidders if* $q(x_1, \boldsymbol{x}_{-1})$ *is nondecreasing in* x_1.
(2) q *is a symmetric, Bayesian incentive compatible, direct mechanism with* I *bidders if* $Eq(x_1)$ *is nondecreasing in* x_1.

The omitted transfer functions are recovered, up to a constant, using the corresponding incentive compatibility characterizations.

In a symmetric mechanism, each bidder's probability function q_i is derived from the single function q by setting $q_i(\boldsymbol{x}) = q(\sigma_i(\boldsymbol{x}))$.

Theorem 1 is this section's main result.

Theorem 1. *The function* q' *is a symmetric, Bayesian incentive compatible mechanism with* I *bidders if and only if there exists a symmetric, dominant-strategy incentive compatible mechanism* q *with* I *bidders that generates the same expected probability of trade, i.e.* $Eq = Eq'$.

Remark 1. *We prove a slightly stronger result: Even if the Bayesian incentive compatible mechanism* q' *is not symmetric, provided* Eq' *is the same for all bidders, there is a* symmetric *dominant-strategy incentive-compatible mechanism* q *so that* $Eq = Eq'$.

Theorem 1 demonstrates that, ceteris paribus, going from dominant strategy to Bayesian implementation does not increase the set of implementable outcomes, when outcomes are defined in terms of *expected* probabilities of trade and expected transfers. This is the appropriate notion of outcome for Bayesian implementation because *expected* probabilities of trade determine bidders' expected payoffs given their private information, and expected transfers up to a constant.

Figure 1 depicts two direct mechanisms in an environment with two bidders whose valuations are uniformly distributed in $[0,1]$. Types are divided in five intervals of equal probability and types in the same interval are treated equally. The left diagram in the figure represents the direct mechanism $q'(x_1,x_2)$. Every type profile (x_1,x_2) belongs to a cell and the number in that cell is the value of $q'(x_1,x_2)$. (Cells without values indicate $q'(x_1,x_2)=0$.) The numbers below the horizontal axis are the expected probability of trade $Eq'(x_1)$—the integral of the function q' for fixed x_1 along the vertical axis. Since $Eq'(x_1)$ is nondecreasing, q' (with its implicit expected transfer) satisfies Bayesian incentive compatibility. (This is IC's classic characterization.) It is clear, however, that q' does not satisfy dominant-strategy incentive compatibility because $q'(x_1,x_2)$ is *not* nondecreasing on x_1 for some x_2, say $x_2 \in [0,1/5]$. The right diagram in the figure represents the mechanism $q(x_1,x_2)$ that is equivalent to q' in that yields the same expected probability of trade, $Eq = Eq'$, but it is also dominant strategy incentive compatible. We go from mechanism q' on the left to mechanism q on the right by "rearranging the cells" in the diagram so that $q(x_1,x_2)$ is nondecreasing on x_1 for fixed x_2.

Care must be exercised so that the "rearrangement of cells" satisfies the symmetry of the mechanism: Given a type profile (x_1,x_2), if $q'(x_1,x_2)$ is the probability that agent 1 gets the object, the probability that bidder 2 gets the object is $q_2'(x_1,x_2)=q'(x_2,x_1)$, and thus $q'(x_1,x_2)+q'(x_2,x_1)\leq 1$. Thus, in both diagrams, the numbers in cells that are symmetric with respect to the diagonal must sum up to no more than one. While focusing on symmetric mechanisms allows us to use a single function q', it also *requires* us to use a single function q. In the example, the required "rearrangement of cells" is straightforward. That the required rearrangement can be carried out for any arbitrary mechanism q is the content of the theorem.

One direction in the proof of Theorem 1 is trivial: if there exists a dominant strategy IC mechanism q with $Eq = Eq'$ then, q' is Bayesian IC. We establish the other direction in three lemmas of independent interest. We will conclude the section with an example. Lemma 4.1 highlights a feasibility constraint.

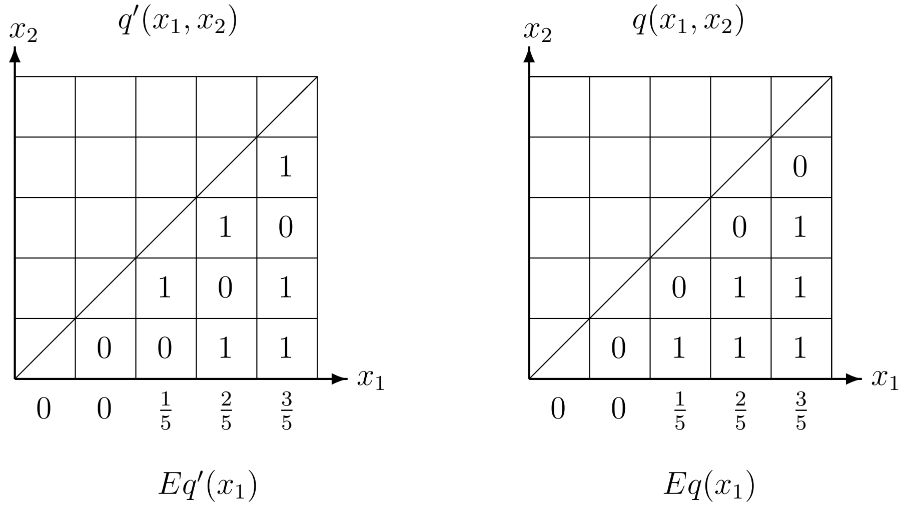

Figure 1. In cells with no values q' and q are 0

Lemma 4.1. *If q is a Bayesian incentive-compatible, symmetric, direct mechanism with I bidders, then $Eq \in W$ where*

$$W = \Big\{ Q \mid Q : X \to [0,1] \text{ is nondecreasing and } \\ B \subseteq X \implies I \int_B Q(x_1)\, d\lambda_b \leq 1 - [\lambda_b(B^c)]^I \Big\} \tag{1}$$

Proof. The standard characterization of Bayesian incentive compatibility implies that Eq is nondecreasing. It remains to show the inequality holds. The integral $\int_B Eq(x_1)\, d\lambda_b$ is the probability that a buyer has type $x_1 \in B$ and wins the object. The left-hand side is therefore the probability that one buyer has type in the specified set and wins the object. The right-hand side is the probability that at least one buyer will have type in B: $\lambda_b(B^c)$ is the probability that a given buyer will not have type in B and $[\lambda_b(B^c)]^I$ is the probability that no buyer will have type in B. Thus the left-hand side cannot exceed the right-hand side. □

The inequality in Lemma 4.1 is a feasibility constraint that must be satisfied by any mechanism not only Bayesian incentive compatible ones. For any specified set of types, the probability that some bidder's type is within the specified set and the bidder obtains the object cannot exceed the probability that some bidder's type is within the specified set.

The feasibility constraint first appears in Matthews (1983) and (1984), and in Maskin and Riley (1984). It plays a key role in our proof. Matthews (1984) conjectured that for any function $Q : X \to [0,1]$, not necessarily nondecreasing, that satisfies the feasibility constraint, there is a symmetric mechanism q with $Eq = Q$, i.e. Q is the expected probability of trade of some symmetric mechanism q. Border (1991) proves Matthews' conjecture for general type spaces. He is not concerned with incentive compatibility; he is interested in determining when the expected probability of trade can be used as the primitive in the analysis. (Lemma 4.1 is a corollary to Lemma 5.1, page 1179 in Border (1991).) Maskin and Riley (1984) (Theorem 7) prove a version of Matthews' conjecture for nondecreasing step functions Q; Matthews (1984) extends their result to arbitrary nondecreasing functions Q. These authors restrict attention to symmetric environments.

Jointly, the remaining two lemmas prove that every *nondecreasing* function Q that satisfies the feasibility condition (1) is the expected probability-of-trade function corresponding to a dominant-strategy incentive compatible mechanism. The main ideas are as follows. Lemma 4.2 characterizes the extreme points of W that are step functions. Since step functions take finitely many values, verifying that a step function is an extreme point is a finite dimensional matter and therefore simpler, in principle, than identifying arbitrary extreme points of the set W.

For each step function that is an extreme point of W, Lemma 4.3 constructs a symmetric, direct mechanism (that satisfies dominant-strategy incentive compatibility) and implements it. Since every function in W is the limit of convex combinations of step functions, the theorem follows.

When applied to a step function, inequality (1) becomes a system of finitely many linear inequalities: A step function defines a partition on the set of types. This partition consists of the level sets of the function, the subsets of the function's domain in which the function takes the same value. Pick a step function and fix the elements of the partition. To be an extreme point (given the fixed partition), the step function must make sufficiently many inequalities bind. (To visualize this point, imagine a set in $\mathbb{R}^2$ defined by finitely many linear inequalities, say a rectangle. The extreme points of the rectangle are its vertices. Vertices are defined by the intersection of sufficiently many lines, more precisely two lines per vertex because the rectangle is a subset of $\mathbb{R}^2$.)

Lemma 4.2. *Let $Q = (\boldsymbol{b}, \bar{\boldsymbol{\beta}}) = \{(b_k, \bar{\beta}_k)\}_{k=1}^K$ be a step function in W. Then $\{(b_k, \bar{\beta}_k)\}_{k=1}^K$ is an extreme point of W if and only if either*

(1) $\bar{\beta}_k = \frac{\left(\sum_{j=1}^k b_j\right)^I - \left(\sum_{j=1}^{k-1} b_j\right)^I}{I b_k}$ for $k = 1, \dots, K$, or

(2) $\bar{\beta}_1 = 0$ and $\bar{\beta}_k$ is as in (1) for $k = 2, \dots, K$.

Proof. The proof has two parts. First, we prove that if $\bar{\boldsymbol{\beta}}$ is as defined in the Lemma's statement, $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W. Second, we prove the converse.

Letting $B = \bigcup_{j=k}^K Q^{-1}(\bar{\beta}_k)$ in the definition of W, its defining inequality becomes

$$\sum_{j=k}^{K} I b_j \beta_j \leq 1 - \left(\sum_{j=1}^{k-1} b_j \right)^I \tag{2}$$

or, in vector notation, $I\boldsymbol{b}_k \cdot \boldsymbol{\beta} \leq 1 - r_k$, where $\boldsymbol{b}_k$ is the vector $(0, \dots, 0, 0, b_k, b_{k+1}, \dots, b_K)$, and $r_k = \left(\sum_{j=1}^{k-1} b_j\right)^I$. Taking $k = 1, \dots, K$, (2) becomes a system of K inequalities.

Recall $\boldsymbol{e}_k$ denotes the vector whose k^{th} coordinate is 1 and all others are zero. Define

$$P = \{\boldsymbol{\beta} \in \mathbb{R}^K : \text{ for } k = 1, \dots, K,\ I\boldsymbol{b}_k \cdot \boldsymbol{\beta} \leq 1 - r_k \text{ and } \boldsymbol{e}_k \cdot \boldsymbol{\beta} \geq 0\} \tag{3}$$

The set $P \subset \mathbb{R}^K$ is defined by $2K$ inequalities; it is the set of all nonnegative vectors $\boldsymbol{\beta} \in \mathbb{R}^K$ (K inequalities), such that $(\boldsymbol{b}, \boldsymbol{\beta})$ satisfies the inequalities (2) (another K inequalities).

A step function $(\boldsymbol{b}, \boldsymbol{\beta}) \in W$ is an extreme point of W if and only if $\boldsymbol{\beta}$ is an extreme point of P (Lemma A.2). A vector $\boldsymbol{\beta} \in P$ is an extreme point of P if and only if the set

$$R(\boldsymbol{\beta}) = \{\boldsymbol{b}_k : k \in \{1, \dots, K\}, I\boldsymbol{b}_k \cdot \boldsymbol{\beta} = r_k\} \cup \{\boldsymbol{e}_k : k \in \{1, \dots, K\}, \boldsymbol{e}_k \cdot \boldsymbol{\beta} = 0\} \tag{4}$$

has K linearly independent elements (Lemma A.1), i.e. if the inequalities defining P are evaluated at $\boldsymbol{\beta}$, they must include K linearly independent equations.

We now prove that if $\bar{\boldsymbol{\beta}}$ is alternative (1) in the Lemma's statement, then $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W. By hypothesis, $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ belongs to W. Therefore, it suffices to demonstrate that $R(\bar{\boldsymbol{\beta}})$ has K linearly independent vectors. We do so in two steps.

First, simple inspection shows that the K vectors $\{\boldsymbol{b}_k\}_{k=1}^K$ are linearly independent.

Second, we demonstrate that $\{\boldsymbol{b}_k\}_{k=1}^K \subset R(\boldsymbol{\beta})$. We must show that for $k = 1, \ldots, K$,

$$I\boldsymbol{b}_k \cdot \bar{\boldsymbol{\beta}} = 1 - r_k$$

Let $\tilde{\boldsymbol{\beta}}$ be a solution to the system of equations $I\boldsymbol{b}_k \cdot \tilde{\boldsymbol{\beta}} = 1 - r_k$, $k = 1, \ldots, K$. (Such a solution always exists because the vectors $\{\boldsymbol{b}_k\}_{k=1}^K$ are linearly independent.) The solution to $I\boldsymbol{b}_K \cdot \bar{\boldsymbol{\beta}} = 1 - r_K$ is $\tilde{\beta}_K = \bar{\beta}_K$. Pick any $k < K$. Subtracting $I\boldsymbol{b}_{k+1} \cdot \tilde{\boldsymbol{\beta}} = 1 - r_{k+1}$ from $I\boldsymbol{b}_k \cdot \tilde{\boldsymbol{\beta}} = 1 - r_k$ yields $\tilde{\beta}_k = \bar{\beta}_k$ as in Lemma 4.2 (1). We have proved that $R(\bar{\boldsymbol{\beta}})$ has K linearly independent vectors and therefore $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W.

Let $\bar{\boldsymbol{\beta}}$ be alternative (2) in the Lemma's statement. We now prove that $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W. For $k = 2, \ldots, K$, $\boldsymbol{b}_k \cdot \bar{\boldsymbol{\beta}} = 1 - r_k$, $\mathbf{e}_1 \cdot \boldsymbol{\beta} = 0$, and the vectors $\{\mathbf{e}_1, \{\boldsymbol{b}_k\}_{k=2}^K\}$ are linearly independent. This completes the first part of the proof.

We now prove the converse, that W has no other extreme points with steps $\boldsymbol{b}$ than those already identified. Let $(\boldsymbol{b}, \tilde{\boldsymbol{\beta}})$ be an extreme point of W. Then $R(\tilde{\boldsymbol{\beta}})$ has K linearly independent vectors. If $\{\boldsymbol{b}_k\}_{k=1}^K$ belong to $R(\tilde{\boldsymbol{\beta}})$, then $\tilde{\boldsymbol{\beta}}$ is as defined in Lemma 4.2 (1). If $e_k \in R(\tilde{\boldsymbol{\beta}})$, then $\tilde{\beta}_k = 0$. This implies that $k = 1$; otherwise $\tilde{\boldsymbol{\beta}}$ does not have K steps. Thus $\{e_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_K\} \subset R(\tilde{\boldsymbol{\beta}})$ and $\tilde{\boldsymbol{\beta}}$ is as in Lemma 4.2 (2).

□

Lemma 4.3 constructs a dominant-strategy incentive compatible mechanism that implements the extreme points identified in Lemma 4.2. Let the step function Q be an extreme point of W. Given a type profile $(x_1, \ldots, x_I)$, $Q(x_i)$ is used to rank bidders. Those bidders with maximum rank, i.e. $\max_i Q(x_i)$, share the object with equal probability; those bidders with less than maximum rank are assigned the object with probability zero. Thus, the associated mechanism q takes values in $\{\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{I}, 0\}$. It depends only on the partition $\{Q^{-1}(b_k)\}_{k=1}^K$ defined by Q and not on the actual values taken by Q.

Lemma 4.3. *Let the step function $Q = \{b_k, \bar{\beta}_k\}_{k=1}^K$ be an extreme point of W. Then, the symmetric, direct mechanism q*

$$q(x_1, x_2, \ldots, x_I) = \begin{cases} \frac{1}{|\{i: Q(x_1) = Q(x_i)\}|} & \text{if } Q(x_1) > 0, Q(x_1) \geq Q(x_i)\ \forall i \\ 0 & \text{otherwise} \end{cases}$$

satisfies dominant-strategy incentive compatibility and $Eq = Q$.

Proof. Since q is nondecreasing in x_1 for any given $\boldsymbol{x}_{-1}$, q satisfies dominant strategy incentive compatibility. We must prove that $Eq = Q$.

Pick an arbitrary x_1. If $Q(x_1) = 0$, then $Eq(x_1) = 0$.

Suppose then that $Q(x_1) = \bar{\beta}_k > 0$. By direct calculation,

$$Eq(x_1) = \sum_{n=1}^{I} \frac{1}{n} \left(\begin{array}{c} I-1 \\ n-1 \end{array} \right) \left(\sum_{j=1}^{k-1} b_j \right)^{I-1-(n-1)} b_k^{n-1}$$

To see this note that $Eq(x_1)$ is the integral of $q(x_1, \ldots, x_I)$ over all x_i with $i \neq 1$. Since q takes finitely many values, its integral is a summation. Each term in the expression above corresponds to a value of $q(x_1, x_{-1})$ as x_{-1} varies. The first factor in a typical term, $\frac{1}{n}$, is the value of q. The second factor is the number of ways in which q may take the value $\frac{1}{n}$: there are $I-1$ variables x_i and exactly $n-1$ of them must be in $Q^{-1}(\beta_k)$. The last two factors represent the probabilities: $\sum_{j=1}^{k-1} b_j$ is the probability that a given x_i is in $\bigcup_{j=1}^{k-1} Q^{-1}(\beta_j)$ and therefore $\left(\sum_{j=1}^{k-1} b_j\right)^{I-1-(n-1)}$ is the probability that $I-1-(n-1)$ of them will be in $\bigcup_{j=1}^{k-1} Q^{-1}(\beta_j)$. Similarly b_k^{n-1} is the probability that $n-1$ variables x_i will be in $Q^{-1}(\beta_k)$.

To show that $x_1 \in Q(\bar{\beta}_k) \implies Eq(x_1) = \bar{\beta}_k$, we must prove that

$$\sum_{n=1}^{I} \frac{1}{n} \binom{I-1}{n-1} \left(\sum_{j=1}^{k-1} b_j \right)^{I-n} b_k^{n-1} = \frac{\left(\sum_{j=1}^{k-1} b_j + b_k\right)^I - \left(\sum_{j=1}^{k-1} b_j\right)^I}{I b_k}$$

Multiply both sides by Ib_k, note that $\frac{I}{n}\binom{I-1}{n-1} = \binom{I}{n}$, and add $\left(\sum_{j=1}^{k-1} b_j\right)^I$ to both sides, to obtain

$$\sum_{n=1}^{I} \binom{I}{n} \left(\sum_{j=1}^{k-1} b_j \right)^{I-n} b_k^n + \left(\sum_{j=1}^{k-1} b_j \right)^I = \left(\sum_{j=1}^{k-1} b_j + b_k \right)^I$$

This is the binomial formula since $\left(\sum_{j=1}^{k-1} b_j\right)^I$ corresponds to the term, missing in the summation, for $n = 0$. □

The fact that the type of mechanism employed in Lemma 4.3 can achieve the bounds in Lemma 4.1 was recognized by Border (1991), Lemma 5.2, page 1180.

The proof of Theorem 1 now follows from Lemmas 4.1, 4.2 and 4.3 plus some continuity arguments, laid out below. (See Border (1991), Lemma 5.4, page 1182, for a detailed presentation of similar arguments.)

Proof of Theorem 1. If Q is an extreme point of W, then there exists a symmetric, direct mechanism q (satisfying dominant-strategy incentive compatibility) such that $Eq = Q$ (Lemmas 4.2 and 4.3). Any step function in W can be expressed as a convex combination of step functions that are extreme points of W. The convex combination of dominant-strategy incentive-compatible mechanisms is also a dominant-strategy incentive compatible mechanism. Therefore for any step function Q^n in W there is a dominant-strategy incentive-compatible mechanism q^n such that $Eq^n = Q^n$.

Pick any $Q \in W$. Since step functions in W are L_∞-dense in W, there exists a sequence of step functions $Q^n \in W$, $Q^n \longrightarrow Q$ in L_∞. For each Q^n, there is a dominant-strategy incentive compatible mechanism q^n, $Eq^n = Q^n$. The set of dominant-strategy incentive-compatible mechanisms as a subset of L_∞ is weak* compact (it is norm-bounded and weak* closed). The function E that maps $q^n \mapsto Eq^n$ is continuous when both domain and range are endowed with their weak* topologies. □

The example below illustrates Lemmas 4.2 and 4.3.

Example 1. *There are two bidders, $i = 1, 2$, $K_1 = K_2 = 4$, and for every i, x_i is uniformly distributed in $X = [0, 1]$.*

Lemma 4.2 states that there are at most two extreme points for any given partition. Fix a partition of $[0, 1]$, say $[0, 1/4]$, $(1/4, 2/4]$, $(2/4, 3/4]$, $(3/4, 1]$. The step function

$$Q = \{(1/4, 1/8), (1/4, 3/8), (1/4, 5/8), (1/4, 7/8)\}$$

is one extreme point of W for the proposed partition. The level sets of Q are the elements of the partition. The second extreme point of W (for the same partition), say Q', is obtained from Q by setting $Q'(x_1) = 0$ for all $x_1 \in [0, 1/4]$, and $Q'(x_1) = Q(x_1)$ elsewhere.

Figure 2 illustrates Lemma 4.3 as it applies to Q. The numbers in the cells of the figure indicate the values of $q(x_1, x_2)$; empty cells indicate $q(x_1, x_2) = 0$ for (x_1, x_2) in the cell. Note that $Eq = Q$.

The mechanism that implements the second extreme point is $q'(x_1, x_2) = 0$ *for* $(x_1, x_2) \in [0, 1/4] \times [0, 1/4]$ *and* $q' = q$ *elsewhere; then* $Eq' = Q'$.

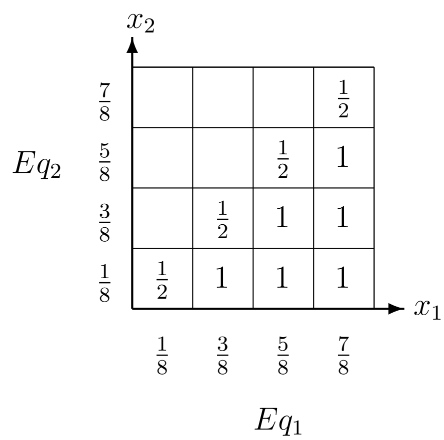

FIGURE 2. An extreme point and its dominant strategy mechanism

5. HETEROGENEOUS BIDDERS

In this section, the I agents are potentially heterogeneous and therefore mechanisms are not required to be symmetric.

Theorem 2. *The collection of functions* $\{q'_i\}_{i\in\mathcal{I}}$ *is a Bayesian incentive compatible mechanism if and only if there exists a dominant strategy, incentive compatible mechanism* $\{q_i\}_{i\in\mathcal{I}}$ *that generates the same expected probability of trade, i.e.* $Eq'_i = Eq_i$ *for* $i \in \mathcal{I}$.

The proof follows closely the proof of Theorem 1. The nontrivial direction proceeds in three lemmas, the analogues of Lemmas 4.1, 4.2, and 4.3.

Lemma 5.1. *If* $\{q_i\}_{i\in\mathcal{I}}$ *is Bayesian incentive compatible, then* $\{Eq_i\}_{i\in\mathcal{I}}$ *is in*

$$W'' = \Big\{ \{Q_i\}_{i\in\mathcal{I}} | \; \forall i, \; Q_i : X_i \to [0,1] \text{ is nondecreasing and } \prod_{i\in\mathcal{I}} B_i \subset \prod_{i=1}^{I} X_i \implies \sum_{i\in\mathcal{I}} \int_{B_i} Q_i \, d\lambda_i \leq 1 - \prod_{i\in\mathcal{I}} \lambda_i(B_i^c) \Big\} \tag{5}$$

Since the proof is analogous to that of Lemma 4.1, we provide only a sketch. Because of Bayesian incentive compatibility Eq_i must be nondecreasing. The probability that bidder i has type in B_i and wins the object is $\int_{B_i} Eq_i \, d\lambda_i$. Hence the left-hand side of the inequality is the probability that one buyer's type is in her specified set and the buyer wins the object. The right-hand side is the probability that at least one buyer has type in her specified set. Therefore the inequality in (5) must hold.

Lemma 5.2 below serves the same purpose as Lemma 4.2 did in the symmetric environment of Section 4. Given a partition of the type space, the lemma identifies all the step functions (obtained from that partition) that are extreme points of W''. Similar arguments are used in the proofs of both lemmas: Identifying the extreme points of W'' is equivalent to finding the solution to a system of equations obtained from the feasibility condition (5). This is what the proof of Lemma 4.2 accomplished.

The differences in details between Lemmas 5.2 and 4.2 arise from the selection of the system of equations and the number of unknowns to be determined. Generally there are more inequalities than necessary to determine an extreme point. To see this, imagine that W'' is a rectangle in $\mathbb{R}^2$. Four inequalities suffice to define the rectangle but each of its extreme points, i.e. each vertex, is determined by only two inequalities; each vertex is a point were two inequalities become binding. To identify a vertex, the inequalities must be chosen judiciously: if two inequalities represented by parallel lines are chosen, no extreme point will be found.

In Lemma 4.2, because the I ex ante identical bidders must be treated symmetrically, the selection of equations to determine the unknowns is trivial: For a fixed partition of the type space, Lemma 4.2 identifies a single "family" of extreme points containing one main extreme point and another one obtained through a small variation (i.e. $\bar{\beta}_1 = 0$).

In Lemma 5.2, the situation is more involved. Since bidders need not be treated symmetrically, even for a fixed partition of the type spaces, the feasible set has many extreme points. Each one of them is identified by a different system of equations. Modulus the selection of the system of equations, however, the arguments used to prove Lemma 5.2 are the same as those used to prove Lemma 4.2. To characterize the different extreme points without listing them individually, we use a labeling system. The labeling system identifies the equations that determine the extreme points of W''. This allows us to prove Lemma 5.2 for a canonical extreme point.

Definition 4. *Let $\{K_i\}_{i=1}^I$ be a collection of I nonnegative integers. A labeling relative to $\{K_i\}_{i\in\mathcal{I}}$ is a function $g : \{0, 1, \ldots, \sum_{i\in\mathcal{I}} K_i\} \to \prod_{i\in\mathcal{I}}\{1, \ldots, K_i + 1\}$ such that*

(1) $g(0) = (K_1 + 1, \ldots, K_I + 1)$,

(2) for $n \geq 1$, $g(n) - g(n-1) = -\boldsymbol{e}_i$ for some $i \in \{1, \ldots I\}$.

For $k \in \{0, \ldots, \sum_{i\in\mathcal{I}} K_i\}$, define $g_i^{-1}(k) = \min\{n : g_i(n) = k\}$.[4]

Example 2 contains three alternative labelings. Example 3 shows that, given a fixed partition of the type space, two of them correspond to extreme points and one of them does not.

Example 2. *There are two bidders, $i = 1, 2$, and $K_1 = K_2 = 4$. Three labeling systems are described in the table below:*

	Labeling (a)	*Labeling (b)*	*Labeling (c)*
$g(0)$	$(5,5)$	$(5,5)$	$(5,5)$
$g(1)$	$(4,5)$	$(5,4)$	$(4,5)$
$g(2)$	$(4,4)$	$(4,4)$	$(3,5)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$g(8)$	$(2,2)$	$(2,2)$	$(1,3)$
$g(9)$	$(1,2)$	$(2,1)$	$(1,2)$
$g(10)$	$(1,1)$	$(1,1)$	$(1,1)$

We will see below that only the first two are compatible with $K_1 = K_2 = 4$.

Before stating Lemma 5.2, we offer an example.

Example 3. *There are two bidders, $i = 1, 2$, $K_1 = K_2 = 4$, and for every i, x_i is uniformly distributed in $X_i = [0, 1]$. Although bidders are ex ante identical, mechanisms are not required to be symmetric.*

[4]Note that $g_i^{-1}(k)$ is well defined: $g_i(0) = K_i + 1$ and $g(\sum_{i\in\mathcal{I}} K_i) = \mathbf{1}$; therefore $g_i(\sum_{i\in\mathcal{I}} K_i = 1)$ and there must be an n' such that $g_i(n') = k$.

Fix a partition of $[0,1]$*, say* $[0,1/4]$*,* $(1/4,2/4]$*,* $(2/4,3/4]$*,* $(3/4,1]$*. Each diagram in Figure 3 corresponds to a pair of step functions* (Q_1,Q_2) *in* W''*. The level sets of these functions generate the proposed partition, and the pair is an extreme point of* W''*.*

The left-hand-side diagram is associated to

$$\begin{aligned} Q_1 &= \{(1/4,1/4),(1/4,2/4),(1/4,3/4),(1/4,4/4)\} \\ Q_2 &= \{(1/4,0),(1/4,1/4),(1/4,2/4),(1/4,3/4)\}. \end{aligned}$$

and is referenced by labeling (a) in Example 2. The right-hand-side diagram illustrates the reciprocal extreme point. It is referenced by labeling (b) in Example 2. (See also Lemma 5.2.)

Numbers in the cells of the diagrams indicate the values of the mechanism $q_1(x_1,x_2)$*; empty cells signify* $q_1=0$*. (See Lemma 5.3.) Note that in both diagrams,* $Eq_i = Q_i$

Finally, Figure 4 shows that labeling (c) in Example 2 is not consistent with the proposed partition for it implies fewer than four steps.

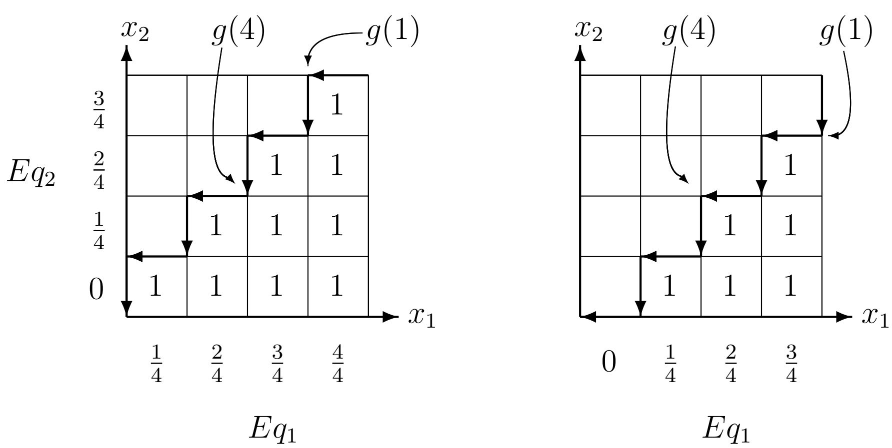

FIGURE 3. Labelings (a) and (b), two extreme points

In a first reading of Lemma 5.2, it may be useful to assume that $K_1 = K_2 = \ldots = K_I$, and that for $n \geq 1$ the difference $g(n) - g(n-1) = -\boldsymbol{e}_{(n \mod I)}$. Under this assumption, the labeling is the natural one, i.e. $g(1) = (K_1, K_2+1, \ldots, K_I+1)$, $g(2) = (K_1, K_2, K_3 + 1, \ldots, K_I + 1)$, etc.

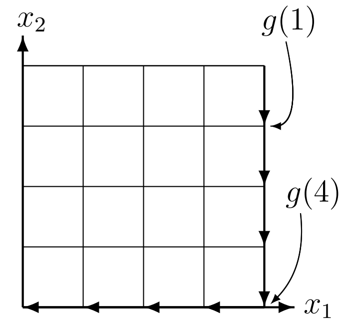

FIGURE 4. Labeling (c)

Lemma 5.2. *Let $\{\{b_k^i, \bar{\beta}_k^i\}_{k=1}^{K_i}\}_{i\in\mathcal{I}}$ be a family of step functions in W''. Then $\{\{b_k^i, \bar{\beta}_k^i\}_{k=1}^{K_i}\}_{i\in\mathcal{I}}$ is an extreme point of W'' if and only if there exists a labeling g relative to $\{K_i\}_{i\in\mathcal{I}}$ such that either*

(1) $\forall i \in \mathcal{I}$ and $k \in \{1,\dots,K_i\}$, $\bar{\beta}_k^i = \prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(g_i^{-1}(k))-1} b_\ell^j$ or

(2) $\bar{\beta}_k^i$ is defined as above for all i and k with the following exceptions: there is i' such that $\bar{\beta}_1^{i'} = 0$, and for every i and k with $k > g_i(g_{i'}^{-1}(1))$, $\bar{\beta}_k^i = 0$.

Proof. Fix $\{\{b_k^i\}_{k=1}^{K_i}\}_{i\in\mathcal{I}}$ as in the Lemma's statement, define

$$K = \prod_{i\in\mathcal{I}}\{1,\dots,K_i+1\}$$
$$\mathcal{K} = \left\{1,2,\dots,\sum_{i\in\mathcal{I}} K_i\right\}$$

and note that g maps $\mathcal{K}\cup\{0\}$ into K.

Any family of step functions $\{\{b_k^i, \beta_k^i\}_{k=1}^{K_i}\}_{i\in\mathcal{I}} \in W''$ must satisfy (5), and therefore

$$\forall \mathbf{k} = (k_1,\dots,k_I) \in K, \quad \sum_{i\in\mathcal{I}}\sum_{k=k_i}^{K_i} b_k^i\beta_k^i \le 1 - \prod_{i\in\mathcal{I}}\sum_{k=1}^{k_i-1} b_k^i \tag{6}$$

It is convenient to use vector notation. To that end, define for $i = 1,\dots,I$,

$$\begin{aligned} \boldsymbol{b}^i &= (b_1^i,\dots,b_{K^i}^i) && \textit{and} \quad \boldsymbol{b} = (\boldsymbol{b}^1,\dots,\boldsymbol{b}^I) \\ \boldsymbol{b}_{k_i}^i &= (0,\dots,0,b_{k_i}^i,b_{k_i+1}^i,\dots,b_{K^i}^i) && \textit{and} \quad \boldsymbol{b}_{\mathbf{k}} = (\boldsymbol{b}_{k_1}^1,\dots,\boldsymbol{b}_{k_I}^I) \\ \boldsymbol{\beta}^i &= (\beta_1^i,\dots,\beta_{K^i}^i) && \textit{and} \quad \boldsymbol{\beta} = (\boldsymbol{\beta}^1,\dots,\boldsymbol{\beta}^I) \end{aligned}$$

Also for every $\mathbf{k} \in K$, define

$$r(\mathbf{k}) = \prod_{i \in \mathcal{I}} \sum_{k=1}^{k_i - 1} b_k^i$$

In vector notation, inequality (6) becomes $\forall \mathbf{k} \in K, \boldsymbol{b}_{\mathbf{k}} \cdot \boldsymbol{\beta} \leq 1 - r(\mathbf{k})$.

Note that we can think of $\boldsymbol{\beta}$ as a vector in $\mathbb{R}^{\sum_{i \in \mathcal{I}} K_i}$. Then, to express nonnegativity constraints in vector form, for any $i \in \mathcal{I}$ and $k \in \{1, \ldots, K_i\}$, let $\boldsymbol{e}_k^i \in \mathbb{R}^{\sum_{i \in \mathcal{I}} K_i}$ such that

$$\boldsymbol{e}_k^i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

where the 1 corresponds to the element k of bidder i. Thus writing $\boldsymbol{e}_k^i \cdot \boldsymbol{\beta} \geq 0$ is equivalent to writing $\beta_k^i \geq 0$.

Define the set containing all nonnegative vectors $\boldsymbol{\beta}$ such that $(\boldsymbol{b}, \boldsymbol{\beta})$ satisfies (6).

$$P'' = \left\{ \boldsymbol{\beta} : [\mathbf{k} \in K \implies \boldsymbol{b}_{\mathbf{k}} \cdot \boldsymbol{\beta} \leq 1 - r(\mathbf{k})] \text{ and } [i \in \mathcal{I}, k \in \{1, \ldots, K_i\} \implies \boldsymbol{e}_k^i \cdot \boldsymbol{\beta} \geq 0] \right\}$$

An element $(\boldsymbol{b}, \boldsymbol{\beta})$ in W'' is an extreme point of W'' if and only if $\boldsymbol{\beta}$ is an extreme point of P'' (Lemma A.2). In turn $\boldsymbol{\beta}$ is an extreme point of P'' if and only if

$$(7) \quad R(\boldsymbol{\beta}) = \{ \boldsymbol{b}_{\mathbf{k}} : \boldsymbol{b}_{\mathbf{k}} \cdot \boldsymbol{\beta} = 1 - r(\mathbf{k}),\ \mathbf{k} \in K \} \cup \left\{ \boldsymbol{e}_k^i : \boldsymbol{e}_k^i \cdot \boldsymbol{\beta} = 0,\ i \in \mathcal{I}, k \in \{1, \ldots, K_i\} \right\}$$

contains $\sum_{i=1}^{I} K^i$ linearly independent vectors (Lemma A.1).

We now prove that for $\bar{\boldsymbol{\beta}}$ defined in Lemma 5.2 (1), $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W''. By hypothesis, $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ belongs to W''. Therefore, it suffices to demonstrate that $R(\bar{\boldsymbol{\beta}})$ has $\sum_{i=1}^{I} K_i$ linearly independent vectors. We do so in two steps.

First, simple inspection shows that the $\sum_{i \in \mathcal{I}} K_i$ vectors $\{\boldsymbol{b}_{g(n)}\}_{n \in \mathcal{K}}$ are linearly independent where g is the labeling used to defined $\bar{\boldsymbol{\beta}}$.

Second, we demonstrate that $\{\boldsymbol{b}_{g(n)}\}_{n \in \mathcal{K}} \subset R(\bar{\boldsymbol{\beta}})$. We must show that

$$(8) \qquad n \in \mathcal{K} \implies \boldsymbol{b}_{g(n)} \cdot \bar{\boldsymbol{\beta}} = 1 - r(g(n))$$

Let $\tilde{\boldsymbol{\beta}}$ be a solution to the system of equations $\boldsymbol{b}_{g(n)} \cdot \tilde{\boldsymbol{\beta}} = 1 - r(g(n))\ \forall n \in \mathcal{K}$. (Such a solution always exists because the vectors $\{\boldsymbol{b}_{g(n)}\}_{n \in \mathcal{K}}$ are linearly independent.) We will show that $\tilde{\boldsymbol{\beta}} = \bar{\boldsymbol{\beta}}$. Pick any $i \in \mathcal{I}$ and $k \in \{1, \ldots, K_i\}$, and let $n' = g_i^{-1}(k)$.

Subtracting $\boldsymbol{b}_{g(n'-1)} \cdot \tilde{\boldsymbol{\beta}} = 1 - r(g(n'-1))$ from $\boldsymbol{b}_{g(n')} \cdot \tilde{\boldsymbol{\beta}} = 1 - r(g(n'))$ yields

$$[\boldsymbol{b}_{g(n')} - \boldsymbol{b}_{g(n'-1)}] \cdot \tilde{\boldsymbol{\beta}} = r(g(n'-1)) - r(g(n'))$$

By definition of n', $g_i(n') = k$, $g_i(n'-1) = k+1$ and for all $j \neq i$, $g_j(n') = g_j(n'-1)$. Therefore $[\boldsymbol{b}_{g(n')} - \boldsymbol{b}_{g(n'-1)}] \cdot \tilde{\boldsymbol{\beta}} = b^i_k \tilde{\beta}^i_k$ and the expression above becomes

$$\begin{aligned}
b^i_k \tilde{\beta}^i_k &= r(g(n'-1)) - r(g(n')) \\
&= \textstyle\prod_{j\in\mathcal{I}} \sum_{\ell=1}^{g_j(n'-1)-1} b^j_\ell - \prod_{j\in\mathcal{I}} \sum_{\ell=1}^{g_j(n')-1} b^j_\ell \\
&= \left(\textstyle\prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b^j_\ell\right) \left[\textstyle\sum_{\ell=1}^{g_i(n'-1)-1} b^i_\ell - \sum_{\ell=1}^{g_i(n')-1} b^i_\ell\right] \\
&= \left(\textstyle\prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b^j_\ell\right) \left[\textstyle\sum_{\ell=1}^{k+1-1} b^i_\ell - \sum_{\ell=1}^{k-1} b^i_\ell\right] \\
&= \left(\textstyle\prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b^j_\ell\right) b^i_k
\end{aligned}$$

Therefore, $\tilde{\beta}^i_k = \prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b^j_\ell = \bar{\beta}^i_k$. This establishes (8).

We have proved that $R(\bar{\boldsymbol{\beta}})$ has $\sum_{i\in\mathcal{I}} K_i$ linearly independent vectors and therefore $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W''.

We now prove that for $\bar{\boldsymbol{\beta}}$ defined in Lemma 5.2 (2), $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W''. Once again, we need to show that $R(\bar{\boldsymbol{\beta}})$ has $\sum_{i\in\mathcal{I}} K_i$ linearly independent vectors.

Let $\bar{\beta}^{i'}_1 = 0 < \prod_{j\in\mathcal{I}\setminus\{i'\}} \sum_{\ell=1}^{g_j(g_{i'}^{-1}(1))-1} b^j_\ell$. (If there is no i' for which this holds, then $\bar{\boldsymbol{\beta}}$ is as in Lemma 5.2 (1) and we are done.)

Define $n' = g_{i'}^{-1}(1)$.

For $n < n'$, $\bar{\beta}^i_{g_i(n)}$ is as defined in Lemma 5.2 (1) and therefore, by (8),

$$\boldsymbol{b}_{g(n)} \cdot \bar{\boldsymbol{\beta}} = \sum_{i\in\mathcal{I}} \sum_{k=g_i(n)}^{K_i} b^i_k \bar{\beta}^i_k = 1 - r(g(n))$$

Therefore for every $n < n'$, $\boldsymbol{b}_{g(n)} \in R(\bar{\boldsymbol{\beta}})$.

For $n \geq n'$,

$$\boldsymbol{b}_{g(n)} \cdot \bar{\boldsymbol{\beta}} = \sum_{i\in\mathcal{I}} \sum_{k=g_i(n)}^{K_i} b^i_k \bar{\beta}^i_k < 1 - r(g(n))$$

This is so because $\bar{\boldsymbol{\beta}}^{i'}_{g_{i'}(n')} = \bar{\boldsymbol{\beta}}^{i'}_1 = 0$ and this variable was strictly positive when (8) applied. Therefore, $\boldsymbol{b}_{g(n')}$ does not belong to $R(\bar{\beta})$ but $\boldsymbol{e}^{i'}_i$ does. For every $n > n'$ the same argument applies: $\bar{\boldsymbol{\beta}}^i_k = 0$ if $k > g_i(n')$, $\boldsymbol{b}_{g(n)} \notin R(\bar{\beta})$, and $\boldsymbol{e}^i_k \in R(\bar{\beta})$.

It is immediate that all vectors in $R(\bar{\beta})$ are linearly independent. Hence $\bar{\boldsymbol{\beta}}$ is an extreme point of P''. Since $(\boldsymbol{b}, \tilde{\boldsymbol{\beta}}) \in W''$, $(\boldsymbol{b}, \tilde{\boldsymbol{\beta}})$ is an extreme point of W''.

We now prove that W'' has no other extreme point than those identified in the Lemma. Recall that $\boldsymbol{b}$ is fixed. Arguing by contradiction, let $(\boldsymbol{b}, \tilde{\boldsymbol{\beta}})$ be an extreme point of W''. Then $\tilde{\boldsymbol{\beta}}$ is an extreme point of P''. Therefore the set $R(\tilde{\boldsymbol{\beta}})$ must have $\sum_{i\in\mathcal{I}} K_i$ linearly independent vectors.

Suppose first that $\tilde{\boldsymbol{\beta}}$ is strictly positive, i.e. $\tilde{\beta}^i_k > 0$ for every i and k. Let $\{\boldsymbol{b}_{\mathbf{k}}\}$ be the collection of linearly independent vectors in $R(\tilde{\boldsymbol{\beta}})$. Each of these vectors satisfies $\boldsymbol{b}_{\mathbf{k}} \cdot \tilde{\boldsymbol{\beta}} = 1 - r(\mathbf{k})$.

Order the $\sum_{i\in\mathcal{I}} K_i$ index vectors from largest to smallest so that $\mathbf{k} > \mathbf{k}' > ....$ If this strict ordering is possible, then there is a labeling g where $g(n)$ is the n^{th} element in the ordered sequence and $g(0) = (K_1 + 1, \dots, K_I + 1)$. Consequently $\tilde{\boldsymbol{\beta}}$ corresponds to one of the extreme points identified by the lemma. Conclude then that the strict ordering described is not possible. Then $\exists \mathbf{k}, \mathbf{k}' \in K$, $\mathbf{k} \neq (\mathbf{k} \wedge \mathbf{k}') \neq \mathbf{k}'$ such that $\boldsymbol{b}_{\mathbf{k}} \cdot \tilde{\boldsymbol{\beta}} = 1 - r(\mathbf{k})$, $\boldsymbol{b}_{\mathbf{k}'} \cdot \tilde{\boldsymbol{\beta}} = 1 - r(\mathbf{k}')$, and $\boldsymbol{b}_{\mathbf{k}\wedge\mathbf{k}'} \cdot \tilde{\boldsymbol{\beta}} = 1 - r(\mathbf{k} \wedge \mathbf{k}')$.

Note that $\mathbf{k} \vee \mathbf{k}' = \mathbf{k} + \mathbf{k}' - \mathbf{k} \wedge \mathbf{k}'$ and thus $\boldsymbol{b}_{\mathbf{k}} + \boldsymbol{b}_{\mathbf{k}'} - \boldsymbol{b}_{\mathbf{k}\wedge\mathbf{k}'} = \boldsymbol{b}_{\mathbf{k}\vee\mathbf{k}'}$. Therefore

$$1 - r(\mathbf{k}) + 1 - r(\mathbf{k}') - 1 + r(\mathbf{k} \wedge \mathbf{k}') = \boldsymbol{b}_{\mathbf{k}\vee\mathbf{k}'} \cdot \tilde{\boldsymbol{\beta}} \leq 1 - r(\mathbf{k} \vee \mathbf{k}')$$

This implies that $r(\mathbf{k} \vee \mathbf{k}') + r(\mathbf{k} \wedge \mathbf{k}') \leq r(\mathbf{k}) + r(\mathbf{k}')$. This is a contradiction because $r(\mathbf{k}\vee\mathbf{k}') + r(\mathbf{k}\wedge\mathbf{k}') \geq r(\mathbf{k}) + r(\mathbf{k}')$, and the inequality is strict except when $\mathbf{k}' \wedge \mathbf{k} \in \{\mathbf{k}', \mathbf{k}\}$.

Finally suppose that $\tilde{\beta}^i_k = 0$ for some i and k, i.e. $\boldsymbol{e}^i_k \cdot \tilde{\boldsymbol{\beta}} = 0$ and thus $\boldsymbol{e}^i_k \in R(\tilde{\boldsymbol{\beta}})$ is one of the linearly independent vectors. Note that $\tilde{\beta}^i_k = 0$ implies $\tilde{\beta}^i_{k-1} = 0$. Therefore, unless $k = 1$ the function will not have the required number of steps. The same argument applied to strictly positive $\tilde{\boldsymbol{\beta}}$ yields the desired result.

□

Lemma 5.3. *Let $\{\{b^i_k, \bar\beta^i_k\}_{k=1}^{K_i}\}_{i\in\mathcal{I}}$ be an extreme point of W'' and let g be its labeling. The two direct mechanisms $\{q_i\}_{i=1}^{I}$ defined below satisfy dominant strategy incentive compatibility and for every i $Eq_i = \{b^i_k, \bar\beta^i_k\}_{k=1}^{K_i}$.*

For $i = 1, \dots, I$, let $\iota_i(x_i) = k : x_i \in Q_i^{-1}(\bar\beta^i_k)$. For alternative (1) in Lemma 5.2, the implementing mechanism is

$$q_i(x_1, \dots, x_I) = \begin{cases} 1 & \text{if } \iota_j(x_j) \le g_j(g_i^{-1}(\iota_i(x_i))) - 1 \;\forall j \neq i \\ 0 & \text{otherwise} \end{cases}$$

For alternative (2) in Lemma 5.2, the implementing mechanism is

$$\begin{aligned} q_{i'}(x_1, \dots, x_I) &= \begin{cases} 1 & \text{if } \iota_j(x_j) \le g_j(g_{i'}^{-1}(\iota(x_{i'}))) - 1 \;\forall j \neq i' \text{ and } \iota_{i'}(x_{i'}) \neq 1 \\ 0 & \text{otherwise} \end{cases} \\ q_i(x_1, \dots, x_I) &= \begin{cases} 1 & \text{if } \iota_j(x_j) \le g_j(g_i^{-1}(\iota(x_i))) - 1 \;\forall j \neq i \text{ and} \\ & \iota_i(x_i) = g_i(n), \text{ for } n \le \min\{n' : n' \in g_i^{-1}(1)\} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The proof is by direct calculation.

Proof. For alternative (1) in Lemma 5.2, pick i and x_i. Let $\iota_i(\boldsymbol{x}_i) = k$ and $g_i^{-1}(\iota(x_i)) = n'$. We must show that $Eq_i(x_i) = \bar\beta^i_k$, i.e, that $Eq_i(x_i) = \prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{k'=1}^{g_j(n')-1} b^j_{k'}$. Using definitions,

$$Eq_i(x_i) = \int_{X_{-i}} q_i(x_i, x_{-i})\, d\lambda_{-i} = \prod_{j\in\mathcal{I}\setminus\{i\}} \int_{\underline{x}_j}^{g_j(n')-1} d\lambda_j = \prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{k'=1}^{g_j(n')-1} b^j_{k'}.$$

For alternative (2), apply the same argument. □

6. Bilateral differential information

The environment in this section has I ex ante identical buyers plus a distinct agent that we call the seller. (We discuss the interpretation of the distinct agent as a seller after Definition 5.)

All agents, including the seller, have private information. To emphasize the difference between the seller and the bidders, the seller's private information is denoted by $y \in Y$, distributed according to a probability distribution λ_s. (Every buyer's private information $x \in X$ is independently distributed according to the same distribution λ_b.)

We require that mechanisms be symmetric with respect to the ex ante identical buyers as in Section 4.

Definition 5. *Let $q : Y \times X^I \to [0,1]$, $q_s : Y \times X^I \to [0,1]$ be such that for every $(y, \boldsymbol{x}) \in Y \times X^I$, $\sum_{i=1}^{I} q(y, \sigma_i(\boldsymbol{x})) + q_s(y, \boldsymbol{x}) \leq 1$.*

If $q(y, \boldsymbol{x})$ and $q_s(y, \boldsymbol{x})$ are nondecreasing in x_1 and y respectively, then (q, q_s) is a symmetric, dominant strategy incentive compatible mechanism with I bidders and a seller.

If $Eq(x_1)$ and $Eq_s(y)$ are nondecreasing, then (q, q_s) is a symmetric, Bayesian incentive compatible mechanism with I bidders and a seller.

The omitted transfer functions are recovered, up to a constant, using the corresponding incentive compatibility characterizations. (See Section 3.)

To interpret the distinct agent as a seller, it suffices to set $Y = [\underline{y}, \overline{y}] \subset \mathbb{R}_-$. A nondecreasing $Eq_s(y)$ becomes nonincreasing as a function of $|y|$.[5]

Fix a profile $(y, \boldsymbol{x})$. While $q_s(y, \boldsymbol{x})$ is the probability that the seller ends up with the object, it is *not* the probability that the object is not given to some buyer. If the probability sum (for the given type profile) is strictly less than one, then the object might not be assigned to either buyers or the seller. This flexibility in the definition of a mechanism increases the set of mechanisms for which the equivalence (between dominant strategy and Bayesian implementation) is obtained. Since we show the equivalence of any mechanism, not just a revenue maximizing one, the additional generality is valuable.

Theorem 3. *The pair of functions (q', q'_s) is a symmetric, Bayesian incentive compatible mechanism with I bidders and a seller if and only if there exists a symmetric dominant-strategy, incentive compatible mechanism (q, q_s) with I bidders and a seller that generates the same expected probability of trade, i.e. $Eq' = Eq$ and $Eq'_s = Eq_s$.*

If there is a single buyer, Theorem 3 is a particular case of Theorem 2 (with two agents, a buyer and a seller). If there are at least two ex ante identical buyers that must be

[5]The seller's preferences, defined as $u_s(y, x) = q_s(y, x)y - t_s(y, x)$ (where $t_s(y, x) \geq 0$ represents transfers from the agent to the mechanism designer) can be written as $u_s(y, x) = -t_s(y, x) - q_s(y, x)|y|$.

treated symmetrically, Theorem 3 does not follow from Theorem 2. The proof, however, is similar to the proofs of Theorems 1 and 2. The nontrivial direction also proceeds in three lemmas. Given the similarities, we state them without proof in Appendix B.

7. Concluding comments

1. An outcome in a game is customarily defined as a distribution on the terminal nodes that results from a strategy profile and nature's moves. For conciseness, suppose there are only two agents in our model and consider the implicit game in a direct revelation mechanism. An outcome is a distribution μ on

$$X_1 \times X_2 \times X_1 \times X_2 \times [0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R}$$

where from left to right we have the type spaces, the action spaces, the probabilities of trade, and the transfers. The assumed preferences imply that the marginal distribution of μ on the first, third, fifth and seventh space ($\mu_{X_1 \times X_1 \times [0,1] \times \mathbb{R}}$) suffices to determine player 1's payoff. (This is the distribution on player 1's own type, own action, own probability of trade and own transfer.) If two outcomes μ and ν generate the same relevant marginal distributions (i.e. $\mu_{X_i \times X_i \times [0,1] \times \mathbb{R}} = \nu_{X_i \times X_i \times [0,1] \times \mathbb{R}}$) for all players), they are equivalent in the sense that players are indifferent between them. We have proved that for any Bayesian Nash equilibrium outcome μ, there is a dominant-strategy equilibrium outcome ν such that the relevant marginal distributions of μ and ν are the same. The actual outcomes μ and ν will generally be different.

   A mechanism design problem is often cast in terms of the maximization of an objective function subject to constraints. If the objective function and constraints depend only on the expected probabilities of trade, then, per our equivalence theorems, there is no loss in requiring dominant-strategy over Bayesian incentive compatibility.

2. Our work is closely related to Border (1991). Border's objective is to identify the functions $Q : X \to [0,1]$ for which there is a symmetric mechanism $q : X^I \to [0,1]$ (with I identical bidders) and $Eq = Q$. He demonstrates that a necessary and sufficient condition for this is that Q satisfy the feasibility inequality in Lemma 4.1). Border assumes ex ante identical bidders and considers only symmetric mechanisms. He does

not require incentive compatibility and therefore his expected probabilities of trade Eq need not be nondecreasing. As a byproduct, Theorem 2 extends Border's result in that it applies to heterogeneous agents (and thus, to bilateral trade) and to nonsymmetric mechanisms.

3. Mookherjee and Reichelstein (1992) demonstrate, as an application of their Proposition 6, that in the symmetric auction environment, as long as bidder distributions satisfy the monotone inverse hazard rate condition, any Bayesian incentive compatible mechanism that maximizes expected seller revenue can be implemented in dominant strategies. An implication of our equivalence results is that, in the symmetric auction framework, the condition on bidder distributions is not necessary. Any mechanism that maximizes expected seller revenue subject to interim individual rationality and interim budget balance can be achieved in a mechanism that is dominant-strategy incentive compatible, and satisfies ex post individual rationality (for the seller and bidders) and ex post budget balance.

   We will prove a stronger claim: Let q' be a symmetric Bayesian incentive-compatible mechanism with I bidders and let t' be its associated transfer. Suppose the expected probability of the lowest type and its expected transfer are both zero, i.e. $Eq'(\underline{x}) = 0 = Et'(\underline{x})$. (In particular, any revenue-maximizing mechanism satisfies these requirements.) Then there is a symmetric dominant-strategy incentive-compatible mechanism q with associated transfers t such that $Eq = Eq'$, $Et = Et'$, and (q, t) satisfies ex post individual rationality and budget balance.

   We outline the argument using bidder 1. (The seller's valuation for the good is assumed to be zero.) By Theorem 1 there is a dominant-strategy incentive-compatible mechanism q with $Eq = Q'$. By hypothesis $Q'(\underline{x}) = 0$, so $Eq(\underline{x}) = 0$. Hence $q(\underline{x}, \boldsymbol{x}_{-1}) = 0$ for most x_{-1}. For fixed $\boldsymbol{x}_{-1}$, transfers are recovered from q using incentive compatibility up to a constant, so by choosing the constant we set $t(\underline{x}, \boldsymbol{x}_{-1}) = 0$ for every $\boldsymbol{x}_{-1}$. This implies $u(\underline{x}, \boldsymbol{x}_{-1}) = 0$, and therefore for every x_{-1}, $u(x_1, \boldsymbol{x}_{-1}) \geq 0$ for all x_1. This shows that ex post individual rationality is satisfied for all bidders.

   Define the seller's revenue as $\sum_{i \in \mathcal{I}} t(\sigma_i(\boldsymbol{x}))$. Hence, by construction, ex post budget balance holds. Finally, $t(\boldsymbol{x}) = q(x_1, \boldsymbol{x}_{-1})x - u(x_1, \boldsymbol{x}_{-1}) \geq 0$ because u is convex on x_1

and q is its gradient with respect to x_1. Therefore the seller's revenue is nonnegative and ex post individual rationality is also satisfied for the seller.

4. In their classic paper on bilateral trade with two-sided private information, Myerson and Satterthwaite (1983) provide conditions on the expected probabilities of trade that ensure that the mechanism is Bayesian incentive compatible and satisfies interim individual rationality and ex post budget balance. An adaptation of the Myerson-Satterthwaite argument (in conjunction with our equivalence result) shows the following: In the bilateral trade framework, any mechanism that is Bayesian incentive compatible and satisfies ex post budget balance and interim individual rationality can be implemented as a dominant-strategy incentive-compatible mechanism satisfying ex post individual rationality and interim budget balance. (Observe that interim individual rationality may be somewhat at odds with ex post incentive compatibility so we strengthen the requirement from interim individual rationality to ex post but weaken the requirement for ex post budget balance to ex ante budget balance.[6] In general, imposing both ex post individual rationality and ex post budget balance would restrict the set of implementable outcomes that satisfy either Bayesian incentive compatibility or dominant-strategy incentive compatibility.)

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[6] Interim budget balance can be interpreted as the presence of a risk neutral underwriter for the trading mechanism.

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Appendix A

The following well-known property is included here for the reader's convenience.

Lemma A.1. *For $j = 1, \dots, J$, let $\mathbf{a_j} \in \mathbb{R}^K$ and let $r_j \in \mathbb{R}$. Let $P = \{\boldsymbol{\beta} \in \mathbb{R}^K : \mathbf{a_j} \cdot \boldsymbol{\beta} \leq r_j, j = 1, \dots, J\}$. Then a vector $\beta \in P$ is an extreme point of P if and only if the set $A_{\boldsymbol{\beta}} = \{\mathbf{a_j} : \mathbf{a_j} \cdot \boldsymbol{\beta} = r_j, j \in \{1, \dots, J\}\}$ contains K linearly independent vectors.*

Proof. See for instance Bertsekas (2003), Proposition 3.3.3, page 184. □

The following lemma adds detail to the proof of Lemmas 4.2 and 5.2. We state it and prove it using W and P used in Lemma 4.2. The result and its proof remain valid for W'' and P'' as used in Lemma 5.2.

Lemma A.2. *Let W be defined as in Lemma 4.1. Let $(\boldsymbol{b}, \boldsymbol{\beta}) \in W$ be a step function with K steps and let P be as defined in (3). Then, $(\boldsymbol{b}, \boldsymbol{\beta})$ is an extreme point of W if and only if $\boldsymbol{\beta}$ is an extreme point of P.*

Proof. First, if $(\boldsymbol{b}, \boldsymbol{\beta})$ is not an extreme point of W, then $\boldsymbol{\beta}$ is not an extreme point of P. This follows applying the definition of extreme point.

Second, if $\boldsymbol{\beta}$ is not an extreme point of P, then $\boldsymbol{\beta} = \boldsymbol{\beta}'/2 + \boldsymbol{\beta}''/2$ for some $\boldsymbol{\beta}', \boldsymbol{\beta}'' \in P$. If $\boldsymbol{\beta}', \boldsymbol{\beta}''$ are both nondecreasing, then $(\boldsymbol{b}, \boldsymbol{\beta}'), (\boldsymbol{b}, \boldsymbol{\beta}'') \in W$ and the proof is complete.

For every $\alpha \in (0, 1)$, $\boldsymbol{\beta} = [\alpha\boldsymbol{\beta}' + (1-\alpha)\boldsymbol{\beta}]/2 + [\alpha\boldsymbol{\beta}'' + (1-\alpha)\boldsymbol{\beta}]/2$. For every α sufficiently small, $\alpha\boldsymbol{\beta}' + (1-\alpha)\boldsymbol{\beta}$ and $\alpha\boldsymbol{\beta}'' + (1-\alpha)\boldsymbol{\beta}$ are nondecreasing members of P. Hence, $\boldsymbol{\beta}$ is not an extreme point of W. □

Appendix B

We discuss here the three Lemmas leading to the proof of Theorem 3. Lemma B.1 follows as a corollary to Lemma 5.1.

Lemma B.1. *If (q, q_s) is a Bayesian incentive-compatible, symmetric, direct mechanism with I bidders and a seller, then (Eq, Eq_s) is in W', where*

$$W' = \Big\{(Q_b, Q_s)|\ Q_b : X \to [0,1],\ Q_s : Y \to [0,1] \text{ are nondecreasing and } [B \subset X, S \subset Y] \implies I\int_B Q_b\, d\lambda_b + \int_S Q_s\, d\lambda_s \leq 1 - \lambda_b(B^c)^I \lambda_s(S^c)\Big\} \tag{9}$$

Ex ante identical buyers must be treated symmetrically by the mechanism.

We proceed to identify the extreme points of W'. Lemma B.2 is the analogue of Lemmas 4.2 and 5.2.[7] Once the domain's partition (for the step function) is determined, these lemmas identify all the step functions (defined by the partition) that are extreme points of the feasible set.

Lemma B.2. *Let $(\{b_k, \bar{\beta}_k\}_{k=1}^{K_b}, \{b_k^s, \bar{\beta}_k^s\}_{k=1}^{K_s})$ be a pair of step functions in W'. Then $(\{b_k, \bar{\beta}_k\}_{k=1}^{K_b}, \{b_k^s, \bar{\beta}_k^s\}_{k=1}^{K_s})$ is an extreme point of W' if there exists a labeling g relative to (K_b, K_s), such that for every $(k_b, k_s) \in \{1, \dots, K_b\} \times \{1, \dots, K_s\}$ either*

[7]In reading the lemma's statement, recall that summations with no terms are assumed to be zero.

(1) $\left\{ \begin{array}{rcl} \bar{\beta}^b_{k_b} &=& \frac{1}{Ib_{k_b}} \left[\left(\sum_{k=1}^{k_b} b_k \right)^I - \left(\sum_{k=1}^{k_b-1} b_k \right)^I \right] \sum_{k=1}^{g_s(g_b^{-1}(k_b))-1} b^s_k \\ \bar{\beta}^s_{k_s} &=& \left(\sum_{k=1}^{g_b(g_s^{-1}(k_s))-1} b_k \right)^I \end{array} \right.$

or

(2) one or both of $\bar{\beta}_1$ *and* $\bar{\beta}^s_1$ *are zero, and all other* $(\bar{\beta}_{k_b}, \bar{\beta}^s_{k_s})$ *are as in* (1).

A direct proof of Lemma B.2 can be obtained following the same steps employed in Lemmas 4.2 and 5.2. Instead of repeating those steps, we make a few heuristic observations that lead to the result.

Suppose that there are only two heterogeneous agents, a buyer and a seller, $i \in \{b, s\}$. From Lemma 5.2 (1), the extreme point mechanism for the buyer is

$$\bar{\beta}^b_{k_b} = \sum_{k=1}^{g_s(g_b^{-1}(k_b))-1} b^s_k \tag{10}$$

for some labeling g.

Suppose instead that there is no seller but that there are I ex ante identical bidders. From Lemma 4.2 (1), the symmetric extreme point mechanism is

$$\bar{\beta}_{k_b} = \frac{\left(\sum_{k=1}^{k_b} b_j \right)^I - \left(\sum_{k=1}^{k_b-1} b_j \right)^I}{Ib_{k_b}} \tag{11}$$

This mechanism assigns the object to one of the I bidders.

Finally, suppose that there are I ex ante identical bidders plus a seller, a heterogeneous agent. The probability that one of the I bidders gets the object is given by (10), $\sum_{k=1}^{g_s(g_b^{-1}(k_b))-1} b^s_k$. This probability must be distributed among the I symmetric bidders; this is done according to (11). The result is precisely Lemma B.2 (1). Similar arguments lead to Lemma B.2 (2).

Lemma B.3. *Let the pair of step functions* $(Q_b, Q_s) = (\{b_k, \bar{\beta}_k\}_{k=1}^{K_b}, \{b^s_k, \bar{\beta}^s_k\}_{k=1}^{K_s})$ *be an extreme point of* W' *and let* g *be its labeling. The symmetric, direct mechanism* (q, q_s)

defined below satisfies dominant strategy incentive compatibility and $(Eq, Eq_s) = (Q_b, Q_s)$.

$$q(y,\boldsymbol{x}) = \begin{cases} \frac{1}{|\{i:\iota_b(x_i)=\iota_b(x_1)\}|} & \textit{if } Q_b(x_1) = \max\{Q_b(x_i)\}_{i=1}^I > 0 \textit{ and} \\ & \quad \iota_s(y) \le g_s(g_b^{-1}(\iota_b(x_1)) - 1 \\ 0 & \textit{otherwise} \end{cases}$$

$$q_s(y,\boldsymbol{x}) = \begin{cases} 1 - \sum_{i\in\mathcal{I}} q(y,\sigma_i(\boldsymbol{x})) & \textit{if } Q_s(y) > 0 \\ 0 & \textit{otherwise} \end{cases}$$

where $\iota_b(x_i) = k : x_i \in Q_b^{-1}(\bar\beta_k)$ *and* $\iota_s(y) = k : y \in Q_s^{-1}(\bar\beta_k^s)$.

Proof. We first show that $Eq = Q_b$. If $\iota_b(x_1) = 1$ and $\bar\beta_1 = 0$, then $Eq(x_1) = 0$ trivially and we are done. Suppose then $\iota_b(x_1) = k_b$ and $\bar\beta_{k_b} \neq 0$. By direct calculation, the expected probability of trade is

$$Eq(x_1) = \left[\sum_{i=1}^I \frac{1}{i}\binom{I-1}{i-1}\left(\sum_{k=1}^{k_b-1} b_k\right)^{I-1-(i-1)} (b_{K-\ell})^{i-1}\right] \sum_{k=1}^{g_s(g_b^{-1}(k_b))-1} b_k^s \tag{12}$$

The argument in Lemma 4.3 yields the desired result.

We now show that $Eq_s = Q_s$. If $\iota_s(y) = 1$ and $\bar\beta_1^s = 0$, then $Eq_s(y) = 0$ trivially. Suppose then $\iota_s(y) = k_s$ and $\bar\beta_{k_s}^s \neq 0$.

Note then that $q_s(y,\boldsymbol{x}) \neq 0 \iff \sum_{i\in\mathcal{I}} q(y,\sigma_i(\boldsymbol{x})) \neq 1 \iff \forall i, q(y,\sigma_i(\boldsymbol{x})) = 0$. It follows from the definition of $q(y,\boldsymbol{x})$ in the Lemma's statement, that $q(y,\sigma_i(\boldsymbol{x})) = 0, \forall i \iff k_s > g_s(g_b^{-1}(\iota_b(x_i))) - 1$, or equivalently, $k_s \ge g_s(g_b^{-1}(\iota_b(x_i)))$. In turn, this is so if and only if $g_s^{-1}(k_s) < g_b^{-1}(\iota_b(x_i)) \iff g_b(g_s^{-1}(k_s)) > \iota_b(x_i) \iff g_b(g_s^{-1}(k_s)) - 1 \ge \iota_b(x_i)$. The probability that x_i is such that $\iota_b(x_i) \le g_b(g_s^{-1}(k_s)) - 1$ is $\sum_{k=1}^{g_b(g_s^{-1}(k_s)-1)} b_k$. This occurs for the I bidders with probability $(\sum_{k=1}^{g_b(g_s^{-1}(k_s)-1)} b_k)^I$. □

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