Ranking Asymmetric Auctions using the Dispersive Order*

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September, 2010†

Abstract

The revenue ranking of asymmetric auctions with two heterogenous bidders is examined. The main theorem identifies a general environment in which the first-price auction is more profitable than the second-price auction. The two propositions in Maskin and Riley's (2000) seminal paper that describe models with this feature are both corollaries. Thus, Maskin and Riley's (2000) results are unified and extended. The use of mechanism design techniques simplifies the problem, allows economic intuition to be developed, and makes extensions possible. Roughly speaking, the first-price auction is more profitable when the strong bidder's distribution is flatter and more disperse than the weak bidder's distribution. These conditions are analogous to standard and intuitive assumptions in monopoly pricing. The main theorem can be extended to rank auctions with reserve prices and other auction formats as well. It also has implications for contest architecture. The revenue ranking is also extended, for the first time, to certain settings with more bidders. The first-price auction is more profitable as long as there is only one strong bidder, or, more generally, if the asymmetry between bidders is "large enough".

JEL Classification Numbers: D44.

Keywords: Asymmetric Auctions, Dispersive Order, Revenue Ranking.

*I would like to thank the Canada Research Chairs programme and the Social Sciences and Humanities Research Council of Canada for funding this research.

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1 Introduction

The interest in asymmetric auctions dates back to the inception of modern auction theory, with Vickrey (1961). In a seminal paper on auctions with two heterogeneous bidders, Maskin and Riley (2000) present three seemingly separate classes of environments in which it is possible to rank the first-price auction and the second-price auction in terms of revenue. In two of the cases – if the strong bidder's type distribution is either a "shifted" or a "stretched" version of the weak bidder's type distribution – the first-price auction yields higher expected revenue than the second-price auction.[1]

In the current paper, I will argue that rather than viewing these results as two separate propositions, as Maskin and Riley (2000) appear to do, they should be viewed as two corollaries of a more general theorem. Roughly speaking, the first-price auction dominates if the strong bidder's distribution is *flatter* and more *disperse* than the weak bidder's distribution; "shifting" and "stretching" are special cases. Importantly, this condition turns out to have an appealing economic interpretation. Maskin and Riley's (2000) third class of models, in which the second-price auction dominates, is unusual in the sense that the weak bidder's distribution is flatter and more disperse than the strong bidder's distribution.[2]

Maskin and Riley (2000) use arguments from mechanism design to demonstrate why the problem is non-trivial. They then abandon the approach with the conclusion that "mechanism design considerations do not settle the matter of which auction

[1] See Greismer et al (1967) for another early study of bidder heterogeneity. As Vickrey (1961) first discovered, and Myerson (1981) and Riley and Samuelson (1981) later proved more generally, the first-price and second-price auctions yield the same expected revenue in the independent private values model if all bidders are homogenous, risk neutral, and not financially constrained. Holt (1980) finds that the first-price auction is more profitable than the second-price auction if bidders are risk averse. Che and Gale (1998) prove that this is also the case if bidders are financially constrained. See also Che and Gale (2006). In Maskin and Riley (2000) and the current paper, bidder heterogeneity produces the same ranking under certain conditions. However, Milgrom and Weber (1982) obtain the opposite ranking in a model with affiliated values.

[2] Maskin and Riley's (2000) paper is the most general treatment of the revenue effects of bidder heterogeneity in the existing literature. In an earlier paper, Maskin and Riley (1985) assume type distributions are discrete and prove that the revenue ranking is also ambiguous in that case. A few other papers provide complementary examples in which the first-price auction is more profitable. These including Vickrey (1961), Lebrun (1996), and Cheng (2006), all of which examine special cases of power distributions (see also Corollary 2 in Section 4). Power distributions have also been examined in the numerical literature, starting with Marshall et al (1994). More recent examples of numerical revenue comparisons include Fibich and Gavious (2003), Li and Riley (2007), and Gayle and Richard (2008).

generates more revenue".[3] Instead, they use the system of differential equations that describe bidding behavior to derive two technical lemmata that quantify revenue in the two auctions. Using these lemmata, Maskin and Riley (2000) are able to prove the superiority of the first-price auction in the two environments described above.

In contrast, I will demonstrate that mechanism design can in fact not only be fruitfully used to address the problem, it also greatly simplifies the analysis itself. Thus, the first contribution of the paper is methodological. More concretely, the key step in developing the main result is to formulate the expected revenue of any mechanism as the expected value of a conditional expectation, where the latter is expected revenue conditional on the weak bidder's type. For the simple case where the bidders' supports have the same lowest end-point, the assumptions in the theorem ensure that this conditional expectation is greater for the first-price auction than the second-price auction for *all* the types in the weak bidder's support.[4] The usefulness of the method used here is evidenced not only by the simplicity of the proof but also by the many extensions that are made possible (and which are described below). An additional advantage of mechanism design is that economic intuition for the results can easily be developed. By providing simple proofs and sufficient conditions with an immediate economic interpretation, it is hoped that the theory of asymmetric auctions and their ranking will be made accessible to a wider audience.

In the following, the main result will be briefly described. Some intuition, also based on mechanism design, will then be presented. Finally, a number of implications and extensions of the main results will be summarized.

Consider an auction with one strong bidder and one weak bidder. The strong bidder draws a type or valuation from a distribution that dominates the distribution of the weak bidder in terms of the reverse hazard rate. With this assumption, some inferences concerning bidding behavior in the first-price auction are possible. For example, the weak bidder bids more aggressively than the strong bidder.[5] For the simple case where the strong bidder's density function is monotonic (his distribution function is convex or concave), the first-price auction is more profitable than the second-price auction if it is also the case that the strong bidder's distribution is flatter and more disperse (more "spread out") than the weak bidder's distribution. The main theorem allows the density function to be non-monotonic by appropriately

[3] To clarify, Maskin and Riley (2000) construct one environment in which they are able to use mechanisms design to prove the second-price auction is more profitable. However, in the two environments where the first-price auction is more profitable their proofs are different.

[4] When the lower end-points are different, the proof is completed by exploiting the fact that the first-price auction extracts more rent from the strong bidder than the second-price auction.

[5] See Lebrun (1999), Maskin and Riley (2000), Milgrom (2004), Hopkins (2007), or Kirkegaard (2009) for different proofs of this and other properties.

strengthening these conditions. As mentioned, two of Maskin and Riley's (2000) propositions are corollaries of the theorem. Indeed, Maskin and Riley's (2000) assumptions are shown to be unnecessarily strong even within each of the two models.

The dispersive order has recently attracted attention in the theoretical auction literature. Jia et al (2010) and Katzman et al (2010) examine comparative statics in symmetric auctions when bidders' distributions become more disperse. Ganuza and Penalva (2010) consider symmetric auctions in which the seller can influence the precision of bidders' information by making their signals more or less disperse. In asymmetric auctions, the dispersive order plays a role in determining the qualitative features of revenue-enhancing interventions into particular auction formats, as demonstrated by Kirkegaard (2010a) and Mares and Swinkels (2010a, 2010b). The results in Mares and Swinkels (2010a, 2010b) are particularly strong when densities are monotonic. Hopkins (2007) describe qualitative features of bidding behavior in auctions where distribution functions cross and one is smaller than the other in the dispersive order. The current paper is the first to use the dispersive order to rank revenue across standard auctions with asymmetric bidders.

Turning to the economic intuition, the compelling interpretation of the auction design problem advanced by Bulow and Roberts (1989) means that more economic intuition can be obtained from this approach than from studying a system of differential equations. It seems particularly important to develop an economic interpretation of the sufficient conditions to assess their plausibility when the revenue ranking is theoretically ambiguous, as with Maskin and Riley's (2000) three propositions.[6]

Following Bulow and Roberts (1989), think of type distributions as capturing demand curves in different markets. Then, the condition that the strong bidder's distribution is flat and disperse relative to the weak bidder's distribution is similar to the assumption used in the textbook explanation of third degree price discrimination that the market with the lowest willingness-to-pay is also the most price sensitive market. Recall that the reason for student discounts, say, is not that students have lower demand or willingness-to-pay per se, but rather that their demand responds more to a price change. The common assumption is that low demand and high price sensitivity go hand-in-hand. It is an assumption of this nature that is made here.

Continuing with the analogy, a second-price auction can be compared to charging identical prices across distinct markets; in both cases, the objects that are purchased are allocated efficiently. In contrast, neither a first-price auction nor price discrimination is efficient. In a first-price auction, the weak bidder wins more often than is

[6] Hafalir and Krishna (2008) show that if the winner of the auction is permitted to resell the object, the first-price auction is unambiguously more profitable than the second-price auction. Resale is ruled out in the current paper.

efficient, just as consumers on a market that receive a discount are allocated more units than is efficient. Crucially, it is possible to establish an upper bound on how much more often the weak bidder wins in the first-price auction. The result that the first-price auction is more profitable than the second-price auction is then equivalent to saying that although the price discount to the weak may be a bit too small or a bit too large compared to the optimal discount, it is better than no discount as all.[7]

The general lesson is that the second-price auction is not a good auction for generating high expected revenue. Mechanisms that "moderately" favour the weak bidder, such as the first-price auction, are more profitable.[8] Of course, it has long been understood that an *optimal* auction generally favours the weak bidder; see e.g. McAfee and McMillan (1989).[9] Based on this property and the weak bidder's tendency to win more often in a first-price auction, Klemperer (1999) argues that "it is plausible that a first-price auction may be more profitable [...] than a second-price auction". This paper provides the to date most general conditions under which this intuition is confirmed, but it goes further by establishing a bound of sorts on the amount of favoritism that can safely be extended to the weak bidder while still guaranteeing revenue above that in the efficient second-price auction. Specifically, any mechanism where the weak bidder wins more often than is efficient but less often than he would in a counterfactual symmetric auction against another weak bidder is more profitable than the second-price auction.

The main result can be extended in several directions. For instance, Mares and Swinkels (2010a) has recently made the point that it is not known whether the revenue ranking is affected by the presence of a reserve price. An application of the main theorem can be used to effortlessly demonstrate that reserve prices in fact do not affect the ranking in any model satisfying the assumptions outlined before.

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[7]As is well understood in the literature, the assumption of reverse hazard rate dominance is responsible for the property that the weak bidder wins more often than is efficient. However, the importance of first order stochastic dominance (which is implied by reverse hazard rate dominance) should not be overlooked. First order stochastic dominance implies that it is possible to bound above how often the weak bidder wins. Combining the two is what leads to the conclusion that the weak bidder wins more often than is efficient, but not too too often. The dispersive order is used to prove that a mechanism with this property is more profitable than the second-price auction.

[8]See Section 5 for a discussion. Mares and Swinkels (2010a) show that the second-price auction may be advantageous when the seller can discriminate between bidders in a specific way. Kirkegaard (2010a) makes the point that different types of discrimination may have very different effects. Interestingly, the dispersive order plays a role in both papers.

[9]In Maskin and Riley's (2000) one model where the second-price auction is superior to the first-price auction, the optimal auction would in fact discriminate *against* the weak bidder. In the current paper, in contrast, the conditions that are imposed imply that an optimal auction favours the weak bidder.

Moreover, it is possible to establish a whole class of auctions that are superior to the second-price auctions. In addition, the result has implications for the selection of bidders into auctions, and, to use Moldovanu and Sela's (2006) terminology, "contest architecture" as well. For example, having asymmetric bidders is undesirable; the seller is better off flipping a coin between two symmetric auctions – one with two strong bidders, the other with two weak bidders – than to accept an asymmetric second-price auction. This result complements Cantillon (2008) who also argues that asymmetry is undesirable, but in a rather different model.

It is also possible, for the first time, to extend the revenue ranking to certain environments with more than two bidders. The first-price auction is shown to be more profitable than the second-price auction when there are many weak bidders, but only one strong bidder. Such a situation occurs when an incumbent monopolist bids against several potential entrants for a licence to operate on a market or use a new technology on the market. However, the methods proposed here and in Maskin and Riley (2000) are not powerful enough to generate a revenue ranking in the general case with many strong bidders. The reason is that the bounds on the "price discount" mentioned before are not sufficiently tight to permit an unambiguous conclusion in this case. Nevertheless, the first-price auction is shown to be more profitable when the asymmetry between bidders is "sufficiently large".

The paper is organized as follows. The two-bidder model is presented in Section 2, and some preliminary observations are made. Section 3 contains the main result, numerous examples, and a discussion of some limitations of the theorem. Section 4 outlines the added insights that can be obtained by reformulating the model. Reserve prices, other auction formats, and contest architecture are examined in Section 5. Section 6 considers auctions with many bidders. Section 7 concludes.

2 Small auctions: Preliminaries

There are two risk neutral bidders. Bidder s is strong and bidder w is weak; bidder s is in some sense more likely to value the object being sold more highly. Bidder i draws a type or valuation from a distribution function, F_i, which is assumed to be continuously differentiable on its support, $S_i = [\beta_i, \alpha_i]$, $i = s, w$. The density, f_i, is strictly positive on $(\beta_i, \alpha_i]$, with $\alpha_i > \beta_i \geq 0$, $i = s, w$. Mass points are ruled out. It is assumed that $\beta_w \leq \beta_s$ and $\alpha_w < \alpha_s$. In the following, it will be necessary to compare the density functions over certain intervals. To facilitate this comparison, let $F_s(v) = f_s(v) = 0$ for all $v \in [\beta_w, \beta_s)$ if $\beta_w < \beta_s$, such that $f_w(v) \geq f_s(v)$ for all $v \in [\beta_w, \beta_s)$. Let $C = S_s \cap S_w$ denote the support that is common to F_s and F_w. Thus, $C = [\beta_s, \alpha_w]$ if the supports overlap, and $C = \emptyset$ otherwise.

The relationship between F_s and F_w will play two distinct roles in the analysis. The first is to determine the properties of the interaction between bidders in the first-price auction (equilibrium strategies). The second is to facilitate a comparison between the revenue that different mechanisms generate. In this respect, the relative strength of the bidders and the relative slope and dispersion of their distributions, respectively, are key. These properties are discussed in more detail in the following.

2.1 Strength and bidding strategies

The sense is which bidder s is stronger than bidder w needs to be made more precise. There are at least four ways in which bidder heterogeneity can be modelled, depending on how F_s and F_w are related on C:

1. *F_s dominates F_w in terms of the likelihood ratio:* $\frac{f_s(v)}{f_w(v)}$ is increasing on C.[10]

2. *F_s dominates F_w in terms of the reverse hazard rate:* $\frac{f_s(v)}{F_s(v)} \geq \frac{f_w(v)}{F_w(v)}$ for all $v \in C$ ($\frac{F_w(v)}{F_s(v)}$ is decreasing).

3. *F_s dominates F_w in terms of the hazard rate:* $\frac{f_s(v)}{1-F_s(v)} \leq \frac{f_w(v)}{1-F_w(v)}$ for all $v \in C$ ($\frac{1-F_w(v)}{1-F_s(v)}$ is increasing).

4. *F_s first order stochastically dominates F_w:* $F_s(v) \leq F_w(v)$ for all $v \in C$.

See Krishna (2002) for an introduction to these four stochastic orders and their use in auction theory. See Shaked and Shanthikumar (2007) for an in-depth treatment of the topic. It can be shown that the first order implies the other orders. The second and third order both imply the fourth.

Maskin and Riley (2000) assume that F_s dominates F_w in terms of the reverse hazard rate (and therefore also in terms of first order stochastic dominance).[11] In the first part of their paper, this assumption enables them to derive rather tight bounds on equilibrium strategies.

[10] In this paper, increasing is taken to mean non-decreasing; decreasing means non-increasing.

[11] Since $F_w(v)/F_s(v)$ is decreasing and strictly larger than one at $v = \alpha_w$, it follows that F_s strictly first order stochastically dominates F_w in the sense that $F_s(v) < F_w(v)$ for all $v \in (\beta_w, \alpha_w]$. Maskin and Riley (2000) assume something slightly stronger than reverse hazard rate dominance. However, reverse hazard rate dominance is strong enough to deliver the key implication on bidding strategies, namely that the weak bidder is at least as aggressive as the strong bidder for comparable types in the first-price auction (see e.g. Kirkegaard (2009)). Hence, reverse hazard rate dominance is assumed in this paper.

To be more specific, let $r(v) = F_s^{-1}(F_w(v))$, $v \in S_w$. By definition, bidder s is just as likely to have a type below $r(v)$ as bidder w is to have a type below v; the two bidders have the same *rank*, or $F_s(r(v)) = F_w(v)$. First order stochastic dominance is equivalent to the condition that $r(v) \geq v$. Given their assumption, Maskin and Riley (2000) show that in a first-price auction, bidder w with type v either submits a non-serious bid (one that is so low that it never wins) or he submits a bid of the same magnitude as a bid submitted by the strong bidder of some type, $k_1(v)$, somewhere in the interval $[v, r(v)]$.[12] In other words, the weak bidder is more aggressive than the strong bidder, but not aggressive enough to make up for the difference in strength.[13]

For the types that submit serious bids, the bid is strictly increasing in type. Moreover, the bidders share the same maximal bid; bidder w with type α_w submits the same bid as bidder s with type α_s, implying $k_1(\alpha_w) = r(\alpha_w) = \alpha_s$. Otherwise, the bidder with the highest bid could lower his bid and still win with certainty.

In a second-price auction, it is a weakly dominant strategy to submit a bid equal to the bidder's type. Since the auction is efficient, bidder w with type v wins if and only if bidder s has a type below $k_2(v) = \max\{\beta_s, v\}$.

2.2 Dispersion, density, and price sensitivity

In the second part of Maskin and Riley (2000), the bounds on bidding strategies help them to infer that the first-price auction is more profitable than the second-price auction if F_s is either a "shifted" or a "stretched" version of F_w. What is much less obvious – but of the utmost importance – is that their additional assumptions in fact *imply* that

$$f_w(v) \geq f_s(x) \text{ for all } x \in [v, r(v)] \text{ and all } v \in S_w. \tag{1}$$

[12] A non-serious bid is made only if bidder w's type, v, is sufficiently far below β_s. A non-serious bid wins with probability $0 = F_s(\beta_s)$. Since $\beta_s \in [v, r(v)]$ when $v \leq \beta_s$, letting $k_1(v) = \beta_s$ for all v that submit non-serious bids implies that $k_1(v) \in [v, r(v)]$ for all $v \in S_w$.

[13] The first part, $k_1(v) \geq v$, is fairly intuitive. Imagine a situation where the two bidders are using the same bidding strategy, $b(v)$. By bidding $b(x)$, the weak bidder with type v earns expected payoff or $(v - b(x))F_s(x)$. The derivative with respect to x is proportional to $(v - b(x))\frac{f_s(x)}{F_s(x)} - b'(x)$. Hence, given reverse hazard rate dominance, the gain to increasing the bid is higher for the weak bidder than for the strong bidder. The reason is that the reverse hazard rate measures the relative increase in winning probability if one decides to outbid a marginally higher rival type. Since the significance of this assumption is by now well understood in the literature, a formal proof is omitted. Using three different techniques, it has been proven in *(i)* Lebrun (1999) and Maskin and Riley (2000), *(ii)* Milgrom (2004) and Hopkins (2007), and *(iii)* Kirkegaard (2009), respectively. The second part, $k_1(v) \leq r(v)$, comes from first order stochastic dominance (see the references mentioned above).

Coupled with first order stochastic dominance, $f_w(v) \geq f_s(v)$ is seen to imply that F_s dominates F_w in terms of the hazard rate as well. Thus, hazard rate dominance is implicit in Maskin and Riley (2000).

In the current paper, I will explicitly assume that (1) is satisfied. As just mentioned, (1) is weaker than the assumptions in Maskin and Riley (2000). Nevertheless, it turns out to be sufficient for the main result. Moreover, the assumption has a clear and intuitive economic interpretation. In contrast, Maskin and Riley (2000) do not provide an economic interpretation of their assumptions.

Bulow and Roberts' (1989) analogy to monopoly pricing is useful to interpret (1) and understand its implications. Thinking of v as a price, the survival function $q_i(v) \equiv 1 - F_i(v)$ has the properties of a demand curve in a market with a continuum of consumers of mass one, distributed on S_i, $i = s, w$. First order stochastic dominance is equivalent to the assumption $q_s(v) \geq q_w(v)$.

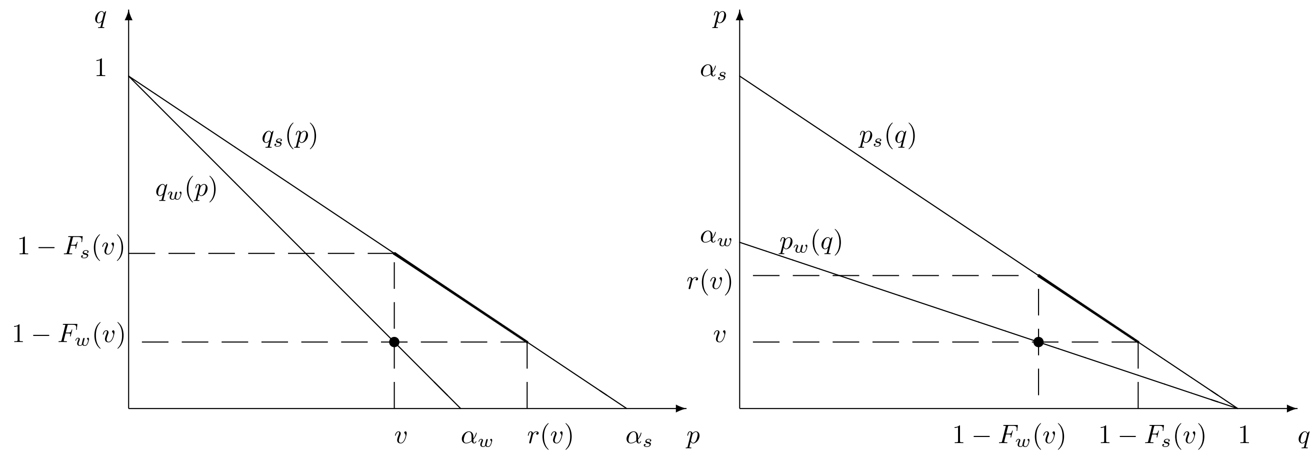
Clearly, the slope of the demand curve is $q_i'(v) = -f_i(v)$, $i = s, w$. Thus, the assumption that $f_w(v) \geq f_s(v)$ means that the weak bidder's demand curve is steeper than the strong bidder's demand curve. At the other end of the $[v, r(v)]$ interval, recall that at price $r(v)$ and v, respectively, demand would be the same in the markets described by demand curves q_s and q_w. Hence, $f_w(v) \geq f_s(r(v))$ means that the weak bidder's demand curve is steeper at comparable quantities as well. Since $f_w(v) \geq f_s(x)$ for all $x \in [v, r(v)]$, the implication is simply that the weak bidder's demand curve at a given price is steeper than the strong bidder's demand curve on a range of prices above v. Alternatively, since the inverse demand curve has slope $-f_i(v)^{-1}$, the weak bidder's inverse demand curve, $p_w(q)$, is flatter than the strong bidder's inverse demand curve, $p_s(q)$.[14] Figure 1 illustrates the assumption.

The economic interpretation of the assumption is therefore that the weak bidder's demand responds more in *absolute* terms to a marginal price change. In a sense, the market with the lowest willingness-to-pay is also the most price sensitive market. The *relative* change in demand following a marginal price increase can be measured by

$$\left|\frac{q_i'(v)}{q_i(v)}\right| = \frac{f_i(v)}{1 - F_i(v)} \text{ and } \varepsilon_i(v) = \left|\frac{v q_i'(v)}{q_i(v)}\right| = \frac{v f_i(v)}{1 - F_i(v)}.$$

By hazard rate dominance, the weak bidder is also more price sensitive than the strong bidder in relative terms, at comparable prices.

[14] Recall that both markets has a set of consumers of mass one, so $q \in [0, 1]$ on both markets. With the assumption that demand in the strong market is more "spread out", or $\alpha_s - \beta_s \geq \alpha_w - \beta_w$, it must necessarily be the case that p_s is steeper than p_w locally, for some q. Thus, (1) can be viewed as a regularity condition, roughly saying that p_s is steeper than p_w globally.



(a) Demand curves (b) Inverse demand curves

Figure 1: Demand curves for $\beta_w = \beta_s = 0$. The strong bidder's demand curve is flatter on the fat segment than the weak bidder's demand curve is at the highlighted point (Panel a). The strong bidder's inverse demand curve is steeper (Panel b).

It is also instructive to examine the implications of the ordering of densities at the same rank, $f_w(v) \geq f_s(r(v))$. To do so, another small detour into order statistics is useful. F_w is said to be smaller than F_s in the *dispersive order* if

$$F_w^{-1}(n) - F_w^{-1}(m) \leq F_s^{-1}(n) - F_s^{-1}(m) \text{ whenever } 0 < m \leq n < 1. \tag{2}$$

In words, F_s is more dispersed if the distance between the types that are at the nth and mth percentile is wider for the strong than the weak bidder. It turns out that (2) is equivalent to the condition that $f_w(v) \geq f_s(r(v))$ for all $v \in S_w$. Thus, (1) implies dispersion, which is therefore also implicit in Maskin and Riley (2000). Condition (2) is also equivalent to the condition that $r(v) - v$ is increasing in v, or $r'(v) \geq 1$, for all $v \in S_w$. Once again, see Shaked and Shanthikumar (2007) for details.

The dispersive order has some intuitive properties. First, if F_s is more disperse than F_w, then it has larger variance. Second, the support is wider for the more disperse distribution, $\alpha_s - \beta_s \geq \alpha_w - \beta_w$. Note also that if $\beta_s - \beta_w = r(\beta_w) - \beta_w \geq 0$ and F_s is more dispersed than F_w, then F_s first order stochastically dominates F_w.

Moreover, the dispersive order has an interesting economic interpretation in terms of relative price sensitivity. First, if F_s is more dispersed than F_w then $p_s(q)$ is steeper than $p_w(q)$ for all $q \in [0, 1]$, as mentioned earlier. Likewise, note that

$$\left|\frac{q_w'(v)}{q_w(v)}\right| = \frac{f_w(v)}{1 - F_w(v)} \geq \frac{f_s(r(v))}{1 - F_s(r(v))} = \left|\frac{q_s'(r(v))}{q_s(r(v))}\right|,$$

meaning that if the same quantity was sold on two markets, the market that is less disperse would react more to a marginal price change in percentage terms.[15] Though it is not needed in this paper, it is commonly assumed that the hazard rate is increasing, in which case the weak bidder is more price sensitive at price v than the strong bidder is at any price below $r(v)$, or

$$\frac{f_w(v)}{1-F_w(v)} \geq \frac{f_s(r(v))}{1-F_s(r(v))} \geq \frac{f_s(x)}{1-F_s(x)} \text{ for all } x \leq r(v).$$

In other words, the weak bidder is more price sensitive at quantity $q_w(v)$ than the strong bidder is at larger quantities, since $q_s(x) \geq q_s(r(v)) = q_w(v)$.

Note that (1) is particularly simple to check if f_s is monotonic. If f_s is increasing (F_s is convex) then (1) is satisfied if and only if $f_w(v) \geq f_s(r(v))$, $v \in S_w$, i.e. if F_s is more disperse than F_w. If f_s is decreasing (F_s is concave), then (1) is satisfied if and only if F_s is flatter than F_w, or $f_w(v) \geq f_s(v)$ for all $v \in S_w$.

For future reference, define marginal revenue evaluated at price v as

$$J_i(v) = v\left[1 - \frac{1}{\varepsilon_i(v)}\right] = v - \frac{1-F_i(v)}{f_i(v)}.$$

The interpretation of J_i as marginal revenue is due to Bulow and Roberts (1989). Myerson (1981) refers to J_i as bidder i's virtual valuation. Hazard rate dominance implies that $J_w(v) \geq J_s(v)$ for all $v \in C$.

3 Small auctions: Analysis and Examples

The general analysis is presented first. Several examples are then presented to illustrate the scope of the main result and its relationship to Maskin and Riley (2000).

[15]Since the economic interpretation of dispersion-related stochastic orders has not yet received much attention, it may be worthwhile to offer an observation on the *star order*. Formally, F_w is said to be smaller than F_s in the star order if $r(v)$ is "starshaped" on $v \in S_w$ ($r(v)/v$ is increasing). Note that

$$\frac{d}{dv}\left(\frac{r(v)}{v}\right) \propto r'(v)v - r(v) = \frac{f_w(v)}{f_s(r(v))}v - r(v) \propto f_w(v)v - f_s(r(v))r(v),$$

implying that

$$\frac{d}{dv}\left(\frac{r(v)}{v}\right) \geq 0 \Longleftrightarrow \varepsilon_w(v) \geq \varepsilon_s(r(v)).$$

In words, the weak bidder's demand is more elastic at all comparable *quantities* if and only if F_w is smaller than F_s in the star order.

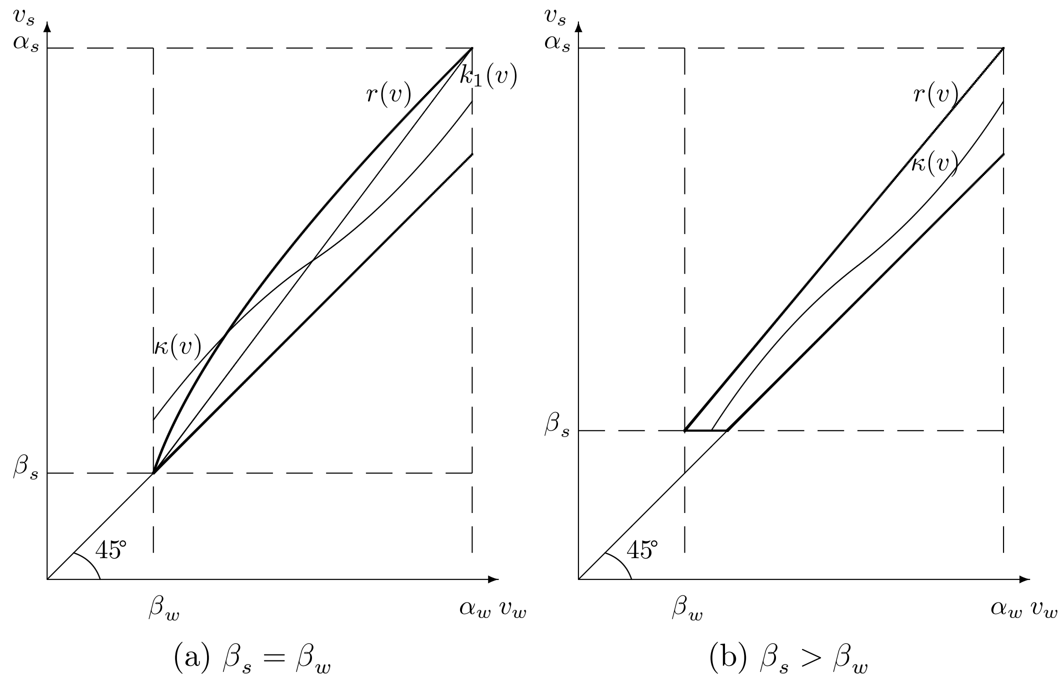


Figure 2: The allocation in the first-price auction, $k_1(v)$, falls between the two thick curves, the lower of which describes the allocation in the second-price auction, $k_2(v)$.

3.1 Analysis

Recalling the textbook intuition supporting third-degree price discrimination, it should be no surprise that a mechanism that "favors" the more price sensitive bidder a bit is more profitable than one that does not (see Bulow and Roberts (1989)). In a second-price auction, it is the bidder with the highest type or willingness-to-pay that wins, which is similar to a monopolist charging the same price on all markets. In contrast, the weak bidder wins more often in a first-price auction, which is roughly equivalent to a monopolist offering a price discount on the weak, price sensitive, market. However, the price discount may be too steep.

Maskin and Riley (2000) discuss this point in their paper, using a figure similar to Figure 2, above. To begin, consider the simpler case with $\beta_w = \beta_s = \beta$, as in Figure 2a. Recall that the weak bidder has higher marginal revenue than the strong bidder for comparable types, or $J_w(v) \geq J_w(s)$ for all $v \in C$. However, when v is close to α_w, it is also the case that $J_w(v) < J_s(r(v))$. For instance, $J_w(\alpha_w) = \alpha_w < \alpha_s = J_s(\alpha_s) = J_s(r(\alpha_w))$. Assume for the moment that J_s is strictly increasing, and let $\kappa(v) = J_s^{-1}(J_w(v))$ (whenever it exists) denote the type the strong bidder must have for his marginal revenue to coincide with that of the weak bidder. By hazard rate dominance, $\kappa(v) \geq v$, but when v is sufficiently large, $\kappa(v) < r(v)$. In an optimal auction (subject to the constraint that the object is sold), the weak bidder should win if and only if the strong bidder's type is below $\kappa(v)$, thereby ensuring that the object is allocated to the bidder with the highest marginal revenue.

In a first-price auction, bidder w with type v wins if his opponent's type is below $k_1(v)$, which is somewhere in the interval $[v, r(v)]$. In contrast, the second-price auction is efficient, $k_2(v) = \max\{\beta_s, v\}$. Thus, *fixing bidder w's type* at v, the difference between a first and second-price auction is that the weak bidder wins in the former but loses in the latter if the strong bidder's type is in $[v, k_1(v)] \subset [v, r(v)]$ (a vertical distance in Figure 2). If $k_1 \leq \kappa$, the allocation in the first-price auction is more profitable; the weak bidder is eating his way into an area where his marginal revenue exceeds the strong bidder's marginal revenue. However, if $k_1 > \kappa$, the trend reverses – the weak bidder is now winning too often, beating the strong bidder even though he has a comparably low marginal revenue. This occurs when v is close to α_w, since $k_1(\alpha_w) = r(\alpha_w) = \alpha_s$, but $\kappa(\alpha_w) < \alpha_s$. Hence, depending on the strong bidder's *actual* type, switching from the second-price auction to the first-price auction may or may not increase the winner's marginal revenue. Therefore, Maskin and Riley (2000) conclude that mechanism design is useless in determining which auction generates more revenue.

However, the concern is with expected revenue. All that is required is to determine which of the two conflicting effects dominates *in expectation*. Mechanism design is eminently suited to address this question.

Myerson (1981) shows that expected revenue in any mechanism is equal to the expected value of the winning bidder's marginal revenue, less any rent earned by the lowest types. Starting with this principle, the key step in the following analysis is simply to formulate expected revenue in a way that perfectly captures the trade-off discussed above. Specifically, the trick is to write the expected value of the winner's marginal revenue as the expected value of a conditional expectation.

Consider some mechanism where bidder w with type v wins if and only if bidder s's type is below $k(v)$, and let $u_i^k(\beta_i)$ denote bidder i's expected utility if his type is β_i, $i = s, w$. If bidder w loses (wins) with probability one, then $k(v) = \beta_s$ ($k(v) = \alpha_s$). Then, expected revenue can be written

$$ER^k = \int_{\beta_w}^{\alpha_w} \left(J_w(v) F_s(k(v)) + \int_{k(v)}^{\alpha_s} J_s(s) dF_s(x) \right) dF_w(v) - u_w^k(\beta_w) - u_s^k(\beta_s). \quad (3)$$

Note that the term in parenthesis is the expected value of the winning bidder's marginal revenue, conditional on the weak bidder's type being v.

Equation (3) is the counterpart to the lemmata (Lemma 4.1 and Lemma 4.2) in Maskin and Riley (2000) in which they derive expressions for revenue in the two particular auctions they study. The proofs of these lemmata are somewhat technical and offer no economic insights.

Whether the auction is a first or second-price auction, the weak bidder wins with probability zero and earns zero rent if his type is β_w. The same is true for the strong bidder with type β_s in the case where $\beta_s = \beta_w$.

However, the strong bidder earns positive rent if $\beta_s > \beta_w$. In this case,

$$u_s^2(\beta_s) = \int_{\beta_w}^{\beta_s} (\beta_s - v) dF_w(v) \tag{4}$$

in the second-price auction, assuming that bidders use the weakly dominant strategy of submitting a bid that equals the bidder's type. In a first-price auction, the strong bidder with type β_s receives expected payoff of

$$u_s^1(\beta_s) = (\beta_s - b_*) F_w(b_*) \tag{5}$$

where b_* is the bid submitted by the strong bidder with type β_s, where $b_* \in (\beta_w, \beta_s)$. Such a bid wins with probability $F_w(b_*)$ because the weak bidder does not bid above b_* if his type is at or below b_*. Thus, the strong bidder with type β_s prefers the second-price auction if $\beta_s > \beta_w$. Subtracting (5) from (4) yields

$$u_s^2(\beta_s) - u_s^1(\beta_s) = \int_{\beta_w}^{b_*} (b_* - v)\, dF_w(v) + \int_{b_*}^{\beta_s} (\beta_s - v)\, dF_w(v). \tag{6}$$

Recall that the weak bidder wins more often in the first-price auction than the second-price auction, or $k_1(v) \geq k_2(v)$. Let

$$D(v|k_1, k_2) = \int_{k_2(v)}^{k_1(v)} (J_w(v) - J_s(x))\, dF_s(x). \tag{7}$$

Given the discussion of Figure 2a, the interpretation of $D(v|k_1, k_2)$ is precisely that it measures the consequences of the change in allocation for a fixed value of v, namely that the seller obtains $J_w(v)$ by sacrificing $J_s(x)$ when he moves from a second-price auction to a first-price auction.

From (3),

$$ER^1 - ER^2 = \int_{\beta_w}^{\alpha_w} D(v|k_1, k_2) dF_w(v) + u_s^2(\beta_s) - u_s^1(\beta_s). \tag{8}$$

The allocation is the same in both auctions if bidder w's type falls below b_* since he loses with probability one ($k_1(v) = k_2(v) = \beta_s$). Thus, $D(v|k_1, k_2) = 0$ for $v \in [\beta_w, b_*]$. Using (6), (8) can then be expanded to

$$\begin{aligned} ER^1 - ER^2 &= \int_{\beta_w}^{b_*} (b_* - v)\, dF_w(v) + \int_{b_*}^{\beta_s} (\beta_s - v + D(v|k_1, k_2))\, dF_w(v) \\ &\quad + \int_{\beta_s}^{\alpha_w} D(v|k_1, k_2) dF_w(v). \end{aligned} \tag{9}$$

The final step of the proof is to show that each term is positive by showing that the function under the integration is positive for every value of v in the relevant range. This is trivially true for the first term. It turns out to also be true for the last term, because (1) implies that D is positive in this case. However, for $v \in [b_*, \beta_s)$, D may be negative. The reason is illustrated in Figure 2b. As is the case in one of Maskin and Riley's (2000) models, it is possible that $J_w(\beta_w) < J_s(\beta_s)$ when $\beta_w < \beta_s$, meaning that the weak bidder should ideally never win if his type is close to β_w, yet he may do so in a first-price auction. However, the extra rent the seller appropriates from the strong bidder with type β_s in the first-price auction is more than enough to compensate for this particular drawback. The case with $\beta_s > \alpha_w$ (no overlap) is handled in a similar manner.

Theorem 1 *Assume that (i) F_s dominates F_w in terms of the reverse hazard rate, and (ii) for all $v \in S_w$, $f_w(v) \geq f_s(x)$ for all $x \in [v, r(v)]$. Then, the first-price auction generates strictly higher expected revenue than the second-price auction.*

Proof. Assume first that $\beta_s \leq \alpha_w$. If $v \in [\beta_s, \alpha_w]$ then $k_1(v) \in [v, r(v)]$ and $k_2(v) = v$, in which case

$$\begin{aligned} D(v|k_1, k_2) &= J_w(v)(F_s(k_1) - F_s(v)) + k_1(1 - F_s(k_1)) - v(1 - F_s(v)) \\ &= -\frac{1 - F_w(v)}{f_w(v)}(F_s(k_1) - F_s(v)) + (k_1 - v)\,(1 - F_s(k_1)) \\ &= \frac{1 - F_s(k_1)}{f_w(v)} \left[f_w(v)(k_1 - v) - \frac{1 - F_w(v)}{1 - F_s(k_1)}\,(F_s(k_1) - F_s(v)) \right] \\ &\geq \frac{1 - F_s(k_1)}{f_w(v)} \left[f_w(v)(k_1 - v) - (F_s(k_1) - F_s(v)) \right] \\ &= \frac{1 - F_s(k_1)}{f_w(v)} \int_v^{k_1} (f_w(v) - f_s(x))\, dx \geq 0, \end{aligned}$$

where the first inequality follows from $F_s(k_1) \leq F_w(v)$ when $k_1(v) \leq r(v)$, and the second inequality from the second assumption in the Theorem. Since $k_1(v) < r(v)$ almost always, the first inequality is strict almost always. If $v \in [b_*, \beta_s]$ then $k_2(v) = \beta_s$ (the weak bidder wins with probability zero in a second-price auction), while $k_1(v) \in [\beta_s, r(v)]$. Hence,

$$\begin{aligned}\beta_s - v + D(v|k_1, k_2) &= \beta_s - v + J_w(v)F_s(k_1) + k_1(1 - F_s(k_1)) - \beta_s \\ &= \frac{1 - F_s(k_1)}{f_w(v)}\left[f_w(v)(k_1 - v) - \frac{1 - F_w(v)}{1 - F_s(k_1)}F_s(k_1)\right] \\ &\geq \frac{1 - F_s(k_1)}{f_w(v)}\int_v^{k_1}(f_w(v) - f_s(x))\,dx,\end{aligned}$$

which is positive, by assumption. The last step uses the fact that $f_s(x) = 0$ for $x \in [v, \beta_s)$. In conclusion, every term in (9) is positive (and strictly positive almost always). Hence, the first-price auction is strictly more profitable than the second-price auction.

If $\beta_s > \alpha_w$ (no overlap),

$$\begin{aligned}u_s^2(\beta_s) - u_s^1(\beta_s) &= \int_{\beta_w}^{\alpha_w}(\beta_s - v)dF_w(v) - (\beta_s - b_*)F_w(b_*) \\ &= \int_{\beta_w}^{b_*}(b_* - v)\,dF_w(v) + \int_{b_*}^{\alpha_w}(\beta_s - v)\,dF_w(v).\end{aligned}$$

Since $k_2(v) = \beta_s$ and $k_1(v) \in [\beta_s, r(v)]$, with $k_1(v) = \beta_s$ for $v \leq b_*$,

$$\begin{aligned}ER^1 - ER^2 &= \int_{b_*}^{\alpha_w} D(v|k_1, k_2)dF_w(v) + u_s^2(\beta_s) - u_s^1(\beta_s) \\ &= \int_{\beta_w}^{b_*}(b_* - v)\,dF_w(v) + \int_{b_*}^{\alpha_w}(\beta_s - v + D(v|k_1, k_2))\,dF_w(v),\end{aligned}$$

and the remainder of the proof is identical to the $\beta_s \leq \alpha_w$ case. Maskin and Riley (2000) point out that if β_s is much larger than α_w, the equilibrium of the first-price auction involves the strong bidder always bidding α_w, and always winning. This case corresponds to $b_* = \alpha_w$. ■

Theorem 1 takes a particularly simple form if f_s is monotonic (F_s is concave or convex).

Corollary 1 *Assume f_s is monotonic on S_s. Then, the first-price auction generates strictly higher expected revenue than the second-price auction if*

1a. F_w *is smaller than* F_s *in the dispersive order,*

1b. $F_w(v) - F_s(v)$ *is increasing on* C *and* $f_w(v) \geq f_s(\beta_s)$ *for all* $v \in S_w \backslash C$.

2. $F_w(v)/F_s(v)$ *is decreasing on* C.

Proof. Condition 1a and 1b implies that $f_w(v) \geq f_s(r(v))$ and $f_w(v) \geq f_s(\max\{\beta_s, v\})$ for all $v \in S_w$, respectively. Since f_s is assumed to be monotonic on S_s and $f_s(v) = 0$ for all $v \notin S_s$, the second part of Theorem 1 is satisfied. Condition 2 is equivalent to the first condition in Theorem 1. ■

Condition 1a is redundant if f_s is decreasing (it is implied by 1b), while condition 1b is redundant if f_s is increasing (it is implied by 1a). Together, condition 1b and condition 2 have an interesting interpretation. In a second-price auction, the difference between the winning probability of bidder s with type v and the winning probability of bidder w with type v is precisely $F_w(v) - F_s(v)$. Condition 1b means that the gap between the winning probabilities grows as the stakes get higher or v increases. That is, the asymmetry between bidders is increasing in "absolute" terms. However, condition 2 says that the absolute asymmetry does not increase too rapidly; in "relative" terms, the asymmetry is decreasing. Under these circumstances, the first-price auction dominates the second-price auction.

The remainder of the Section is devoted primarily to examples. The purpose is, first, to prove that Maskin and Riley's (2000) two propositions in which the first-price auction dominates are corollaries of Theorem 1 and, second, to illustrate the much larger scope of Theorem 1. A discussion of some limitations of Theorem 1 and examples violating the conditions of Theorem 1 is also included.

3.2 Examples

Recall that the two assumptions in the Theorem have quite different roles. The first ensures that $k_1(v) \in [v, r(v)]$. The second ensures that mechanisms with this property are more profitable than a second-price auction. It may be of interest to find conditions that imply both of the assumptions. Several examples are listed below, in three groups. Examples 1 and 3 are somewhat generalized versions of the models examined in Maskin and Riley (2000). The remaining models are new to the literature. Examples 2 and 4 complements examples 1 and 3, respectively. Examples 5 and 6 make the point that the two assumptions in the Theorem are more easily satisfied when the asymmetry between bidders is "sufficiently large", a point that will be revisited in Section 6. The Section concludes with a discussion of limitations of Theorem 1.

3.2.1 Horizontal and vertical shifts

Example 1 (horizontal shifts): As in Maskin and Riley (2000), assume that F_w is convex, and assume that F_s is obtained by shifting F_w to the right, onto the support $[\beta_w + a, \alpha_w + a]$, $a > 0$. That is, $F_s(v) = F_w(v - a)$, for $v \in [\beta_w + a, \alpha_w + a]$. Clearly, $f_w(v) = f_s(r(v))$ (this reflects the fact that the dispersive order is location free). By convexity, (1) is therefore satisfied. Although F_w is convex, assume that it is not too convex. Specifically, assume that F_w is strictly log-concave, meaning that the reverse hazard rate is strictly decreasing, directly implying that

$$\frac{f_w(v)}{F_w(v)} < \frac{f_w(v-a)}{F_w(v-a)} = \frac{f_s(v)}{F_s(v)}.$$

Hence, both assumptions of the theorem are satisfied. Note how each of the two assumptions (convexity and log-concavity) separately of each other imply one of the two assumptions. ▲

Remark 1: Maskin and Riley (2000) impose the assumption that $J_w(v) \leq 0$ for all $v \in [\beta_w, \beta_w + a]$ in order to make Example 1 work. This assumption is not necessary with the method employed here; the bounds on revenue are tighter because (9) makes better use of the fact that the seller appropriates more rent from the strong bidder with type β_s.

In addition, Maskin and Riley's (2000) assumption on J_w implies that F_s is not shifted too far to the right because $J_w(\beta_w + a) \approx \alpha_w > 0$ when $\beta_w + a$ is close to α_w. Hence, their assumption necessitates that the supports overlap. Nevertheless, one of their concrete motivating examples is of a case where the supports do not overlap (see also Example 5, below). On the other hand, Maskin and Riley (2000) allow for a mass point at β_i, such that $F_i(\beta_i) \geq 0$. △

Example 2 (vertical shifts): Consider some continuously differentiable, strictly increasing, and concave function, $G(v)$, $v \in [0, \alpha_s]$. Assume $G(\alpha_s) = 1$ and $G(\beta_s) = 0$ for some $\beta_s > 0$. Let $F_s(v) = G(v)$ on $[\beta_s, \alpha_s]$. Assume that F_w is a vertical shift of G, such that it can be written $F_w(v) = \gamma + G(v)$ for some $\gamma > 0$, where $(\gamma, \beta_w, \alpha_w)$ satisfies $F_w(\beta_w) = 0$ and $F_w(\alpha_w) = 1$. On the overlap of $[\beta_s, \alpha_s]$ and $[\beta_w, \alpha_w]$, if any, it holds that

$$\frac{F_w(v)}{F_s(v)} = \frac{\gamma}{G(v)} + 1$$

is decreasing. Hence, reverse hazard rate dominance applies. Moreover, due to the concavity of G, $f_w(v) = G'(v) \geq f_s(x)$ for all $x \in [v, \alpha_s]$, implying that (1) is satisfied as well. ▲

REMARK 2: In the light of (1), Examples 1 and 2 each represent an extreme case. Example 1 satisfies $f_w(v) = f_s(r(v))$ while Example 2 satisfies $f_w(v) = f_s(v)$ (on the overlap of the supports). That is, (1) is satisfied "with equality" at one of the endpoints of the interval $[v, r(v)]$. The monotonicity assumptions on f_s then imply that these properties are sufficient for $f_w(v) \geq f_s(x)$ to hold on the entire interval. See also Corollary 1. △

3.2.2 Truncations and Compressions

EXAMPLE 3 (TRUNCATIONS AND STRETCHES): Assume that F_w is a truncation of F_s. That is, $F_w(v) = \frac{F_s(v)}{F_s(\alpha_w)}$, $v \in [\beta_w, \alpha_w]$, where $\alpha_w < \alpha_s$ and $\beta_w = \beta_s$. It is seen that F_w/F_s is constant on $v \in [\beta_w, \alpha_w]$, meaning that the first assumption in the Theorem is satisfied. Assume that F_s is log-concave, as in Example 1. Then,

$$\frac{f_w(v)}{F_w(v)} = \frac{f_s(v)}{F_s(v)} \geq \frac{f_s(x)}{F_s(x)}$$

for any $x \in [v, r(v)]$. For any x in this range, it is also the case that $F_s(x) \leq F_w(v)$. The above inequality then necessitates that $f_s(x) \leq f_w(v)$ for all $x \in [v, r(v)]$. Hence, the second assumption of the Theorem is satisfied as well. ▲

REMARK 3: Maskin and Riley's (2000) set-up is slightly different. They say that F_s is obtained by "stretching" F_w, so that $F_s(v) = \lambda F_w(v)$ on $[\beta_w, \alpha_w]$, for some $\lambda \in (0, 1)$. This leaves the problem of what F_s looks like on $[\alpha_w, \alpha_s]$, and they are then forced to make restrictive assumptions on this as well (compare (4.13) in their paper to (1) in the current paper). Unfortunately, their conditions rule out the much-studied class of convex power distributions, where $F_s(v) = (v/\alpha_s)^{\gamma_s}$ for some $\gamma_s > 1$, $v \in [0, \alpha_s]$, and all other distribution functions for which $f_s(\beta_s) = 0$. In contrast, the power distribution satisfies all the assumptions in Example 3 of the current paper. The power distribution is also discussed in Section 3.1.4 and Section 4. △

EXAMPLE 4 (COMPRESSIONS): Assume that F_s is concave, and write F_w as a transformation of F_s, such that $F_w(v) = G(F_s(v))$, $v \in [\beta_w, \alpha_w]$, with $\beta_w = \beta_s$. Assume the transformation "compresses" F_s in the sense that $G'(\cdot) > 1$ and that it is concave, $G''(\cdot) \leq 0$. Since

$$\frac{f_s(v)}{f_w(v)} = \frac{1}{G'(F_s(v))}$$

is increasing, F_s dominates F_w in terms of the likelihood ratio, and hence in terms of the reverse hazard rate as well. Moreover, $f_w(v) = G'(F_s(v))f_s(v) > f_s(v) \geq f_s(x)$,

$x \in [v, r(v)]$, meaning that (1) is satisfied as well. The first inequality comes from $G'(\cdot) > 1$, while the second comes from the concavity of F_s. ▲

Remark 4: The transformations from F_s to F_w in Examples 3 and 4 are related. Obviously, $\beta_w = \beta_s$ in both. Moreover, the transformation in Example 3, $G(F_s) = \frac{F_s}{F_s(\alpha_w)}$, is a special case of the transformation in Example 4 since it satisfies $G'(\cdot) > 1$ and $G''(\cdot) \leq 0$. However, since the transformation is linear, it is possible in Example 3 to weaken the assumption that F_s is concave and instead assume only that it is log-concave. △

3.2.3 Large asymmetries

Example 5 (no overlap): Assume there is no overlap between the supports, in which case the first assumption in the Theorem is trivially satisfied. Assume further that F_s is convex and more dispersed than F_w. Then, both assumptions of the Theorem are satisfied. ▲

Example 6 (large differences in size): Assume that F_s dominates F_w in terms of the reverse hazard rate and $\min_{v\in[\beta_w,\alpha_w]} f_w(v) \geq \max_{v\in[\beta_s,\alpha_s]} f_s(v)$. The latter assumption applies to situations where the support of F_s is much wider than the support of F_w. Three special cases are reported below. Clearly, both assumptions in the Theorem are satisfied. ▲

Example 6.1 (concave vs. convex): F_w is concave and F_s is convex, with $f_w(\alpha_w) \geq f_s(\alpha_s)$. The curvature assumptions imply likelihood ratio dominance, which in turn implies reverse hazard rate dominance. Finally, $\min_{v\in[\beta_w,\alpha_w]} f_w(v) = f_w(\alpha_w) \geq f_s(\alpha_s) = \max_{v\in[\beta_s,\alpha_s]} f_s(v)$. ▲

Example 6.2 (normal distributions): This example involves distribution functions that are not necessarily globally concave or convex. Assume F_i is obtained by taking a normal distribution with variance σ_i^2 and mean μ_i and truncating it to fit on the support $[0, \alpha_i]$, $i = s, w$. Assume also that $\sigma_s^2 = \sigma_w^2$ and $\mu_s \geq \mu_w$.[16] Then, it can be verified that

$$\frac{d}{dv}\left(\frac{f_s(v)}{f_w(v)}\right) \propto \mu_s - \mu_w \geq 0.$$

[16] Recall that μ_i and σ_i^2 refers to the mean and variance of the "parent distribution" that is truncated. Truncation changes both the mean and variance. As mentioned earlier, the variance of F_s is no smaller than the variance of F_w if the former is more disperse than the latter. First order stochastic dominance implies that F_s has the higher mean.

Once again, likelihood ratio dominance implies reverse hazard rate dominance. If the difference between α_s and α_w is sufficiently large, then it must also be the case that $\min_{v\in[\beta_w,\alpha_w]} f_w(v) \geq \max_{v\in[\beta_s,\alpha_s]} f_s(v)$. ▲

Example 6.3 (no overlap and large differences in size): Combining Example 5 and 6, assume that the supports do not overlap and that $\min_{v\in[\beta_w,\alpha_w]} f_w(v) \geq \max_{v\in[\beta_s,\alpha_s]} f_s(v)$. Note that no curvature assumptions are imposed. For example, F_s may be obtained by first stretching F_w and then shifting it far to the right. Both assumptions in Theorem 1 are trivially satisfied. ▲

3.2.4 Limitations

Maskin and Riley's (2000) examples of "shift" and "stretch" each push one of the two conditions in Theorem 1 as far as it can go. In the former case, $f_w(v) \geq f_s(r(v))$ holds with equality while in the latter case $F_w(v)/F_s(v)$ is constant. Thus, a small perturbation of either example could very well leave Theorem 1 useless. Two related but slightly more general observations are presented next.

Assume the support of F_s and F_w overlap, $\alpha_w \geq \beta_s$. Then, the first condition in Theorem 1 requires that $F_w(v)/F_s(v)$ be decreasing (or non-increasing). At $v = \alpha_w$, the ratio is $1/F_s(\alpha_w)$. Thus, $F_w(v)/F_s(v) \geq 1/F_s(\alpha_w)$ on $[\beta_s, \alpha_w]$. In other words, the second condition in Theorem 1 implies or necessitates that

$$F_w(v) \geq \frac{F_s(v)}{F_s(\alpha_w)},\ v \in [\beta_s, \alpha_w]\,. \tag{10}$$

The implication is that if $F_w(v)$ first order stochastically dominates a truncated version of F_s, then Theorem 1 is not applicable. Thus, Example 3 is right on the boundary of Theorem 1's domain. More precisely, Theorem 1 does not apply if bidder w becomes just a bit stronger.[17]

Assume now that F_s is convex and $\beta_s > \beta_w$, as in Example 1. Then, the second condition in Theorem 1 is equivalent to the requirement that $f_w(v) \geq f_s(r(v))$. Recall that dispersion implies $\alpha_s - \beta_s \geq \alpha_w - \beta_w$. Beginning with the distribution function F_s, consider the left-ward shift $F_s(v + \beta_s - \beta_w)$, $v \in [\beta_w, \beta_w + (\alpha_s - \beta_s)]$. Consider now a type v for which F_w coincides with and is strictly flatter than this shifted version of F_s. In this case, $r(v) = v + \beta_s - \beta_w$ and

$$f_w(v) < f_s((v + \beta_s - \beta_w)) = f_s(r(v)),$$

[17] It is a fundamental problem in auction theory to understand behavior when reverse hazard rate dominance does not apply. Hopkins (2007) and Kirkegaard (2009) infer some qualitative features of bidder interaction in such environments.

thereby violating dispersion. Thus, to satisfy dispersion it is necessary that $F_w(v) \geq F_s(v + \beta_s - \beta_w)$, $v \in [\beta_w, \alpha_w]$. In other words, Theorem 1 in not applicable if F_w first order stochastically dominates a left-ward shift of F_s. Thus, Examples 1 is also right on the boundary of Theorem 1's domain.

Obviously, the conditions in the Theorem are sufficient, but not necessary. Cheng (2006) provides an example in which the first-price auction dominates the second-price auction, even though (1) is not satisfied. In Cheng's (2006) example, distribution functions are power distributions of the form $F_i(v) = (v/\alpha_i)^{\gamma_i}$, $v \in [0, \alpha_i]$, with $\gamma_s > \gamma_w > 0$ and $\alpha_s > \alpha_w > 0$. In the next section, it will be shown that Theorem 1 applies whenever $\alpha_s\gamma_w \geq \alpha_w\gamma_s$. Cheng (2006) considers a specific case in which this restriction is not satisfied. However, by imposing another restriction on the parameters, he is able to characterize bidding strategies explicitly and a direct revenue comparison is therefore possible (see Example 7 in Section 4.2). Cheng (2006) also extends the revenue ranking to situations with many weak and strong bidders.

Lebrun (1996) considers a similar model, but assumes $\alpha_s = \alpha_w$. In this case it is immediately clear that Theorem 1 does not apply; the combination of first order stochastic dominance and dispersion necessitates $\alpha_s > \alpha_w$. In fact, when $\alpha_s = \alpha_w$, Lebrun (1996) shows that it is always the case that $D(v|k_1, k_2)$ is negative for v close to $\alpha_s = \alpha_w$, but positive for small values of v. Thus, the first-price auction does not dominate the second-price auction type for type, as is required for Theorem 1 to work. Remarkably, in his specific model, Lebrun (1996) is able to prove that, on balance, the first-price auction is more profitable than the second-price auction whenever $2\gamma_s\gamma_w \geq 1$. However, his proof relies heavily on the specific functional form of the distribution functions, and does unfortunately not generalize.

Finally, it should be noted that Maskin and Riley's (2000) example of a setting where the second-price auction dominates the first-price auction obviously violates (1). In that model, the two distribution functions share the same support, F_w has a mass point at β_w, and it is flatter than F_s. Returning to the monopoly analogy, F_w's mass point in a sense implies that it is no longer the case that two "markets" with the same mass of consumers are being compared; the mass at β_w never wins the auction and so might as well not have been there. In Figure 1(b), where $\beta_w = 0$, the weak bidder's inverse demand curve would be steeper and hit the horizontal axis before the strong bidder's inverse demand curve. That is, the market with the lowest willingness-to-pay is the market that is the least price sensitive. Favoring the weaker market would then be a mistake, which explains why the first-price auction performs poorly in that set-up. In this example, $D(v|k_1, k_2)$ is in fact negative for all v. Thus, one auction dominates the other type for type as in Theorem 1.

4 A reformulation

So far, the distribution functions F_s and F_w have been treated as the primitives of the model. Knowing these, the function r can then be inferred. An alternative is to consider the distribution function F_s and the function r to be the primitives of the model, in which case the weak bidder's distribution, $F_w(v) = F_s(r(v))$, follows immediately. Pursuing the latter approach provides another perspective on the scope and limitations of the model and results.

First, at any type where the supports overlap, it must be the case that

$$\frac{d}{dv}\left(\frac{F_w(v)}{F_s(v)}\right) = \frac{d}{dv}\left(\frac{F_s(r(v))}{F_s(v)}\right) \propto \frac{f_s(r(v))}{F_s(r(v))}r'(v) - \frac{f_s(v)}{F_s(v)}. \tag{11}$$

Recall that first order stochastic dominance necessitates $r(v) \geq v$ and dispersion necessitates $r'(v) \geq 1$. Imagine now that the reverse hazard rate of F_s is increasing – or, equivalently, that F_s is log-convex – on an interval that includes $[v, r(v)]$. Then, (11) must necessarily be positive, thereby violating reverse hazard rate dominance. In other words, the conditions of Theorem 1 are less likely to be simultaneously satisfied if F_s is locally log-convex. Recall that Maskin and Riley (2000) assume F_s is not "too convex", or more precisely that it is log-concave (Examples 1 and 3). Of course, local log-convexity would not be a problem if the supports do not overlap (Example 5). Bagnoli and Bergstrom (2005) point out that most commonly used distributions are in fact log-concave.

However, reverse hazard rate dominance is not guaranteed even if F_s is log-concave. The reason is that if r is very steep, the product in the first term of (11) may still exceed the second term. This argument already reveals that r cannot be too step, which immediately raises the question of whether a more precise statement be made? An answer to this question will be pursued in the remainder of this Section.

Throughout, it will be assumed that F_s is convex, or f_s is increasing. Imposing the requirement that $r'(v) \geq 1$ (dispersion) then guarantees that the second condition in Theorem 1 is satisfied. Thus, the problem is purely to ensure that (11) is negative.

As mentioned, dispersion is equivalent to the condition that $\delta(v) \equiv r(v) - v$ is increasing. That is, the weak bidder faces an increasing gap between his own type (v) and type of the strong bidder with the same rank ($r(v)$), the larger his type is. In other words, there is a widening gap between the willingness-to-pay of bidders at comparable percentiles, the higher up the distribution one goes. A possible interpretation is that the auction becomes more asymmetric in an "absolute" sense the higher the stakes are. In the following, it is possible to obtain results by assuming that gap between $r(v)$ and v does not increase too rapidly. Specifically, it is assumed that the "relative" asymmetry is decreasing, in the sense that $\rho(v) \equiv r(v)/v$ is decreasing.[18] Note the similarities to Corollary 1, where another measure of asymmetry is also increasing in absolute terms but decreasing in relative terms.

Proposition 1 *The first-price auction generates strictly higher expected revenue than the second-price auction if*

(i) $\delta'(v) \geq 0$ *and* $\rho'(v) \leq 0$*, for all* $v \in S_w$*, and*

(ii) F_s *is convex and satisfies*

$$\frac{f_s(v)}{F_s(v)} v \text{ decreasing in } v. \tag{12}$$

Proof. By assumption, $\rho'(v) \propto r'(v)v - r(v) \leq 0$ or $r'(v)v \leq r(v)$. Then, from (11),

$$\begin{aligned} \frac{d}{dv}\left(\frac{F_s(r(v))}{F_s(v)}\right) &\propto \frac{f_s(r(v))}{F_s(r(v))} r'(v)v - \frac{f_s(v)}{F_s(v)} v \\ &\leq \frac{f_s(r(v))}{F_s(r(v))} r(v) - \frac{f_s(v)}{F_s(v)} v \leq 0, \end{aligned}$$

where the first inequality comes from $r'(v)v \leq r(v)$ and the second from (12) and $r(v) \geq v$. Thus, the first assumption in Theorem 1 is satisfied. Since F_s is convex and more dispersed than F_w, the second assumption is satisfied as well. ■

Proposition 1 can be strengthened in the special case where F_s is a convex power distribution, $F_s(v) = (v/\alpha_s)^{\gamma_s}$, $v \in [0, \alpha_s]$, with $\gamma_s \geq 1$. In this case, the second part of Proposition 1 is trivially satisfied since $f_s(v)v/F_s(v)$ is constant. The power distribution includes the uniform distribution as a special case, $\gamma_s = 1$. It is the most thoroughly examined specific distribution function in the theoretical literature. Plum (1992) derive closed form expressions for strategies when both bidders draw types from power distributions with the same exponent but stretched over different intervals. However, he does not compute revenue (see Remark 3). Maskin and Riley (2000) use the uniform distribution as a specific example of distribution stretches in the opening part of their paper. As mentioned, Cheng (2006) is able to derive strategies for other parameter constellations, while Lebrun (2002) studies the general case where $\alpha_s = \alpha_w$.

[18] Thus, the combination of assumptions is that F_w is smaller than F_s in the dispersive order ($\delta'(v) \geq 0$) and yet F_s is smaller than F_w in the star order ($\rho'(v) \leq 0$).

Corollary 2 *Assume F_s is a convex power distribution, $F_s(v) = (v/\alpha_s)^{\gamma_s}$, $v \in [0, \alpha_s]$, with $\gamma_s \geq 1$. Then, the conditions of Theorem 1 are satisfied if and only if $\delta'(v) \geq 0$ and $\rho'(v) \leq 0$ for all $v \in S_w$.*

Proof. Proposition 1 establishes the "if" part. For the other direction, note that (11) becomes

$$\frac{d}{dv}\left(\frac{F_s(r(v))}{F_s(v)}\right) \propto \frac{r'(v)}{r(v)} - \frac{1}{v} \propto r'(v)v - r(v).$$

Thus, the first part of Theorem 1 is violated if $\rho'(v) > 0$ for some v. If $\delta'(v) < 0$ for some v, then dispersion is violated. ■

If F_s is a convex power distribution, Corollary 1 means that the approach in this paper will allow a revenue ranking only if $r(v)$ satisfies the assumption in the first part of Proposition 1. Otherwise, other methods must be used to compare revenue, as in Cheng (2006) and Lebrun (2002).

Example 7 (power distribution): Assume that $F_s(v) = (v/\alpha_s)^{\gamma_s}$, $v \in [0, \alpha_s]$, with $\gamma_s \geq 1$. The second part of Proposition 1 is satisfied. Thus, if F_w can be written $F_s(r(v))$ for any function $r(v)$ for which $r(0) = 0$, $r'(v) \geq 1$, and $r(v)/v$ is decreasing (e.g., r is concave), then the first-price auction is more profitable than the second-price auction. For example, if $\alpha_s = 1$, $r(v) = 4ve^{-v}$ satisfies all the requirements, with $\alpha_w = 0.357$. In this case, $F_w(v) = (r(v))^{\gamma_s}$ has a non-monotonic density when $\gamma_s = 2$ (F_w is first convex, then concave). As another example, let $r(v) = \alpha_s (v/\alpha_w)^c$, where $c \in (0, 1]$ and $\alpha_w \in (0, \alpha_s)$. Clearly, $r(v)/v$ is decreasing and $r(\alpha_w) = \alpha_s$. Since $r(v)$ is concave, the requirement that $r'(v) \geq 1$ is satisfied if $r'(\alpha_w) \geq 1$, or $\alpha_s c/\alpha_w \geq 1$. Note that $F_w(v) = (r(v)/\alpha_s)^{\gamma_s} = (v/\alpha_w)^{\gamma_w}$, where $\gamma_w = c\gamma_s \in (0, \gamma_s]$ (note that F_w may be concave). In other words, with two power distributions the first-price auction generates strictly more revenue than the second-price auction whenever

$$\frac{\alpha_s}{\alpha_w}\frac{\gamma_w}{\gamma_s} \geq 1.$$

Example 3 implies the same revenue ranking only for the special case where $\gamma_s = \gamma_w$ (or $c = 1$). Lebrun (2002) proves the ranking also holds if $\alpha_w = \alpha_s = 1$ and $2\gamma_s\gamma_w \geq 1$, while Cheng (2006) proves that it holds if

$$\left(\frac{\alpha_s}{\alpha_w}\frac{\gamma_w}{\gamma_s}\right)\frac{1 + \gamma_s}{1 + \gamma_w} = 1.$$

See Section 3.1.4 for further discussion of Lebrun and Cheng's results. ▲

The first condition in Proposition 1 requires either that $\beta_w = \beta_s = 0$ or $\beta_s > \beta_w \geq 0$. Otherwise, that is if $r(\beta_w) = \beta_w > 0$, it must be the case that $\rho(\beta_w) = 1$, and ρ can then not be decreasing without violating dispersion. An example with $\beta_w = \beta_s = 0$ has already been provided. The case with $\beta_s > \beta_w \geq 0$ is discussed in more detail in the following.

In order for (12) to be satisfied it is necessary that F_s is log-concave (the reverse hazard rate must be decreasing). The condition is equivalent to assuming that the ratio of "marginal probability", $f_s(v)$, to "average probability", $F_s(v)/v$, is decreasing. If f_s is differentiable, note that

$$\begin{aligned} \frac{d}{dv}\left(\frac{f_s(v)}{F_s(v)}v\right) &= v\frac{d}{dv}\left(\frac{f_s(v)}{F_s(v)}\right) + \frac{f_s(v)}{F_s(v)} \\ &= \frac{f_s(v)}{F_s(v)}\left[v\frac{f_s'(v)F_s(v) - (f_s(v))^2}{F_s(v)f_s(v)} + 1\right]. \end{aligned} \tag{13}$$

The first term in the bracket is strictly negative under the assumption that F_s is strictly log-concave, or $f_s'(v)F_s(v) - (f_s(v))^2 < 0$. Now imagine shifting F_s to the right, such that F_s is replaced by $G_s(v) = F_s(v-a)$ where $a > 0$ and $v \in [\beta_s + a, \alpha_s + a]$. For any $v \in [\beta_s, \alpha_s]$,

$$\frac{d}{dv}\left(\frac{g_s(v+a)}{G_s(v+a)}(v+a)\right) = \frac{f_s(v)}{F_s(v)}\left[(v+a)\,\frac{f_s'(v)F_s(v) - (f_s(v))^2}{F_s(v)f_s(v)} + 1\right].$$

Naturally, as a increases, the first term in brackets must eventually dominate the second term, and so (12) is satisfied. In other words, if F_s is convex yet log-concave, the second condition in Proposition 1 is satisfied as long as the strong bidder's support is located far enough to the right, i.e., when the stakes are known to be very high.

In summary, when the stakes are high for the strong bidder, Proposition 1 says that the first-price auction is superior when (i) F_s is more dispersed than F_w, but not too much more dispersed, and (ii) F_s is convex, but not too convex (it is log-concave). Indeed, if the stakes are high for the strong bidder *and* the difference in size is large (Example 6), F_s does not need to be convex.

Example 8 (shifting and re-scaling distributions): Let $r(v) = \gamma v + a$, where $\gamma \geq 1$ and $a \geq 0$ with at least one strict inequality. Clearly, $r(v)$ satisfies the first part of Proposition 1. The horizontal shift in Example 1 corresponds to $\gamma = 1$, $a > 0$. When $\gamma > 1$, F_w is a "scaled down" version of F_s. That is, F_w has the same shape as F_s, but it is more compressed to fit onto its narrower support. If $\gamma > 1$ and $a > 0$ then F_w is a rescaled and shifted version of F_s. The second part of Proposition

1 is satisfied if the stakes are high enough for the strong bidder and F_s is convex and log-concave. ▲

The conditions in Proposition 1 are sufficient but obviously not necessary. This point is illustrated in the next example, for which it is also the case that $\beta_w = \beta_s$.

EXAMPLE 3 (CONTINUED): It has already been established that the first-price auction dominates the second-price auction in the case of truncations. Note that F_w can be written $F_w(v) = \lambda F_s(v)$ or $F_w(v) = F_s(r(v))$. Thus, $F_s(r(v)) = \lambda F_s(v)$ and so

$$\frac{r'(v)v}{r(v)} = \left(\lambda \frac{f_s(v)}{f_s(r(v))}\right) \frac{v}{r(v)} = \frac{f_s(v)v}{F_s(v)} \frac{F_s(r(v))}{f_s(r(v))r(v)}$$

exceeds one if (12) is satisfied. In other words, F_w is smaller than F_s in the star order, or $\rho'(v) \geq 0$. See footnote 14 for an implication. ▲

5 Extensions and implications

Theorem 1 can be extended to auctions with reserve prices and to certain other auction formats. These extensions are presented below. Implications of Theorem 1 for contest architecture are also discussed. Extensions to auctions with more bidders are postponed until Section 6.

5.1 Reserve prices

It has been assumed the good is sold with probability one. However, reserve prices are often employed in practice. As Mares and Swinkels (2010a, footnote 33) recently note, the "degree to which Maskin and Riley's ranking depends on the absence of a reserve price is open". The method developed in Section 3 can be used to close the question. In the following, let τ denote the reserve price.

To begin, assume that $\beta_s = \beta_w$. Then, the strength of the proof of Theorem 1 is that it establishes the superiority of the allocation in the first-price auction for all values of the weak bidder's type. The only thing a reserve price does is to "shut out" some types, but this effect is the same in both types of auctions.[19] For those types

[19]Bidders with type exceeding the reserve price will also change their bidding strategy in the first-price auction. Thus, $k_1(v)$ is a function of τ, although that dependence will be suppressed here. The important point is that the reserve price does not change the property that $k_1(v) \in [v, r(v)]$ for all $v \in [\tau, \alpha_w]$. See Lebrun (1999). In a first-price auction it is possible that the strong bidder

that are not shut out, it remains the case that the first-price auction dominates type for type. Thus, the revenue ranking is intact.[20]

More generally, assume that $\beta_w \leq \beta_s < \alpha_w$, such that the supports overlap. The first-price auction is then strictly more profitable than the second price auction for a fixed reserve price as long as the reserve price has "bite" in the sense that $\tau \in [\beta_s, \alpha_w)$. The two auctions are revenue equivalent if $\tau \geq \alpha_w$ because in that case the winner is the same in the two auctions and $u_s^1(\beta_s) = u_s^2(\beta_s) = 0$.

Proposition 2 *Assume that (i) F_s dominates F_w in terms of the reverse hazard rate, (ii) for all $v \in [\tau, \alpha_w]$, $f_w(v) \geq f_s(x)$ for all $x \in [v, r(v)]$, and (iii) $\tau \in [\beta_s, \alpha_w)$. Then, the first-price auction with a reserve price of τ generates strictly higher expected revenue than the second-price auction with the same reserve price.*

Proof. Because $\tau \geq \beta_s \geq \beta_w$, $u_i^k(\beta_i) = 0$ for $i = s, w$ and $k = 1, 2$. A bidder stays out of the auction if and only if his type is below τ. By modifying (3), expected revenue in the mechanism is then

$$\begin{aligned} ER^k(\tau) &= \int_{\beta_w}^{\tau} \left(J_w(v) \times 0 + \int_{\tau}^{\alpha_s} J_s(s) dF_s(x) \right) dF_w(v) \\ &+ \int_{\tau}^{\alpha_w} \left(J_w(v) F_s(k(v)) + \int_{k(v)}^{\alpha_s} J_s(s) dF_s(x) \right) dF_w(v). \end{aligned}$$

Hence,

$$ER^1(\tau) - ER^2(\tau) = \int_{\tau}^{\alpha_w} D(v|k_1, k_2) dF_w(v),$$

which, as shown in the proof of Theorem 1, is positive since $\tau \geq \beta_s$. ■

5.2 Other auction formats

The conclusion in Theorem 1 is made possible because the allocation in the second-price auction can be described precisely, while the possible allocations in the first-price auction can be narrowed down to a relatively small set. It is not necessary to know the exact allocation in the first-price auction.

(but not the weak bidder) bids τ for a mass of types. In this case, $k_1(\tau)$ denotes the highest type that bids τ, such that types in $[\tau, k_1(\tau)]$ pool their bids at τ. If the weak bidder bids higher than τ he will therefore win with a probability no smaller than $F_s(k_1(\tau))$.

[20] If the reserve price exceeds α_w then all the weak bidder's types are excluded. In this case, the two auctions are revenue equivalent (i.e., the revenue ranking is not strict).

Aside from the issue of how much rent is extracted from β_i types, Theorem 1 therefore really says that the second-price auction is a poor auction format if the objective is to generate high expected revenue.[21] For instance, if $\beta_w = \beta_s$ and $u_w(\beta_w) = u_s(\beta_s) = 0$, any auction with $k(v) \in [v, r(v)]$ is more profitable than the second-price auction if (1) is satisfied. In other words, it is profitable to design an auction that favours the weak bidder moderately. One advantage of mechanism design is that such insights are made possible.

To make the same point differently, Theorem 1 should not necessarily be interpreted as saying that the first-price auction is a particularly good mechanism, just that it is better than the second-price auction. Consider some other mechanism which also has the property that $k(v) \in [v, r(v)]$, and assume that $k(v)$ and $k_1(v)$ cross once in the interior. It is not possible to rank the first-price auction and the new mechanism. The reason is that in Figure 2a, $k(v)$ will be closer than $k_1(v)$ to $\kappa(v)$ for some v, but not for others. Consequently, one auction does not dominate the other for all v, and the method of proof in Theorem 1 does therefore not apply.

Define an auction as a winner-pay auction if the winner pays a proportion γ of his own bid and $(1-\gamma)$ of the losing bid, and the loser does not pay, $\gamma \in [0,1]$. The first-price auction corresponds to $\gamma = 1$, the second-price auction to $\gamma = 0$.

Proposition 3 *Assume that (i) F_s dominates F_w in terms of the reverse hazard rate, (ii) for all $v \in S_w$, $f_w(v) \geq f_s(x)$ for all $x \in [v, r(v)]$, and (iii) $\beta_s = \beta_w$. Then, the second-price auction yields strictly the lowest expected revenue of all winner-pay auctions.*

Proof. Consider $\gamma \in (0,1]$, i.e. an auction that is not a pure second-price auction. In this case, the two bidders must share the same maximal bid, $\overline{b}$. Let $\phi_i(b)$ denote bidder i's inverse bidding strategy, $i = s, w$, where $b \in [\beta_w, \overline{b}]$. Assume for the moment the bidding strategy is strictly increasing and differentiable. If bidder i has type v, his problem is

$$\max_b \int_{\beta_w}^{b} [v - (\gamma b + (1-\gamma)\, x)]\, dF_j(\phi_j(x)),$$

where $j \neq i$ denotes bidder i's rival. The first order condition is

$$\frac{f_j(\phi_j(b))}{F_j(\phi_j(b))}\phi_j'(b)) = \frac{\gamma}{v-b}.$$

[21] Of course, this conclusion is based on the assumption of independent private values. In a more general model, but under the assumption of bidder symmetry, Milgrom and Weber (1982) show that the second-price auction is more profitable than the first-price auction.

In equilibrium, bidder i bids b if his type is $v = \phi_i(b)$. Substituting into the first order conditions produces the system of differential equations

$$\frac{f_w(\phi_w(b))}{F_w(\phi_w(b))}\phi_w'(b) = \frac{\gamma}{\phi_s(b) - b}, \quad \frac{f_s(\phi_s(b))}{F_s(\phi_s(b))}\phi_s'(b) = \frac{\gamma}{\phi_w(b) - b}.$$

The only difference from the first-price auction is that $\gamma \in (0, 1]$ (the boundary conditions are the same). The proofs in Maskin and Riley (2000) can then be repeated to conclude that the auction has the same features as a first-price auction, $k_\gamma(v) \in [v, r(v)]$ for all $\gamma \in (0, 1]$. Since bidders with type β_i earns zero rent for all $\gamma \in [0, 1]$, Theorem 1 applies directly. ■

Not all auctions have the property that $k(v) \in [v, r(v)]$. The most prominent example is probably the all-pay auction for which $k(v) < v$ when v is small. The reason is that a weak bidder with a low type is deterred from bidding (which is a sunk cost in an all-pay auction) when facing a rival he perceives as strong. Thus, it is not possible to rank the second-price auction and the all-pay auction using the method developed in this paper.

5.3 An order statistics result

Theorem 1 can be used to derive a new and interesting result in order statistics, the implications of which are discussed momentarily. Let $E[M_{i,j}]$ denote the expected value of the second highest type from one draw from F_i and one draw from F_j, where $i, j \in \{s, w\}$ may or may not be identical. Note that $E[M_{i,j}]$ equals the expected revenue in a second-price auction with bidders i and j.

Proposition 4 *Assume $f_w(v) \geq f_s(x)$ for all $x \in [v, r(v)]$ and all $v \in S_w$. Then,*

$$\frac{1}{2}E[M_{s,s}] + \frac{1}{2}E[M_{w,w}] \geq E[M_{s,w}].^{22} \tag{14}$$

Proof. Consider an auction with a strong and a weak bidder. Define a *rank-symmetric* mechanism as a mechanism in which $k(v) = r(v)$ and $u_i^k(\beta_i) = 0$, $i = s, w$. In such a mechanism, bidder i with type v wins with probability $F_i(v)$. From Theorem 1, the expected revenue in a rank-symmetric mechanism is strictly greater than expected revenue in a second-price auctions, where it equals $E[M_{s,w}]$ (the right hand side of (14)). To see this, let $b_* = \beta_w$ in Theorem 1 and replace $k_1(v)$ with

[22] The assumption that $\beta_s \geq \beta_w$ and $f_w(v) \geq f_s(v)$ implies that $F_w(v) \geq F_s(v)$. The inequality in (14) is strong as long as the two distributions are not identical (i.e., whenever F_s first order stochastically dominates F_w).

$k(v) = r(v)$. Note that $b_* = \beta_w$ implies $u_s^k(\beta_s) = 0$ because the strong bidder with type β_s wins with probability zero.

In the rank-symmetric mechanism, bidder s wins with probability $F_s(v)$, just as he would in a second-price auction facing another strong bidder. In such a balanced second-price auction, the expected payment from each bidder would be $\frac{1}{2}E[M_{s,s}]$. In both mechanisms, the bidder earns zero rent if his type is β_s. By the Revenue Equivalence Theorem, the expected payment from bidder s in the rank-symmetric mechanism is therefore exactly $\frac{1}{2}E[M_{s,s}]$. The same argument proves that bidder w's expected payment in the rank-symmetric mechanism is precisely $\frac{1}{2}E[M_{w,w}]$. Thus, the expected revenue of the proposed mechanism is $\frac{1}{2}E[M_{s,s}] + \frac{1}{2}E[M_{w,w}]$ (the left hand side of (14)). ■

Consider a seller who has some limited control over the composition of a two-bidder auction. Proposition 4 implies that he would be better off flipping a coin between two symmetric second-price auctions – one with two strong bidders, the other with two weak bidders – than to settle for an asymmetric second-price auction with one weak and one strong bidder. This result can be seen as another manifestation of the fact that asymmetric second-price auctions are relatively unprofitable. It is better to gamble on symmetric auctions, even at the risk of ending up with one consisting of two weak bidders. By the Revenue Equivalence Theorem, this result holds for any efficient mechanism, but it may not hold for other auction formats. Cantillon (2008) also argues that bidder asymmetry is unprofitable, but her alternative symmetric auction is different.

Proposition 4 is directly relevant for "contest architecture". Moldovanu and Sela (2006) show that with symmetric bidders it is more profitable to stage one grand auction than to stage smaller, same-sized, auctions in which a fraction of the total prize is up for grabs in each. Order statistics play a dominant role in the analysis in that paper and a related paper on contests for status, Moldovanu et al (2007). The symmetry assumption makes it easier to apply known order statistics results.

Proposition 4 can be seen to complement Moldovanu and Sela (2006). Suppose the contest designer is forced to stage smaller auctions (a grand auction may be unmanageable). Then, Proposition 4 means that it is better to stage two symmetric auctions, one with two weak bidders, the other with two strong bidders, than to stage two asymmetric (but efficient) auctions.

6 Larger auctions

It is straightforward to extend the revenue ranking to allow for the presence of more weak bidders. However, allowing more strong bidders is considerably more difficult. As explained below, extending the ranking to this case is possible if the asymmetry between bidders is "large enough". When the asymmetry is small, neither the mechanism design approach nor Maskin and Riley's (2000) approach is powerful enough to analytically produce a ranking.

Let $m \geq 1$ and $n \geq 1$ denote the number of strong and weak bidders, respectively. With symmetric and monotonic strategies within each group, the winner of the auction is then either the strong bidder with the highest type or the weak bidder with the highest type. In other words, it is straightforward to modify (3) to

$$ER^k = \int_{\beta_w}^{\alpha_w} \left(J_w(v) F_s(k(v))^m + \int_{k(v)}^{\alpha_s} J_s(s) dF_s(x)^m \right) dF_w(v)^n - n u_w^k(\beta_w) - m u_s^k(\beta_s),$$

where the term in parenthesis is expected revenue conditional on the *highest* type among the n weak players being equal to v. The counterpart to (7) is

$$D_m(v|k_1, k_2) = \int_{k_2(v)}^{k_1(v)} \left(J_w(v) - J_s(x) \right) dF_s(x)^m.$$

The method in the first part of Maskin and Riley (2000) or in Kirkegaard (2009) can be used to prove that $k_1(v) \in [v, r(v)]$ holds in larger first-price auctions as well. The proof is omitted.

However, if there are several strong bidders who are much stronger than the weak bidder(s), the former may compete so hard among themselves that the latter would be content to submit relatively low bids, even with types close to α_w. Let $\overline{b}_i$ denote the bid submitted by a bidder of strength i, if his type is α_i, $i = s, w$. When $m = 1$, all bidders share the same maximal bid, $\overline{b}_w = \overline{b}_s$ or $k_1(\alpha_w) = \alpha_s$, but when $m > 1$ it is possible that $\overline{b}_w < \overline{b}_s$ or $k_1(\alpha_w) < \alpha_s$ (see Lebrun (2006)). Another difference is that when $m > 1$, a strong bidder with type β_s earns zero rent because he is certain to be outbid by another strong bidder with a higher type.

6.1 One strong bidder, more weak bidders

Assume that $n \geq m = 1$. As before, $u_w^k(\beta_w) = 0$ in both the first-price and second-price auction. For the strong bidder, (6) is as before, but with $F_w(v)^n$ in place of $F_w(v)$. Hence, $ER^2 - ER^1$ takes the exact same form as (9), with $F_w(v)^n$ in place of $F_w(v)$. Thus, the proof of Theorem 1 applies to the situation with $n > 1$ weak bidders as well.

Proposition 5 *Assume that (i) F_s dominates F_w in terms of the reverse hazard rate, and (ii) for all $v \in [\beta_w, \alpha_w]$, $f_w(v) \geq f_s(x)$ for all $x \in [v, r(v)]$. Then, the first-*

price auction generates strictly higher expected revenue than the second-price auction when $n \geq m = 1$.

6.2 More strong bidders

It is immediately clear that the *strict* revenue ranking in Theorem 1 does not generalize to auctions with $m \geq 2$ strong bidders. To see this, consider the case of no overlap (Example 2). In both auctions, competition between bidders ensure that the winning bid must be at least β_s. Thus, the winner is the strong bidder with the highest type in both auctions. Moreover, $u_i^k(\beta_i) = 0$ in both auctions, $i = s, w$. Hence, the first-price auction and the second-price auction are revenue equivalent when $\beta_s \geq \alpha_w$.

Assume for the remainder of the section that $\beta_s < \alpha_w$. Assume, for now, that the asymmetry is so small that $\bar{b}_w = \bar{b}_s$, or $k_1(\alpha_w) = \alpha_s$. Since $J_w(\alpha_w) = \alpha_w$, evaluating D_m at $v = \alpha_w$ then yields

$$\int_{\alpha_w}^{\alpha_s} \left(J_w(\alpha_w) - J_s(x)\right) dF_s(x)^m = \alpha_w \left(1 - F_s(\alpha_w)^m\right) - \int_{\alpha_w}^{\alpha_s} J_s(x) dF_s(x)^m.$$

The last term on the right is equal to the expected value of an auction among m strong bidders with a reserve price of α_w. Clearly, such an auction would yield revenue in excess of α_w if it results in a sale, which occurs with probability $1 - F_s(\alpha_w)^m$. Hence, $D_m(\alpha_w|k_1, k_2) < 0$ when $m \geq 2$ (when $m = 1$, the auction with reserve price α_w generates revenue of exactly α_w if the object is sold, implying that $D_1(\alpha_w|k_1, k_2) = 0$). It follows that the approach in this paper and in Maskin and Riley (2000), which is based on showing the superiority of the first-price auction for *all* v, will not work in general. Thus, the complication identified by Lebrun (2002) for the case with $\alpha_w = \alpha_s$ and $n = m = 1$ is endemic to auctions with many strong bidders, even if $\alpha_w < \alpha_s$.

In the preceding example, the root cause of the problem is in a sense that the bidders are not sufficiently asymmetric. Specifically, it was the assumption that all bidders share the same maximal bid that led to the negative conclusion.

In the following, I will demonstrate that a revenue ranking can sometimes be obtained for auctions in which the asymmetry is "large", such that $\bar{b}_s > \bar{b}_w$. Kirkegaard (2010b) presents an example in which the size of the asymmetry needed to generate $\bar{b}_s > \bar{b}_w$ can be quantified.

6.2.1 Small overlap

Assume that α_w is "close" to β_s such that there is little overlap between the supports. As a starting point, if $\beta_s = \alpha_w$ it would clearly be the case that (1) $J_w(\alpha_w) = \alpha_w = \beta_s > J_s(\beta_s)$ and (2) $\bar{b}_s > \bar{b}_w$ in a first-price auction (a strong bidder with type β_s bids $\beta_s = \alpha_w \geq \bar{b}_w$, and his strategy is strictly increasing). If α_w is "slightly above" β_s, is must remain the case that $\bar{b}_s > \bar{b}_w$, or $k_1(\alpha_w) < \alpha_s$, with $J_w(\alpha_w) > J_s(x)$ for all $x \in [\beta_s, k_1(\alpha_w)]$. Moreover, by continuity,

$$J_w(v) > J_s(x) \text{ for all } v \in [\beta_s, \alpha_w] \text{ and } x \in [\beta_s, k_1(\alpha_w)]\,, \tag{15}$$

when β_s and α_w are sufficiently close.[23] In the following, when the overlap is said to be "small", it should be taken to mean that (15) is satisfied.

In this case, the first-price auction yields higher expected revenue than the second-price auction because the weak bidders are winning more often against strong bidders with inferior marginal revenue. Recall that the two are revenue equivalent if there is no overlap.

Proposition 6 *Assume that (i) F_s dominates F_w in terms of the reverse hazard rate and (ii) the overlap is small. Then, the first-price auction generates strictly higher expected revenue than the second-price auction when $m \geq 2$, $n \geq 1$.*

Proof. Both auctions ensure that $u_i^k(\beta_i) = 0$, $i = s, w$. A weak bidder with type below β_s loses both auctions (competition between the strong bidders ensures that any serious bid must be at least β_s). By (i), a weak bidder with type $v \in (\beta_s, \alpha_w]$ wins more often in the first-price auction than in the second-price auction. By (ii) or (15), the winner's marginal revenue is no lower in the first-price auction, and may be higher. In other words, D_m is positive. This concludes the proof. ■

6.2.2 Large stretches

Assume that the asymmetry between bidders is so large that $\bar{b}_s > \bar{b}_w$. Define $\overline{\alpha}_s \equiv k_1(\alpha_w)$ as the highest strong type that competes with the weak bidders. A strong bidder outbids the weak bidders with probability one if his type exceeds $\overline{\alpha}_s$. By assumption, $\overline{\alpha}_s < \alpha_s$.

Consider the consequences of "stretching" the strong bidder's distribution, transforming F_s with support $[\beta_s, \alpha_s]$ to F_s^λ with support $\left[\beta_s, \alpha_s^\lambda\right]$, $\alpha_s^\lambda > \alpha_s$, such that

[23] F_s need not be more disperse then F_w. For instance, the former could have a smaller support than the latter. It is a general property that $J_i(\alpha_i) = \alpha_i$ and $J_i(\beta_i) < \beta_i$, $i = s, w$.

$F_s^\lambda = \lambda F_s$ on the subinterval $v \in [\beta_s, \alpha_s]$, with $\lambda \in (0,1)$. More concisely, F_s is a truncation of F_s^λ. Importantly, F_s and F_s^λ have the same reverse hazard rate on $[\beta_s, \alpha_s]$ and therefore on $[\beta_s, \overline{\alpha}_s]$. Thus, if F_s dominates F_w in terms of the reverse hazard rate, so does F_s^λ. Likewise, the system of first order conditions from the bidders' maximization problems is unchanged at bids below $\overline{b}_w$.[24] The implication is that weak bidders regardless of type and strong bidders with type below $\overline{\alpha}_s$ use the exact same strategy in either case. Consequently, k_1 is the same in both environments.

For types in $[\beta_s, \overline{\alpha}_s]$, the strong bidders' marginal revenue is

$$J_s^\lambda(v) = v - \frac{1 - F_s^\lambda(v)}{f_s^\lambda(v)} = v - \frac{\frac{1}{\lambda} - F_s(v)}{f_s(v)}$$

as a function of λ. The important property is that J_s^λ decreases without bound as F_s is stretched more and more (that is, as λ decreases and goes to zero). Thus,

$$J_w(v) \geq J_s^\lambda(x) \text{ for all } v \in [\beta_w, \alpha_w] \text{ and } x \in [\beta_s, \overline{\alpha}_s] \tag{16}$$

when F_s is stretched sufficiently much. In the following, when F_s is said to be stretched "a lot", it should be taken to mean that (16) is satisfied.

For reasons similar to those in Proposition 6, the first-price auction is more profitable than the second-price auction.

Proposition 7 *Assume that (i) F_s dominates F_w in terms of the reverse hazard rate and (ii) F_s is stretched a lot. Then, the first-price auction generates strictly higher expected revenue than the second-price auction when $m \geq 2$, $n \geq 1$.*

Proof. The proof is identical to the proof of Proposition 6. ■

6.2.3 Examples

Compare Theorem 1 and Proposition 5 ($m = 1$) to Propositions 6 and 7 ($m \geq 2$). Examples 5 and 6 demonstrate that large asymmetries allow the conclusion that the first-price auction is superior in the former case. Propositions 6 and 7 suggest a similar conclusion, although the precise definition of "large asymmetries" differ in all three results.

Maskin and Riley's (2000) models can be used to illustrate Propositions 6 and 7.

[24] This can be seen by examining the systems in Maskin and Riley (2000) or Lebrun (2006). Kirkegaard (2010b) discusses this point more fully. See also footnote 12.

Example 1, continued: Consider a many-bidder extension of Example 1, with $m \geq 2$. If F_s is shifted far to the right such that there is no overlap between supports $(a \geq \alpha_w - \beta_w)$, then the two auctions are revenue equivalent. The same is true if $a = 0$, in which case bidders are homogenous. Proposition 2 then states that the first-price auction is superior for large "interior" values of a. A comparison cannot be made for small values. Recall that Proposition 6 does not require F_s to be a "shifted" version of F_w.

Example 3, continued: Proposition 3 applies directly if F_w is a truncation of F_s, in which case $F_s(v) = \lambda F_w(v)$ on $v \in [\beta_w, \alpha_w]$. As with Example 1, the two auctions are revenue equivalent if the bidders are homogenous, or $\lambda = 1$. A comparison cannot be made if λ is close to one, or the asymmetry is small. By Proposition 3, however, the first-price auction is superior when λ is close to zero. Note that Proposition 7 does not require F_s and F_w to be related in any way other than through reverse hazard rate dominance (it does not imply one is a truncation of the other), nor does it require log-concavity.

6.3 More kinds of bidders

Any bidder was assumed to be either strong or weak, ex ante. When more kinds of bidders are added to the auction, the problems discussed in the beginning of this Section multiply. Thus, in general, it becomes harder to rank the auctions. The following (perhaps degenerate) example illustrates that a revenue ranking is sometimes obtainable.

Assume there are three bidders; a weak, a strong, and an extremely strong bidder (indexed by e). Assume that $\alpha_w < \beta_s \leq \beta_e$. Then, the weak bidder earns zero payoff and never wins, regardless of whether the auction is a first-price or second-price auction. In other words, the auction reduces to a competition between the two stronger bidders. Assuming F_s and F_e satisfies the assumptions in Section 2, the first-price auction is then more profitable than the second-price auction.

7 Conclusion

In this paper, mechanism design methods are used to simplify the analysis and ranking of asymmetric auctions. The first-price auction is shown to be more profitable than the second-price auction under more general conditions than those proposed by Maskin and Riley (2000). Moreover, these conditions have an intuitive economic interpretation. Roughly speaking, the first-price auction is more profitable than the second-price auction if the weak bidder is more "price sensitive" than the strong bidder. To obtain this ranking, it is assumed that the strong bidder's distribution is more disperse than the weak bidder's distribution. If the former distribution is convex and dominates the latter in terms of the reverse hazard rate, then dispersion is sufficient. The central role played by the dispersive order complements recent findings by Mares and Swinkels (2010a, 2010b) in other asymmetric auction settings. Thus, the dispersive order may prove to be as useful for auction design as the usual stochastic orders of strength, such as reverse hazard rate dominance.

The use of mechanism design techniques has the advantage that extensions and generalizations are made easier. For example, the revenue ranking is robust to the inclusion of reserve prices. The underlying reason for the revenue ranking is that the second-price auction is not a particularly good auction for generating high revenue. Mechanisms that moderately favour the weak bidder, among them the first-price auction, are more profitable. Thus, it is possible to identify a whole class of auctions that dominates the second-price auction. Likewise, generalizations to certain larger auctions are possible. Specifically, the revenue ranking can painlessly be extended to the case with many weak bidders, but only one strong bidder. When there are many strong bidders, the same revenue ranking holds if the asymmetry between bidders is "large enough".

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