

# Auctions with Resale and Bargaining Power

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## Abstract

We establish the bid-equivalence between an independent private-value (IPV) first-price auction model with resale and a model of first-price common-value auction, when the resale market satisfies a weak efficiency property and the common value is defined by the transaction price. The buyer-speculators model of auction with resale is bid-equivalent to the Wilson tract model. With an application of the Coase Theorem, we show two polar cases in which auctions with resale have opposite properties. We examine the effects of bargaining power on the revenue and efficiency of first-price auctions with resale. This is done for three types of bargaining models: (a) bargaining with commitment, (b) bargaining with delay costs, and (c) k-double auctions resale market.

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# 1 Introduction

In this paper we study the way resale opportunities after the auction may affect bidders's behavior in the auction. When resale is allowed after the auction, we refer to it as an auction with resale, and represent it by a two-stage game. It is intuitively understood in the profession that resale is an important source of common-value among the bidders. In the survey for their book, Kagel and Levin (2002, page 2) said that "There is a common-value element to most auctions. Bidders for an oil painting may purchase for their own pleasure, a private-value element, but they may also bid for investment and eventual resale, reflecting the common-value element". Haile (2001)<sup>1</sup> studied the empirical evidence of the effects of resale in the U.S. forest timber auctions. In spectrum auctions held by many governments, there are often restrictions on resale. For example, in the British 3-G spectrum auctions<sup>2</sup> of 2000, resale restrictions were imposed despite economists' recommendation to the contrary. It is not clear why the restrictions were imposed. It is possible that the government may look bad when the bidders can turn around and resell for quick profits after the auction<sup>3</sup>. Bidders, however, find ways to circumvent such restrictions in the form of a change of ownership control. For example, a month after the British 3-G auction, Orange, the winner of the license E, was acquired by France Telecom, yielding a profit of 2 billion pounds to Vodafone<sup>4</sup>. The winner of the most valuable licence A is TIW (Telesystem International Wireless). In July 2000, Hutchison then sold 35% of its share in TIW to KPN and NTT DoCoMo with an estimated profit of 1.6 billion pounds<sup>5</sup>.

Although resale is sometimes conducted so that parties who did not or could not participate in the auction has a chance to acquire the object sold during the

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<sup>1</sup>His model of resale is different from our specifications here. In his model, there is no asymmetry among bidders before auctions, and trade occurs after the auction because of information differences after the auction. In our model, bidders are asymmetric before auctions. Haile, Hong, and Shum (2003) studied the U.S. forest lumber auctions and found the bidding data to conform to private-value auctions in some and common-value auctions in others. An explanation may be due to the presence or lack of resale.

<sup>2</sup>The third-generation technology allows high speed data access to the internet. It was held on Mar 6, 2000, and concluded on April 27, 2000, raising 22.5 billion pounds (2.5% of the GNP of UK). This revenue is seven times the original estimate. Five licenses A,B,C,D,E were offered. Licence A was available for bidding by non-incumbent operators only. A more detailed account is given in Klemperer (2004).

<sup>3</sup>Beyond the political and legal reasons, resale may facilitate collusions in the English auction as is shown in Garrat, Troger and Zheng (2007).

<sup>4</sup>Orange paid 4 billion pounds for the licence. In May of 2000, France telecom paid 6 billion pounds more than the price Mannesmann had paid for it in October 1999 before the auction. Orange was the number three UK mobile group at the time. The reason for the resale is due to a divestment agreement by Vodafone with the British government after acquiring Mannesmann.

<sup>5</sup>TIW was a Canadian company based in Montreal and largely owned by Hutchison Whampoa. The Hong Kong conglomerate) gained the upper hand when NTL Mobile, a joint venture of the UK cable operator and France Telecom, withdrew from the bidding. The price of the licence is 4.4 billion pounds. The profit is based on the implicit valuation of the license at 6 billion in the transaction.

auction. We will focus on resale among bidders of the original auction. There are situations in which resale opportunities to a third party only affect a bidder's valuation of the object, and thus can be indirectly represented by a change in valuations. In this case, our model may allow third party participation. However third party participation may give rise to issues that are not explored here. The main idea of the paper is that it is the resale price, not the private valuation, which determines the bidder behavior, and we expect this idea to apply to a more general model with broader participation.

By focusing on the resale among bidders, we study an interesting interaction between resale in the second stage and bidding behavior in the first stage. Hortascu and Kastl (2008) showed that bidding data for 3-month treasury bills are more like private-value auctions, but not so for 12-month treasury bills. The difference may be due to the relevance of resale in long term treasury bills. When an asset is held in a longer period, there may be a need for resale, and this may affect bidding behavior. We will in fact show that in private-value auctions with resale, the bidding data will behave as if it is a common-value auction. This has been observed In Gupta and Lebrun (1999), and Lebrun (2007) in cases when the resale market is a simple monopoly or monopsony market. We will provide a theoretical examination for this intuition in more general resale environments.

We describe the phenomenon by the term "bid-equivalence". It means that the bidding behavior of the an independent private-value auction with resale is the same as a pure common-value auction in which the common-value is defined by the transaction price. The two auctions have the same equilibrium bid distributions. The auctioneer has no way of knowing the difference between the two from the bidding behavior in the auctions, nor can an econometrician from the bidding data. The concept of bid-equivalence is similar to the observational equivalence used in Green and Laffont (1987). We prefer to use a different term because the observational equivalence concept is often associated with the identification problem in econometrics. Here we want to focus on the theoretical implications. This concept is different from equilibrium equivalence as the auction with resale is a two-stage game, while the common-value auction is a one-stage game. Furthermore, the equilibrium payoffs of the two auctions for the bidders are in general different. Laffont and Vuong (1996) showed that for any fixed number of bidders in a first-price auction, any symmetric affiliated values model is observationally equivalent to some symmetric affiliated private-values model. In a symmetric model, there is no incentive for resale. In our paper, we look at the asymmetric IPV auctions with resale. We show that when bidders anticipate trading activities after the auction, the bidding data is observationally equivalent to a common-value auction.

We assume that there are only two bidders in the auction, except in the case of buyer-speculators model. This is partly due to the substantial complexity of auctions with resale when there are more than two bidders, a model which is still not well-developed in the literature. This assumption is however justified

here as we are looking at the issue of bargaining power effects in auctions with resale, and the single seller and buyer framework in the resale game gives us a clear setting to address this issue. We adopt an general description of the resale stage game. The framework is flexible enough to include a resale process in which bidders make sequential offers, or simultaneous offers in the bargaining. The main assumption is a weak efficiency property which says that trade should be efficient (occur with probability one) when the trade surplus is the highest possible. It is easily satisfied in most Bayesian bargaining equilibrium with sequential or simultaneous offers. It rules out the no-trade equilibrium in which there cannot be bid-equivalence. Our formulation of the weak efficiency property is equivalent to the sure-trade property in Hafalir and Krishna (2008)<sup>6</sup>.

The bid-equivalence result may be somewhat surprising. One would expect that resale only contributes a common-value "component" to the bidding behavior, and there is still a private-value component. Our result however says that the bidding behavior is the same as if it is a pure common-value model. What happens to private-value component? The answer is that bid-equivalence is true only in equilibrium, hence the private-value is still relevant out-of-equilibrium. Furthermore, the private-value is incorporated in the definition of the common-value, and is therefore indirectly affecting the equilibrium bidding behavior. In equilibrium, the payoffs of bidders in the auction with resale is different from the payoffs in the common-value auction, even though the bidding behavior is the same. Therefore, it is not correct to say that we have equilibrium equivalence.

Auction with resale in general can be a very complicated game. The resale game may involve potentially complicated sequences of offers, rejections, and counter-offers. In-between the auction stage and the resale stage, there may be many possible bid revelation rules that affect the beliefs of the bargainers in the resale stage. We cannot deal with so many issues at the same time. For the bid revelation rule, we adopt the simplest framework of minimal information, i.e. that is there is no bid revelation in-between the two stages. Despite the lack of bid information, the bidders update their beliefs after winning or losing the auction. Since bidders with different valuations bid differently in the first stage, the updated beliefs depend on the bid in the first stage. For this reason, there are heterogeneous beliefs in the resale stage. Furthermore, a bidder may become a seller or a buyer in the resale stage depending on the bidding behavior in the first stage. This distinguishes the bargaining in the resale stage from the standard bargaining game in the literature in which there is a fixed seller or buyer, and the beliefs are homogeneous. The outcome of the bargaining in general is different. For instance, the updating may improve the efficiency of the bargaining compared to the standard homogenous model, as bargainers have better information.

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<sup>6</sup>Hafalir and Krishna (2008) use it to show the bidding symmetry property of weak and strong bidders in equilibrium. We use it to show the bid-equivalence result. In more general models (such as affiliated signals or more than three bidders), bid equivalence may hold even though the symmetry property typically fails.

One important implication of the bid equivalence result is that the equilibrium analysis of the auction with resale is reduced to the simpler equilibrium analysis of the one-stage common-value model. The revenue of the auction with resale is completely determined by the common-value function, and the efficiency of the auction with resale is determined by the trading set as well as the common-value function. We will apply the bid-equivalence result to address some of these questions. More applications of this approach can be found in Cheng and Tan (2007). Gupta and Lebrun (1999) showed that there is a reversal of revenue ranking between the first-price and second-price auctions with resale for the maximum and minimum (common-value) case. We show that there is also a reversal of revenue ranking between the first-price auction with and without resale for the two cases. Gupta and Lebrun (1999) assume that there is complete information during the resale stage, and the maximum case represents the monopoly market, while the minimum case represents the monopsony market. We combine the bid-equivalence result and the Coase Theorem to argue that in repeated bargaining, when there is commitment problems in the offers, and the buyer is sufficiently patient, then the seller loses all the bargaining power, and in the limit, the common-value function converges to the maximum function in the limit. Thus the maximum case provides the upper bound of the revenue of all possible revenue of the auction with resale. Similarly, the minimum case provides the lower bound of the revenue of all possible revenue of the auction with resale. In this sense, the two cases provide polar cases of the revenue of the auction with resale.

Three kinds of questions are of interest regarding auctions with resale. Is it a good idea to allow resale<sup>7</sup>? Is it more efficient to allow resale? For first-price and second-price auctions with resale, which one gives a higher revenue? Some of these questions have been investigated in Hafalir and Krishna (2007,2008). Examples of the application of the bid-equivalence to these questions are given in section 6. We look at the three types of resale markets: bargaining with commitment, sequential bargaining with delay costs, and k-double auctions in sections 6.1,6.2,6.3 respectively. These relatively simple applications also raise new interesting questions regarding bargaining with heterogeneous beliefs. For instance, we would expect the seller and the buyer to choose an efficient mechanism in the resale stage. In this case, we would like to know how to characterize efficient bargaining mechanisms with heterogeneous beliefs such as extensions of Williams (1987) result to this case. Other applications can be found in Cheng and Tan (2007).

In section 6.1, we look at the issue of delay costs and bargaining power in a two-period sequential bargaining model of Sobel and Takahashi (1983). The weak bidder has very high delay costs, while the strong bidder has little delay costs. As a result, the revenue of the first-price auction with resale is substan-

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<sup>7</sup>According to Myerson (1981), the optimal auction can be achieved by selecting the optimal reserve price, one for each bidder. No resale is needed for the highest revenue. However, this is not easy to do in practice. Here we assume no reservation price or a single reservation price low enough to make little difference.

tially depressed so that it is lower than the auction without resale. In section 6.2, we examine the bargaining power in the linear k-double auction resale market of Chatterjee and Samuelson (1983). We show that the auctioneer's revenue is an increasing function of the weak bidder's bargaining power. Furthermore, we show that there is a trade-off between efficiency and revenue. As the revenue gets higher, the efficiency of the auction with resale is lower. We also touch on the question of how bid revelation affects the outcome. It is interesting to note that in the k-double auction model of section 6.3, if the auctioneer announces the winning bid as often is the case in the real world auctions, the revenue of the auctioneer is higher, and the revelation is profitable for the monopolist. The revenue is lower if the losing bid is announced. Thus the auctioneer has the incentive to reveal the winning bid, but not the losing bid. The bargaining position of the monopolist is enhanced by information disclosure of the monopolist's valuation.

There are interesting implications of the idea offered in this paper on some policy issues regarding auction design. Higher prices in resale will raise the bid in the auction and improve the revenue of the auctioneer. The fact that resale prices were high after the British 3-G auction meant that it would have been better for the government to allow resale. Since then the British government has become more receptive to the idea of allowing resale (Klemperer (2004)).

Section 2 illustrates the ideas of the paper with a simple discrete model. Section 3 presents the common value model, while section 4 formulates the auction with resale model and gives the bid equivalence result. Section 5 applies the bid-equivalence result and the Coase Theorem to provide two polar cases of the auctions with resale model. Section 6 gives applications to three types of bargaining problems.

## 2 An Illustrative Example

In this section, we shall use a simple discrete model to illustrate the important issue of bargaining power to the study of auctions with resale.

Assume that there are two bidders in an independent private-value asymmetric auction. Bidder one valuation is either 0 or 1 with probability 0.7 for 0. Bidder two valuation is either 0 or 2 with probability 0.4 for 0. We call bidder one the weak bidder, and bidder two the strong bidder.

Let  $H_i$  denote the equilibrium cumulative bid distribution of bidder  $i$ . Consider the first-price auction without resale. Bidder one with valuation 1 chooses  $b$  to maximize

$$H_2(b)(1 - b).$$

The first-order condition is

$$\frac{H_2'(b)}{H_2(b)} = \frac{1}{1 - b}.$$

Similarly, the first-order condition for bidder two is

$$\frac{H_1'(b)}{H_1(b)} = \frac{1}{2-b}.$$

The boundary conditions are  $H_1(0) = 0.7, H_2(0) = 0.4$ . These probabilities have been chosen so that the other boundary condition is satisfied:  $H_1(b^*) = H_2(b^*) = 1$  for some  $b^*$ .

We have the following simple equilibrium without resale:

$$H_1(b) = \frac{1.4}{2-b}, H_2(b) = \frac{0.4}{1-b} \text{ for } b \in [0, 0.6].$$

Now we consider the first-price auction with resale. The winner of the auction acts as a monopolist in the resale game, and chooses an optimal monopoly price. The optimal monopoly price is 2 when bidder one wins the auction. There is no resale if bidder two has valuation 0, or if bidder two wins the object and has valuation 2. The payoff of bidder one with valuation 1 bidding  $b > 0$  is

$$(H_2(b) - 0.4)(2 - b) + 0.4(1 - b) = H_2(b)(2 - b) - 0.4.$$

The first-order condition for this maximization problem is

$$H_2'(b)(2 - b) - H_2(b) = 0,$$

or

$$\frac{H_2'(b)}{H_2(b)} = \frac{1}{2-b}. \tag{1}$$

For a bidder one with valuation 0, the payoff from bidding  $b > 0$  is

$$(H_2(b) - 0.4)(2 - b) + 0.4(0 - b) = H_2(b)(2 - b) - 0.8,$$

and the first-order condition is also (1). We have the same first-order condition for a bidder one with different valuations, because the resale opportunity has changed his valuation to 2 rather than 0 or 1.

For bidder two with valuation 2, there is no profit in the resale whether or not there is resale. The payoff from bidding  $b > 0$  is

$$H_1(b)(2 - b),$$

and the first-order condition is also given by (1). From the boundary conditions,  $H_1(0) = 0.4 = H_2(0)$ , we get the auction with resale equilibrium<sup>8</sup>

$$H_1^r(b) = H_2^r(b) = \frac{0.8}{2-b} \text{ for } b \in [0, 1.2].$$

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<sup>8</sup>This equilibrium is unique for any tie-breaking rule adopted. For the auction without resale model, the equilibrium is also unique for this example. In auctions with resale, bidder one with valuation 0 may bid positive amount because of future resale.

This is a symmetric equilibrium in bid distributions. The symmetry of the equilibrium bid distribution was first discovered in Engelbrecht-Wiggans, Milgrom, and Weber (1983) for the Wilson track model and proved more generally in Parreiras (2006) and Quint (2006) with independent signal (see equations (4)). This property also holds in first-price auctions with resale in Hafalir and Krishna (2008). The identical first-order conditions strongly suggest that there is some equivalence relationship between the auction with resale equilibrium and the common-value auction properly defined. We will in fact show that the equilibrium bid distributions of the auction with resale is the same as a common-value auction in which the common-value of the two bidders is the transaction price 2 when bidder two receives a high signal regardless of the signal of bidder one. This idea has great generality and is formulated in the paper.

If we let the loser of the auction make offers, so that the resale is a monopsony game. The optimal monopsony price to offer is 1 when bidder two loses the auction. There is resale if the bidder two has valuation 2 and loses the auction. When bidder two with valuation 2 bids  $b > 0$  and the payoff is

$$H_1(b)(2 - b) + (1 - H_1(b))(2 - 1) = H_1(b)(1 - b) + 1.$$

The first-order condition is

$$\frac{H_1'(b)}{H_1(b)} = \frac{1}{1 - b}. \quad (2)$$

In other words, bidder two bids as if his valuation is 1. When bidder one with valuation 1 bids  $b > 0$ , the payoff is

$$0.4(1 - b) + (H_2(b) - 0.4)(1 - b) = H_2(b)(1 - b),$$

and we have the same first-order condition (2). The model is bid equivalent to a common-value model in which both bidders value the object at either 0 or 1. The only equilibrium with full extraction of surplus, determined by the boundary conditions  $H_1(0) = 0.7 = H_2(0)$ , by the monopsonist is given by<sup>9</sup>

$$H_1^r(b) = H_2^r(b) = \frac{0.7}{1 - b} \text{ for } b \in [0, 0.3]. \quad (3)$$

The equilibrium bid distributions are the same as that of a common-value auction in which the common-value is 1 when both bidders have high signals, and 0 otherwise.

Now, we want to compare the revenues with and without resale. In the auction without resale, bidder one has a positive profit only if the valuation is

1, and in this case the profit is 0.4. Bidder two has a positive profit only if the valuation is 2, and in this case the profit is 1.4. The revenue is just the realized surplus minus the expected profit of the bidders.

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<sup>9</sup>This is an equilibrium when ties are broken in favor of bidder two. Other equilibria exist. This particular equilibrium has the lowest revenue among all possible equilibria and the only equilibrium with full extraction of surplus by the monopsonist.

For the equilibrium (2) in the auction with monopoly resale, bidder one has a positive profit only if the valuation is 1, and in this case the profit is 0.4. Note that this is the same for the auction without resale. The reason is that even though resale promises more profit to bidder one, it also makes the bidder two bid more aggressively, and the two effects happen to cancel each other in this example. Bidder two has a positive profit only if the valuation is 2, and in this case, the profit is 0.8. This lower profit is due to the more aggressive bidding behavior of bidder one, while bidder two gets no benefit from resale. Note that for each bidder the expected profit is the same or smaller in the auction with resale, and the realized surplus is higher in the auctions with resale because it is efficient. Therefore it is clear intuitively that the resale opportunity increases the revenue and is beneficial to the auctioneer when the resale is a monopoly market.

For the equilibrium (3) in the auction with monopsony resale, bidder one has a positive profit only if the valuation is 1, and in this case the profit is 0.7. Bidder two has a positive profit only if the valuation is 2, and in this case the profit is 1.7. Although the realized trade surplus is higher than that of the auction with resale, the revenue is lower with resale. In fact, the equilibrium bid distribution of the auction without resale first-order stochastically dominates that of the auction with resale as

$$\frac{1.4}{2-b} \frac{0.4}{1-b} \leq \frac{0.49}{(1-b)^2}.$$

This example illustrates the idea that if the resale market is the seller's market, allowing resale benefits the auctioneer, but the opposite is true when the resale market is the buyer's market. Again this reversal is quite general, and will be proved later.

More generally, different trading rules in the resale game change the transaction price of the object in resale, which is used to define the common-value function of the common-value auction. Auction with resale can then be shown to be bid equivalent to the common-value auction with a specifically defined common-value function.

### 3 The Common-Value Model

There are two risk neutral bidders in an auction for a single object. Our model begins with an independent private-value (IPV) model. A pure common-value model will be constructed from the IPV model and the resale process. To make the notations more compatible with the common-value model, we shall adopt a signal representation of the IPV model. This representation (called a distributional approach) is first proposed in Milgrom and Weber (1985) and discussed extensively in Milgrom (2004).

In this representation, a bidder is described by an increasing valuation function  $v_i(t_i) : [0, 1] \rightarrow [0, a_i]$ , with the interpretation that  $v_i(t_i)$  is the private

valuation of bidder  $i$  when he or she receives the private signal  $t_i$ . We can normalize the signals so that both signals are uniformly distributed over  $[0, 1]$ . The two signals are assumed to be independent. The word “private” refers to the important property that bidder  $i$ ’s valuation is not affected by the signal  $t_j$  of the other bidder, while in the common-value model, this is not the case. The function  $v_i(t_i)$  induces a distribution on  $[0, a_i]$ , whose cumulative probability distribution is given by  $F_i(x_i) = v_i^{-1}(x_i)$  when  $v_i$  is one-to-one. The standard IPV model is often described by  $F_i$ ’s rather than  $v_i$ ’s. If  $v_1(t) \leq v_2(t)$  for all  $t$ , we say that bidder one is the weak bidder, and bidder two is the strong bidder, and the pair is a weak-strong pair of bidders<sup>10</sup>. This is equivalent to saying that  $F_1$  is first-order stochastically dominated by  $F_2$ . We assume that  $F_i$  are continuously differentiable.

To define a pure common-value auction model, we need to define a common value function  $V = w(t_1, t_2)$ . This common value function represents the (expected) common value  $V$  of both bidders when both signals  $t_1, t_2$  are known. The common value can be defined through  $v_1(t_1), v_2(t_2)$ , and in this case we write  $V = p(v_1(t_1), v_2(t_2))$ . For instance, in a double auction between the seller and the buyer,  $V = p(v_1(t_1), v_2(t_2))$  may be defined as the transaction price when the seller offer is lower than the buyer offer. We will assume that the function  $w$  is continuously differentiable and non-decreasing in each  $t_i$ .

Let  $b_i(t_i)$  be the strictly increasing equilibrium bidding strategy of bidder  $i$  in the first-price common-value auction, and  $\phi_i(b)$  be its inverse. The following first order condition is satisfied by the equilibrium bidding strategy

$$\frac{d \ln \phi_i(b)}{db} = \frac{1}{p(v_1(\phi_1(b)), v_2(\phi_2(b))) - b} \text{ for } i = 1, 2. \quad (4)$$

with the boundary conditions  $\phi_i(0) = 0, \phi_1^{-1}(1) = \phi_2^{-1}(1)$ . The ordinary differential equation system with the boundary conditions determine the equilibrium inverse functions. Hafalir and Krishna (2008) have shown the same first-order condition for auctions with resale. We will also prove it in our formulation.

The function  $h(t)$  satisfies the single crossing condition if  $h(t) \geq 0$  when  $h(t') \geq 0$ , and  $t > t'$ . The function  $h(t)$  satisfies the strict single crossing condition if  $h(t) > 0$  when  $h(t') \geq 0$ , and  $t > t'$ . The function  $g(b, t)$  satisfies the strict single crossing differences condition if for all  $x' > x$ , the function  $h(t) = g(x', t) - g(x, t)$  satisfies the strict single crossing condition. The function  $g(b, t)$  satisfies the smooth single crossing differences condition if it satisfies the single crossing differences condition, and for all  $b, g_1(b, t) = 0, \delta > 0$ , we must have  $g_1(b, t + \delta) \geq 0$ , and  $g_1(b, t - \delta) \leq 0$ .

<sup>10</sup>Here we only require that  $F_1$  is dominated by  $F_2$  in the sense of the first order stochastic dominance. Note that this concept is weaker than that of Maskin and Riley (2000a), in which conditional stochastic dominance is imposed.

We use the single crossing property to show the existence of equilibrium in common value auctions. Note that the equilibrium strategies in the following theorem depends only on the value of  $w$  on the diagonal  $\{(t, t) : t \in [0, 1]\}$ . Our approach shows that the equilibrium is not affected when  $w$  is modified outside the diagonal.

**Theorem 1** *The pair of bidding strategies*

$$b_i(t_i) = \frac{1}{t_i} \int_0^{t_i} w(s, s) ds, t_i > 0, b_i(0) = 0$$

*is an equilibrium in the first-price common-value auction.*

**Proof.** Since  $w(s, s)$  is a continuous function,  $b_i$  is continuously differentiable in  $(0, 1]$ . By the L'Hopital rule,  $b_i(t_i) \rightarrow 0$  as  $t_i \rightarrow 0$ . Hence  $b_i$  is continuous on  $[0, 1]$ . Let  $\phi_i = b_i^{-1}$  be the inverse bidding function. Then  $\phi_i$  is also continuously differentiable in  $(0, 1]$ . We have

$$t_i b_i(t_i) = \int_0^{t_i} w(s, s) ds \quad (5)$$

Taking derivative of (6), we have

$$t_i b_i'(t_i) + b_i(t_i) = w(t_i, t_i).$$

Since  $\phi_i'(b(t_i)) = \frac{1}{b_i'(t_i)}$ , we have

$$\frac{t_i}{\phi_i(b_i(t_i))} = w(t_i, t_i) - b_i(t_i)$$

or

$$\frac{\phi_i(b)}{\phi_i'(b)} = w(\phi_1(b), \phi_2(b)) - b \text{ for all } b > 0,$$

and we know that the first-order conditions (6) are satisfied by  $\phi_i, i = 1, 2$ . We want to apply the Sufficiency Theorem (Theorem 4.2 of Milgrom (2004)). Let

$$g(b, t_1) = \int_0^{\phi_2(b)} (w(t_1, t_2) - b) dt_2$$

This function is continuously differentiable in  $(b, t_1)$ . Let  $b' > b$ . We have

$$g(b', t_1) - g(b, t_1) = -(b' - b)\phi_2(b) + \int_{\phi_2(b)}^{\phi_2(b')} (w(t_1, t_2) - b') dt_2.$$

Since  $w$  is non-decreasing in  $t_1$ , the single crossing differences condition is satisfied. The partial derivative of  $g$  with respect to  $b$  is

$$g_1(b, t_1) = (w(t_1, \phi_2(b)) - b)\phi_2'(b) - \phi_2(b).$$

Since  $w$  is monotonic in  $t_1$ , the smooth single crossing differences condition is satisfied. The Sufficiency Theorem then says that  $b_1(t_1)$  is an equilibrium strategy of bidder one. The proof for bidder two is the same. ■

## 4 Auctions with Resale

The first-price auction with resale is a two-stage game. The bidders participate in a standard sealed-bid first-price auction in the first stage. In the second stage, there is a resale game. At the end of the auction and before the resale stage, some information about the submitted bids may be available. The disclosed bid information in general changes the beliefs of the valuation of the other bidder. This may further change the outcome of the resale market. We shall adopt the simplest formulation in which no bid information is disclosed<sup>11</sup>. We call this the minimal information case. It should be noted that there is valuation updating even if there is no disclosure of bid information, as information about the identity of the winner alone leads to updating of the beliefs.

We shall adopt a general formulation of the resale mechanism as in Hafalir and Krishna (2008). We assume that given the updated belief after the auction stage, there is a Bayesian Nash equilibrium in the resale stage. Using the revelation principle, there is a direct trade mechanism in the resale stage in which truth-telling is incentive compatible and individually rational and the equilibrium outcome is the same. In the direct trade mechanism, given the reported valuations  $(x_1, x_2)$ , let  $q(x_1, x_2)$  be the probability of trade, and  $p(x_1, x_2)$  be the price of the transaction. Let  $Q$  be the set of  $(x_1, x_2)$  such that  $q(x_1, x_2) = 1$  and assume that  $p$  is nondecreasing in each variable continuously differentiable on  $Q$ . Note that the determination of the resale outcome depends on the bidding strategies in the first stage as the bidding strategies determine the belief system in the second stage and the belief system affects the outcome of the resale game. Let  $\phi_i, i = 1, 2$  be the inverse bidding strategies of the bidders in the first stage auction, and  $h(t_1) = \phi_2(b_1(t_1))$ . For a seller  $i$ , let  $F_i|_x$  be the conditional distribution of  $F_i$  over the support  $[x, a_i]$ , and for a buyer  $j$ , let  $F_j|_y$  be the conditional distribution of  $F_j$  over the support  $[0, y]$ . When bidder one with signal  $t_1$  bids  $b$  and wins the auction, he becomes a seller, and the updated belief about the buyer is described by  $F_2|_{v_2(\phi_2(b))}$ . Therefore different types of bidder one have different updated beliefs. Similarly, when bidder two bids  $b$  and loses the auction, she becomes the buyer, and her updated belief about the seller (bidder one) is described by  $F_1|_{v_1(\phi_1(b))}$ , where it is understood that if  $b > b_1^*$ , we mean  $\phi_2(b) = a_1$ . Because of the difference in updated beliefs among different types of bidders, the resale game after the auction here differs from the standard bilateral bargaining model. In the standard bilateral bargaining, the beliefs of different types of players are the same. This will make the equilibrium behavior in the second stage resale game different from the standard bargaining models with homogeneous beliefs. Furthermore, each player in the resale bilateral trade game may be a seller or a buyer depending on whether the player wins or loses the auction in the first stage. This feature is also absent from the standard

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<sup>11</sup>Although the equivalence result may be established in a broader context with disclosure of different bid information, it is sufficient to restrict ourselves to the resale market with no disclosure of bid information in this paper. We shall deal with a more general formulation of the observational equivalence result in a later paper.

bilateral trade with homogeneous beliefs. We shall assume that the resale game satisfies the following weak efficiency property  $q(\phi_1(b), \phi_2(b)) = 1$  if either bidder one wins the auction and  $\phi_1(b) \leq \phi_2(b)$ , or bidder two wins the auction and  $\phi_1(b) \geq \phi_2(b)$ . Note that the pair  $(\phi_1(b), \phi_2(b))$  has the highest trade surplus. Efficiency for the pair means that transaction must take place for this pair. It is a weak efficiency property for the resale mechanism because it only requires efficiency for the pair with the highest trade surplus. By Proposition 3 (section 5.1) of Hafalir and Krishna (2008), the following first-order condition and symmetry property hold for the equilibrium of auction of resale.

**Proposition 2** *If the inverse equilibrium bidding functions  $\phi_i, i = 1, 2$  are differentiable in  $(0, b^*]$ , then the following first-order conditions are satisfied*

$$\frac{\phi'_i(b)}{\phi_i(b)} = \frac{1}{w(\phi_1(b), \phi_2(b)) - b}, i = 1, 2, b \in (0, b^*]. \quad (6)$$

and we have  $\phi_1(b) = \phi_2(b)$  for all  $b \in [0, b^*]$ .

In most interesting bilateral trade models, such as  $k$ -double auctions or sequential bargaining models, it is often the case that trade takes place with probability 0 or 1. For the standard bilateral trade model, Myerson and Satterthwaite (1983) and Satterthwaite and Williams (1989) have shown that trade occurs with probability 1 or 0 when they are incentive efficient. In this case,  $Q$  is the set of pairs for which trade occurs for sure, and outside  $Q$  there is no trade.

From Proposition 2, we have  $h(t_1) = t_1$  in an equilibrium of auction with resale. The description of the resale process includes most of the well-known equilibrium models of bilateral bargaining between the seller and the buyer. For example, the monopoly resale of Hafalir and Krishna (2008) is a special case in which the winner of the auction is the monopolist seller in the resale game. The monopolist makes a take-it-or-leave-it offer, and the transaction price is the optimal monopoly price. Assume that bidder one is the weak bidder. Bidder one with signal  $t_1$  has the valuation  $v_1(t_1)$  and the belief that bidder two's valuation is  $F_2|_{h(t)}$ . Bidder one is the seller when  $h(t_1) \geq t_2$ . Assume that there is a uniquely determined optimal offer (equilibrium) price  $P(t_1)$  of the seller. In this case,  $Q = \{(t_1, t_2) : v_2(t_2) \geq P(t_1), h(t_1) \geq t_2\}$ , and the pricing function  $w(t_1, t_2) = P(t_1)$  is defined for  $(t_1, t_2) \in Q$ . Hence trade occurs if and only if  $(t_1, t_2) \in Q$ , and the trading price is the optimal offer price. The weak efficiency property must be satisfied in this case, as we know  $P(t_1) \leq v_2(h(t_1))$  when  $h(t_1) \geq t_2$ .

Similarly, in a monopsony resale mechanism with a take-it-or-leave-it offer by the buyer, the buyer chooses an optimal monopsony price higher than the lowest possible valuation of the seller. The offer is accepted when the seller has the lowest valuation, hence the weak efficiency property also holds, and the transaction price is the optimal monopsony price.

Another possibility is to designate one of the bidder, say bidder one, as the offer-maker. When it is not a weak-strong pair, bidder one may become a seller or a buyer depending on the realized signals. Thus it is a mixture of the monopoly and the monopsony market. The choice of the offer-maker or the market type affects the bargaining power of the bidders and the outcome of the resale. In the case of the monopoly market mechanism, the choice of the offer-maker is not fixed in the beginning, and is contingent on the outcome of the auction. More generally, there can be simultaneous offers by both, or repeated offers with delay costs in a sequential bargaining model of resale.

Now we show how the multiple-offer bargaining with a discount factor  $\delta$  can be represented by  $w(t_1, t_2)$  on  $Q$  satisfying our assumptions. Consider a bargaining model with two rounds of offers by the seller. Assume that signals are independent, and we have a weak-strong pair. The seller with the signal  $t_1$  and the valuation  $v_1(t_1)$  has the belief  $F_2|_{h(t_1)}$  and makes an offer  $P_1$  in the first period. This offer is either accepted or rejected, with the threshold of acceptance represented by  $Z$ , *i.e.* a buyer accepts the first offer if and only if his or her valuation is above  $Z$ . If the first offer is accepted, the game ends. If it is not accepted, the seller makes a second offer  $P_2$  which is a take-it-or-leave-it offer. Let  $P_1(t_1), P_2(t_1), Z(t_1)$  denote the equilibrium first-period, second-period prices and threshold level in this bargaining problem. The equilibrium prices in the bargaining model can be used to define the pricing function  $w(t_1, t_2)$ . Given the reported  $(v_1(t_1), v_2(t_2))$ , bidder one is the seller if  $h(t_1) > t_2$ . There is no trade if  $v_2(t_2) < P_2(t_1)$ . Trade occurs (with probability one) with the transaction price  $p(v_1(t_1), v_2(t_2)) = P_1(t_1)$  if  $v_2(t_2) \geq Z(t_1)$ , and the transaction price  $p(v_1(t_1), v_2(t_2)) = \delta P_2(t_2)$  if  $P_2(t_1) \leq v_2(t_2) < Z(t_1)$ . The set  $Q$  is

$$Q = \{(t_1, t_2) : h(t_1) \geq t_2, v_2(t_2) \geq Z(t_1) \text{ or } P_2(t_1) \leq v_2(t_2) < Z(t_1)\}$$

There is no trade when the pair is not in  $Q$ . The weak efficiency property is satisfied because we must have  $Z(t_1) < v_2(t_2)$ , and we have  $p(v_1(t_1), v_2(t_2)) = P_1(t_1)$ . The weak efficiency property holds in a monopoly resale mechanism with many rounds of offers from the seller, if the equilibrium first offer is lower than the highest valuation of the buyer. This is true if the monopolist has a strictly positive payoff in the equilibrium.

The resale market may allow simultaneous offers made by both the buyer and the seller similar to a double auction game. We now give a resale game with simultaneous offers to illustrate the formulation of the model. Assume that the signals are independent and  $v_1(t) = t, v_2(t) = 2t$  so that  $F_1(x) = x, F_2(x) = \frac{x}{2}$ . The first stage is a first-price auction. In the resale game, let  $p_s, p_b$  be the offer price by the seller and buyer respectively. The transaction takes place if and only if  $p_s \leq p_b$ , and the transaction price is given by

$$p = \frac{p_s + p_b}{2}.$$

Let the inverse bidding strategy in the first-price auction with resale be  $\phi_1, \phi_2$  and in equilibrium we have  $\phi_2(b) = 2\phi_1(b)$  by the symmetry property. To

find an equilibrium with linear strategies in the resale game, let  $p_s(v_1) = c_1 v_1 + d_1, p_b(v_2) = c_2 v_2 + d_2$  be the equilibrium strategies as functions of valuations. Bidder one with valuation  $v_1$  chooses  $p \leq 2c_2 v_1 + d_2$  to maximize

$$\int_{\frac{p-d_2}{c_2}}^{2v_1} \left[ \frac{p + c_2 v_2 + d_2}{2} - v_1 \right] dv_2.$$

The derivative of the payoff with respect to  $p$  is given by

$$\begin{aligned} & -\frac{p - v_1}{c_2} + \frac{1}{2} \int_{\frac{p-d_2}{c_2}}^{2v_1} dv_2 \\ &= \frac{1}{c_2} \left( -\frac{3}{2}p + (1 + c_2)v_1 + \frac{1}{2}d_2 \right) \end{aligned}$$

which is decreasing in  $p$ . Therefore the payoff function is concave. The first-order condition of optimality gives us

$$p_s(v_1) = \frac{2}{3}(1 + c_2)v_1 + \frac{1}{3}d_2. \quad (7)$$

For the bidder two with valuation  $v_2$ , the price offer  $p \geq \frac{v_2}{2}c_1 + d_1$  maximizes

$$\int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} \left[ v_2 - \frac{c_1 v_1 + d_1 + p}{2} \right] dv_1.$$

The first-order condition for the optimal offer is

$$\frac{v_2 - p}{c_1} - \frac{1}{2} \int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} dv_1 = 0$$

or

$$v_2 - p - \frac{c_1}{2} \left( \frac{p - d_1}{c_1} - \frac{v_2}{2} \right) = 0,$$

and we have the optimal offer of the buyer

$$p_b(v_2) = \frac{4 + c_1}{6}v_2 + \frac{1}{3}d_1.$$

To be an equilibrium, we must have

$$\begin{aligned} d_1 &= \frac{1}{3}d_2, d_2 = \frac{1}{3}d_1 \\ c_1 &= \frac{2}{3}(1 + c_2), c_2 = \frac{4 + c_1}{6} \end{aligned}$$

Solving the equations, we have

$$d_1 = d_2 = 0, c_1 = \frac{5}{4}, c_2 = \frac{7}{8}.$$

The (piecewise) linear equilibrium in the resale game is then given by

$$\begin{aligned} p_s(v_1) &= \frac{5}{4}v_1, v_1 \in [0, 1], \\ p_b(v_2) &= \frac{7}{8}v_2 \text{ for } v_2 \leq \frac{10}{7}, \\ &= \frac{5}{4} \text{ for } v_2 > \frac{10}{7}. \end{aligned}$$

The transaction price in the direct mechanism corresponding to this resale game equilibrium is given by

$$\begin{aligned} p(v_1(t_1), v_2(t_2)) &= \frac{1}{2}\left(\frac{5}{4}v_1(t_1) + \frac{7}{8}v_2(t_2)\right) = \frac{5}{8}t_1 + \frac{7}{8}t_2 \text{ if } v_2(t_2) \leq \frac{10}{7}, \\ &= \frac{5}{8}t_1 + \frac{5}{8} \text{ if } v_2(t_2) > \frac{10}{7}. \end{aligned}$$

Here  $Q = \{(t_1, t_2) : t_1 \geq t_2, \min(\frac{7}{8}v_2(t_2), \frac{5}{4}) \geq \frac{5}{4}v_1(t_1)\} = \{(t_1, t_2) : t_1 \geq t_2, \min(t_2, \frac{5}{7}) \geq \frac{5}{7}t_1\}$ , or  $Q = \{(t_1, t_2) : t_1 \geq t_2 \geq \frac{5}{7}t_1\}$ . Trade occurs with probability one if and only if  $(t_1, t_2) \in Q$ , and there is no trade outside  $Q$ . Trade

occurs if and only if  $2v_1 \geq v_2 \geq \frac{10}{7}v_1$ .

From Proposition 2, we get the equilibrium bidding strategies:

$$\begin{aligned} b_1(t_1) &= \frac{1}{t_1} \int_0^{t_1} w(t, t) dt = \frac{1}{t_1} \int_0^{t_1} \left(\frac{5}{8}t + \frac{7}{8}t\right) dt = \frac{3}{4}t_1, \text{ for } t_1 \leq \frac{5}{7} \\ &= \frac{1}{t_1} \int_0^{\frac{5}{7}} \frac{3}{2}t dt + \frac{1}{t_1} \int_{\frac{5}{7}}^{t_1} \left(\frac{5}{8}t + \frac{5}{8}\right) dt = \frac{5}{8}\left(1 + \frac{1}{2}t_1 - \frac{5}{14t_1}\right), \text{ for } t_1 \geq \frac{5}{7} \end{aligned}$$

and the same formula applies to  $b_2(t_2)$ . Hence

$$\phi_1(b) = \phi_2(b) = \frac{4}{3}b \text{ for } b \leq \frac{15}{28}.$$

**Remark 3** *With homogeneous beliefs of the traders, the optimal offer functions are  $p_s(v_1) = \frac{2}{3}v_1 + \frac{1}{2}$ ,  $v_1 \in [0, 1]$  for and seller, and  $p_b(v_2) = \frac{2}{3}v_2 + \frac{1}{6}$ ,  $v_2 \in [\frac{1}{2}, \frac{3}{2}]$ ;  $= \frac{1}{2}$  when  $v_2 \leq \frac{1}{2}$ ;  $= \frac{7}{6}$  when  $v_2 \geq \frac{3}{2}$  for the buyer. Trade offers if and only if  $v_2 \geq v_1 + \frac{1}{2}$ . Since  $v_2 \geq v_1 + \frac{1}{2}$  implies  $v_2 \geq \frac{10}{7}v_1$ , trade is less efficient in the homogeneous case. This is because the updating of beliefs improves efficiency of trade in our model.*

**Remark 4** *Another important difference from the standard  $k$ -double auction bilateral trade model is that here we have the boundary conditions  $p_s(0) = p_b(0) = 0$ . It can be shown that any differentiable equilibrium of the  $k$ -double auction must satisfy*

$$p'_s(0) = \frac{5}{4}, p'_b(0) = \frac{7}{8}.$$

*Therefore there is only one differentiable strictly monotone equilibrium by the theory of differential equations. In the standard  $k$ -double auction model with homogeneous beliefs, Satterthwaite and Williams (1989, Th. 3.2) have shown that there is a two-parameter family of such equilibria.*

We assume that the pricing function  $p(v_1(t_1), v_2(t_2))$  can be extended from  $Q$  to the set of all pairs  $(x_1, x_2)$  monotonically and smoothly so that the extension is continuously differentiable on the unit square. This extended function is denoted by  $w(t_1, t_2)$ . Consider a common-value auction in which the common value is defined by  $w(t_1, t_2)$ . From Theorem 1, we know that the equilibrium is the same for all different smooth monotonic extensions of  $p$ , as the bidding strategy only depends on the value of  $w$  on the diagonal  $\{(t, t) : t \in [0, 1]\}$  which belongs to  $Q$  by the weak efficiency property.

The following bid equivalence result applies to all such extensions. Intuitively this says that the common value is defined by the transaction price in the resale game when trade occurs with probability one. When there is possibility of no trade, the common value can be defined arbitrarily without affecting the equilibrium.

**Theorem 5** *Assume that there is no disclosure of bid information in between the auction stage and the resale stage, and the resale process satisfies the weak efficiency property. Assume that there is a strictly monotone differentiable equilibrium bidding strategy profile  $b_i(t), i = 1, 2$  in the auction with resale. Let  $w(t_1, t_2)$  be a monotonic extension of the pricing function  $p(v_1(t_1), v_2(t_2))$  on  $Q$ . Then  $b_i(t), i = 1, 2$  is also an equilibrium of the common-value auction with the common value function  $w$ , and we have bid-equivalence between the auction with resale and the common-value auction.*

Proof of Theorem 5:

From Proposition 2, the solution of the first-order condition gives us the symmetric bidding strategy

$$b_i(t_i) = \frac{1}{t_i} \int_0^{t_i} p(v_1(s), v_2(s)) ds, t_i > 0, b_i(0) = 0. \quad (8)$$

This strategy only depends on the value of  $w$  on the diagonal  $(t, t)$ , hence is the same for all monotonic smooth extension of  $p$ . According to Theorem 1,

the equilibrium bidding strategy in (8) is an equilibrium of the common-value auction.

For the proof in Theorem 1 to work, we require that the monotonic extension to be smooth. It is convenient to allow the extension to have kinks. Because the equilibrium is unaffected by different smooth extensions, we can use those smooth extensions to approximate a monotonic extension with possible kinks and the equilibrium property holds in the limit. This means that the bid equivalence result still holds even if we allow the extension to have kinks.

Kinked extensions are easier to describe. The following method is one way to extend the definition of  $w$  to all pairs. For  $(t_1, t_2)$  outside  $Q$ ,  $t_2 < t_1$ , let  $k_1(t_1) = \inf\{t_2 : (t_1, t_2) \in Q\}$ . we define  $w(t_1, t_2) = w(t_1, k_1(t_1))$  for  $(t_1, t_2) \notin Q$ . For  $t_2 > t_1$ , we can just let  $w(t_1, t_2) = w(t_2, t_2)$ . In this extension, we have kinks on the boundary of  $Q$ . We have the property  $w(t_1, t_2) \geq x$  when  $v_1(t_1) = v_2(t_2) = x$ . We can use the resale game example with simultaneous offers in the last section to illustrate the extension. On  $Q$ , we have

$$\begin{aligned} w(t_1, t_2) &= \frac{5}{8}t_1 + \frac{7}{8}t_2 \text{ if } t_2 \leq \frac{5}{7}, \\ &= \frac{5}{8}t_1 + \frac{5}{8} \text{ if } t_2 > \frac{5}{7}. \end{aligned}$$

For  $(t_1, t_2) \notin Q$ ,  $t_2 < \frac{5}{7}t_1$ , we define  $w(t_1, t_2) = w(t_1, \frac{5}{7}t_1) = \frac{5}{4}t_1$ . If  $t_2 > t_1$ , we let  $w(t_1, t_2) = w(t_1, t_1) = \frac{3}{2}t_1$  when  $t_1 \leq \frac{5}{7}$ , and  $w(t_1, t_1) = \frac{5}{8}t_1 + \frac{5}{8}$  when  $t_1 > \frac{5}{7}$ . When  $v_1(t_1) = v_2(t_2) = x$ , we have  $t_2 = \frac{1}{2}t_1$ , and in this case, we have  $w(t_1, t_2) = \frac{5}{4}t_1 = \frac{5}{4}x > x$ .

Another alternative definition of the common value outside  $Q$  is to let  $k_2(t_2) = \sup\{t_1 : (t_1, t_2) \in Q\}$ , and then define  $w(t_1, t_2) = w(k_2(t_2), t_2)$ . In this definition, we may have  $w(t_1, t_2) \leq x$  when  $v_1(t_1) = v_2(t_2) = x$ . For the same example, when  $(t_1, t_2) \notin Q$ ,  $t_2 < \frac{5}{7}t_1$ , we define  $w(t_1, t_2) = w(\frac{7}{5}t_2, t_2) = \frac{7}{4}t_2$  when  $t_2 \leq \frac{5}{7}$ . and  $w(\frac{7}{5}t_2, t_2) = \frac{7}{8}t_2 + \frac{5}{8}$  when  $t_2 > \frac{5}{7}$ . When  $v_1(t_1) = v_2(t_2) = x$ , we have  $t_2 = \frac{1}{2}t_1$ , and in this case, we have  $w(t_1, t_2) = \frac{7}{4}t_2 = \frac{7}{8}x < x$  for  $t_2 \leq \frac{5}{7}$ .

In a bilateral trade equilibrium, trade need not occur with probability one even when the trade surplus is the highest possible amount. In this case, the weak efficiency property fails, and the bid equivalence result need not hold. We shall give such an example with discrete valuations. Note that the above bid equivalence result can be easily extended to the case of discrete valuations. To see this, assume that there are two bidders in an independent private-value asymmetric auction. Bidder one valuation is either 0 or 1 with probability 0.7 for 0. Bidder two valuation is either 0 or 2 with probability 0.4 for 0. We call bidder one the weak bidder, and bidder two the strong bidder. We can adopt a signal representation of the IPV model. This representation is called a distributional approach) is first proposed in Milgrom and Weber (1985) and discussed extensively in Milgrom (2004). In this representation, both bidders

are described by step functions  $v_i(t_i) : [0, 1] \rightarrow \{0, 1\}$ ,

$$v_1(t_1) = 0, \text{ when } t_1 \in [0, 0.7); = 1 \text{ when } t_1 \in [0.7, 1]$$

$$v_2(t_2) = 0, \text{ when } t_2 \in [0, 0.5); = 1 \text{ when } t_2 \in [0.5, 1]$$

with the interpretation that  $v_i(t_i)$  is the private valuation of bidder  $i$  when he or she receives the private signal  $t_i$ . The two signals are assumed to be independent.

The bidding strategies are mixed strategies which can be represented by  $b_i(t_i), i = 1, 2$ . The inverse bidding function  $\phi_i(b)$  is the cumulative bid distribution of bidder  $i, i = 1, 2$ . The beliefs after the first-stage auction is described conditional distribution of the signal  $t_j$  over  $[0, \phi_j(b)]$  when bidder  $i$  bids  $b$  and wins the auction. When bid  $i$  loses the auction after bidding  $b$ , the belief about bidder two is described by the conditional distribution of  $t_j$  over  $[\phi_j(b), 1]$ . In the space of valuations, such distributions are also discrete distributions. In the bilateral trade, a direct mechanism is a report of the signal rather valuation. A direct trade mechanism then is represented by the probability of trade  $q(t_1, t_2)$ , and the trade price  $p(t_1, t_2)$ . Thus we can adopt the model with a continuum of valuations to the model with a continuum of signals. The first-order condition of the equilibrium in the auction with resale holds except at  $t_i$  when there is a change in valuation. The symmetry property continues to hold with the same proof. The trading price on  $Q$  can be extended monotonically to the whole range to define a common value function. The equilibrium in the common-value auction satisfies the same first-order condition except a finite number of points. The equilibrium bidding strategy is given by the same formula, except that it is piecewise continuously differentiable rather than continuously differentiable. There is a bid equivalence between the equilibrium in the auction with resale and the common value auction. For the example in the last paragraph, when the resale is a monopoly, the optimal monopoly price is 2 when bidder one wins the auction. There is no resale if bidder two has valuation 0, or if bidder two wins the object and has valuation 2. The payoff of bidder one with valuation 1 bidding  $b > 0$  is

$$(\phi_2(b) - 0.4)(2 - b) + 0.4(1 - b) = \phi_2(b)(2 - b) - 0.4.$$

The first-order condition for this maximization problem is

$$\phi_2'(b)(2 - b) - \phi_2(b) = 0,$$

or

$$\frac{\phi_2'(b)}{\phi_2(b)} = \frac{1}{2 - b}. \quad (9)$$

For a bidder one with valuation 0, the payoff from bidding  $b > 0$  is

$$(H_2(b) - 0.4)(2 - b) + 0.4(0 - b) = H_2(b)(2 - b) - 0.8,$$

and the first-order condition is also (9). We have the same first-order condition for a bidder one with different valuations, because the resale opportunity has changed his valuation to 2 rather than 0 or 1.

For bidder two with valuation 2, there is no profit in the resale whether or not there is resale. The payoff from bidding  $b > 0$  is

$$H_1(b)(2 - b),$$

and the first-order condition is also given by

$$\frac{\phi_1'(b)}{\phi_2(b)} = \frac{1}{2 - b}.$$

From the boundary conditions,  $\phi_1(0) = 0.4 = \phi_2(0)$ , we get equilibrium<sup>12</sup> inverse bidding functions in the auction with resale:

$$\phi_1(b) = \phi_2(b) = \frac{0.8}{2 - b} \text{ for } b \in [0, 1.2].$$

and the equilibrium bidding strategies

$$\begin{aligned} b_i(t) &= 2 - \frac{0.8}{t} \text{ when } t \geq 0.4 \\ &= 0 \text{ when } t \leq 0.4. \end{aligned}$$

We have the set of trade

$$Q = \{(t_1, t_2) : t_1 < 0.4\} \cap [0.4, 1] \times [0.4, 1] \cap \{(t_1, t_2) : t_1 \geq t_2, 0 \leq t_i \leq 1, i = 1, 2\}.$$

and the pricing function on  $Q$  :  $p(t_1, t_2) = 0$  if  $t_1 < 0.4$ ;  $= 2$  if  $(t_1, t_2) \in [0.4, 1] \times [0.4, 1]$ . The common value function is a monotonic extension of  $p$ . For instance, we can have the extension  $w(t_1, t_2) = 0$  if  $t_1 < 0.4$ ;  $= 2$  if  $t_1 \geq 0.4$  for all pairs in  $[0, 1] \times [0, 1]$ . The equilibrium bidding strategy of the common value function is

$$\begin{aligned} b_i(t) &= \frac{1}{t} \int_0^t p(s, s) ds = \frac{2}{t}(t - 0.4) = 2 - \frac{0.8}{t} \text{ if } t \geq 0.4 \\ &= 0 \text{ if } t < 0.4, \end{aligned}$$

which is the same as the bidding strategy of the auction with resale.

Our counter example is a variation of the above example in which the resale market is a monopoly with trade impediments. This means that the winner of the auction makes offers, but trade can only occur with probability 0.5 when the buyer accepts the offer. There is no trade when the buyer does not accept the offer. With this trade mechanism, the optimal offer by the seller is the same as the monopoly price without restriction. Hence the optimal offer price is 2

<sup>12</sup>This equilibrium is unique for any tie-breaking rule adopted. In auctions with resale, bidder one with valuation 0 may bid positive amount because of future resale.

whenever there is a positive probability that the buyer has valuation 2. Let  $\phi_i$  denote the equilibrium cumulative bid distribution of bidder  $i$ . The weak bidder with valuation 1 chooses  $b$  to maximize the following payoff

$$0.5(\phi_2(b) - 0.4)(2 - b) + 0.5(\phi_2(b) - 0.4)(1 - b) + 0.4(1 - b) = \phi_2(b)(1.5 - b) - 0.2.$$

The first-order condition is

$$\frac{\phi_2'(b)}{\phi_2(b)} = \frac{1}{1.5 - b}. \quad (10)$$

Bidder one with valuation 0 chooses  $b$  to maximize

$$0.5(\phi_2(b) - 0.4)(2 - b) + 0.5(\phi_2(b) - 0.4)(0 - b) + 0.4(0 - b) = \phi_2(b)(1 - b) - 0.4$$

and the first-order condition is

$$\frac{\phi_2'(b)}{\phi_2(b)} = \frac{1}{1 - b}. \quad (11)$$

Bidder two with valuation 2 chooses  $b$  to maximize the payoff

$$\phi_1(b)(2 - b)$$

with the first-order condition

$$\frac{\phi_1'(b)}{\phi_1(b)} = \frac{1}{2 - b}. \quad (12)$$

Let  $b^*$  be the common maximum bid of both bidders. We have the following boundary conditions

$$\phi_2(0) = 0.4, \phi_2(b^*) = \phi_1(b^*) = 1.$$

Note that the first-order conditions of the two bidders are not symmetric, and we don't expect the equilibrium bid distributions of the two bidders to satisfy the symmetry property. If the symmetry property fails, then the equilibrium of the auction with this restricted monopoly resale market cannot be bid equivalent to any equilibrium of a common-value auction. To find the equilibrium, let  $\phi_1(c) = 0.7$ . We have

$$\phi_1(b) = \frac{A}{2 - b} \text{ for } b \in [0, b^*],$$

where  $A = 2 - b^*$ ,  $0.7(2 - c) = 2 - b^*$ . For bidder two, we have

$$\phi_2(b) = \frac{B}{1.5 - b} \text{ for } b \in [c, b^*],$$

hence  $B = 1.5 - b^*$ . For  $b \in [0, c]$ , we must have  $\phi_2(0) = 0.4$ , hence

$$\phi_2(b) = \frac{0.4}{1 - b} \text{ for } b \in [0, c].$$

We must have

$$\phi_2(c) = \frac{0.4}{1-c} = \frac{B}{1.5-c}$$

or

$$\frac{0.4(1.5-c)}{1-c} = B = 1.5 - b^* = 1.5 - (2 - 0.7(2-c)) = 0.9 - 0.7c$$

We have the following quadratic equation in  $c$ :

$$(0.9 - 0.7c)(1 - c) - 0.4(1.5 - c) = 0.7c^2 - 1.2c + 0.3 = 0$$

and the solution is  $c = 0.30386$ . From this number, we have  $b^* = 0.68730$ ,  $A = 1.31270$ ,  $B = 0.81270$ . Hence we have the following equilibrium cumulative bid distributions when ties are broken in favor of bidder one:

$$\phi_1(b) = \frac{1.3127}{2-b}, b \in [0, 0.6873],$$

$$\begin{aligned} \phi_2(b) &= \frac{0.4}{1-b}, b \in [0, 0.30386] \\ &= \frac{0.8127}{1.5-b}, b \in [0.30386, 0.6873]. \end{aligned}$$

Clearly, this equilibrium does not satisfy the symmetry property, and the bid equivalence result fails.

## 4.1 Buyer-Speculators Model of Auction with Resale

A special interesting case is worth mentioning here. If bidder one is a regular buyer, and bidder two is a speculator having no value for the object, but participates in the auction for resale, we call this the buyer-speculator model of auction with resale. Assume that the resale market is a monopoly. From our bid equivalence result, the equilibrium bidding behavior of this model is the same as the Wilson Drainage Tract common value Model in which there is one neighbor and one non-neighbor. The common value is the defined by the monopoly price offered by the speculator.

Since the Wilson tract model is often discussed in the context of one neighbor and many non-neighbors, we shall extend the bid-equivalence result to this case. Consider an auction with resale model with  $m$  speculators and one regular buyer, Bidder one is a regular buyer with a valuation distribution  $F(x)$  over  $[0, a_1]$ . All other bidders have zero valuation of the object, and participate in the auction only because of the resale opportunity. Now consider only symmetric equilibrium in which all speculators use the same bidding strategy  $b_s(t)$ , and let the bidding strategy of the regular buyer be  $b_r(x)$ . There is no resale if bidder one wins the auction, and resale will occur if one of the speculator wins the

auction. Let  $\phi_s, \phi_r$  be the inverse bidding functions, where  $\phi_s$  is the cumulative bid distribution. If bidder one with valuation  $x$  bids  $b$ , and loses the auction, the highest bidder among the speculators wins the auction, and sells the object to bidder two. The price depends on  $\phi_r(b_s)$  because the winning speculator believes that bidder one valuation is distributed over  $[\phi_r(b_s), a_1]$ . Let  $p(x)$  be the optimal monopoly price when the speculator makes offers to bidder one with valuation distributed over  $[x, a_1]$ . Let  $h(t) = \phi_r(b_s(t))$ . The profit of bidder one can be written as

$$\int_{\phi_s(b)}^1 [x - p(h(t))] dt^m + \phi_s(b)^m (x - b).$$

The first-order condition is given by

$$m\phi_s'(b)\phi_s(b)^{m-1}[p(\phi_r(b)) - b] - \phi_s(b)^m = 0$$

or

$$\frac{d \ln \phi_s(b)^m}{db} = \frac{1}{p(\phi_r(b)) - b}.$$

For the speculator, the profit from bidding  $b$  is

$$\int_0^{\phi_r(b)} [p(x) - b] dF(x).$$

The first-order condition is

$$\phi_r'(b)f(\phi_r(b))[p(\phi_r(b)) - b] - \phi_r(b) = 0,$$

or

$$\frac{d \ln F(\phi_r(b))}{db} = \frac{1}{p(\phi_r(b)) - b}.$$

From the first-order conditions, we know that  $F(\phi_r(b)) = \phi_s(b)^m$ . From this, we have the bidding strategy

$$b_r(x) = \frac{1}{F(x)} \int_0^x p(x) dF(x).$$

This is the same (symmetric) equilibrium in the Wilson tract model with one neighbor and many non-neighbors, and the common value is given by  $p(F^{-1}(t_1))$  (see, for example, Milgrom (2004), Theorem 5.3.2). Therefore we have bid-equivalence between the auction with resale and the Wilson tract model for the symmetric equilibrium.

It is an interesting question to ask whether bidders behave the same way in both models in practice or in experiments, given the evidence that winner's curse is observed in practice, but we don't expect similar irrationality in the auction with resale.

Assume that the buyer makes offers in the speculator-buyer model. The optimal monopsony price is 0, and the bid is always 0 (ties are broken in favor of the non-speculator). This is an individually rational incentive compatible direct bilateral trade mechanism in the resale game which yields an efficient outcome. In this efficient outcome, the auctioneer has 0 revenue. Let  $R^m$  be the auctioneer's revenue when the resale market is a monopoly market. Any revenue in the interval  $[0, R^m]$  is a possible revenue in the auction with resale. To show this, we can simply use a stochastic mechanism to determine who is the offer-maker after the auction and before the resale occurs. Let  $\pi$  be the probability that the speculator makes offers. The common value function in this case is given by  $w(t_1, t_2) = \pi p(t_1)$ . The revenue of the auction with resale is  $\pi R^m$ . As  $\pi$  varies between 0 and 1, the auctioneer revenue can take any value in the interval  $[0, R^m]$ . In this simple model, bargaining power is determined by who makes offers. Higher bargaining power means that there is a higher probability  $\pi$  that the speculator makes offers. The auctioneer's revenue is an increasing function of the bargaining power proxy  $\pi$ , but the efficiency of the auction is a decreasing function of  $\pi$ . We have a trade-off between the revenue and the efficiency of the auction with resale.

The above result holds in a model with one-sided uncertainty in which bidder one has a publicly known value  $a_1 \in [0, a_2]$ . The auction with a resale which is efficient and incentive compatible has the lowest value. There is a trade-off between efficiency and revenue as bidder one has a higher probability of making offers (higher bargaining power).

## 5 Two Polar Cases

The existence and uniqueness of the equilibrium in the first-price common-value auctions have been studied in the literature<sup>13</sup>. There is an explicit formula for the equilibrium in Parreiras (2006). In the pure common-value model with independent signals, it is well-known that in equilibrium, the winning probabilities of the two bidders are the same when they bid the same amount<sup>14</sup>. The symmetric property of the winning probabilities is the property that both bidders have identical bidding strategies (as functions of  $t$ ). In other words, we have  $b_1(t) = b_2(t)$ . Note that there is asymmetry in the signals as  $v_1, v_2$  are different, and bidding strategies as functions of  $v_i$  are not symmetric. However, bidding strategies in terms of  $t$  are symmetric.

The symmetry property of the equilibrium bidding strategy gives us very simple formulas for the bidding strategy and the revenue. The following propo-

<sup>13</sup>The existence of a non-decreasing equilibrium in the common value model is established in Athey (2001). The existence of a strictly increasing equilibrium has been shown in Rodriguez (2000). The uniqueness of equilibrium of the first price auction of the common value model can be found in Lizzeri and Persico (1998) and Rodriguez (2000).

<sup>14</sup>This can be found in Engelbrecht-Wiggans, Milgrom, and Weber (1983) for the Wilson track model and more generally in Parreiras (2006) and Quint (2006). This property also holds in first-price auctions with resale in Hafalir and Krishna (2007).

sition summarizes the formulas in Gupta and Lebrun (1999) in the context of auction with resale and in Parreiras (2006) in the context of common-value auctions. We adopt a different notation, as we use the signal representation (or distributional) approach<sup>15</sup>. In this representation, the formulas take a very simple form.

**Proposition 6** *With two bidders, the revenue of the first-price common-value auction is*

$$R^F = 2 \int_0^1 (1-t)p(v_1(t), v_2(t))dt.$$

We shall first apply the formulas to the two polar cases:  $w(t_1, t_2) = \max\{v_1(t_1), v_2(t_2)\}$ , or  $w(t_1, t_2) = \min\{v_1(t_1), v_2(t_2)\}$ . These common-value functions correspond to two types of resale markets. In Gupta and Lebrun (1999), they assume that after the first-stage auction, the valuations of both bidders become common knowledge (with no particular explanation of how this is achieved). In the monopoly resale market, the complete information leads to the transaction price  $\max\{v_1(t_1), v_2(t_2)\}$ . In the monopsony resale market, the complete information leads to the transaction price  $\min\{v_1(t_1), v_2(t_2)\}$ . We have the following equilibrium strategies and revenues

$$b_{\max}(t) = \frac{1}{t} \int_0^t \max(v_1(r), v_2(r))dr, R_{\max}^F = 2 \int_0^1 (1-t) \max(v_1(t), v_2(t))dt,$$

for the maximum case, and

$$b_{\min}(t) = \frac{1}{t} \int_0^t \min(v_1(r), v_2(r))dr, R_{\min}^F = 2 \int_0^1 (1-t) \min(v_1(t), v_2(t))dt,$$

for the minimum case. Note that the equilibrium bidding strategies are strictly increasing. The two values give us the upper bound and the lower bound of the revenue of the auction with resale.

Another simple example without perfect information in the resale stage in which the two polar cases represent the seller's and buyer's market is the one in section two. In that example, the common value functions are  $w(t_1, t_2) = \max\{v_1(t_1), v_2(t_2)\}$ , or  $w(t_1, t_2) = \min\{v_1(t_1), v_2(t_2)\}$  respectively, even though there is still incomplete information about the valuations. Furthermore, even if all the bids are announced in that example, the equilibrium still holds. In the maximum case, if bidder two lowers the bid to 0, the profit is lower or the same

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<sup>15</sup>We want to thank Jeremy Bulow for pointing out that the bidding formula can also be obtained from the theorem in Milgrom and Weber (1982) by using symmetric signals but asymmetric common value functions.

(when ties are broken in favor of bidder two). Therefore there is no incentive for bidder two to lower the bid to signal low valuation in this case. In the minimum case, If bidder one with valuation 0 bids  $b > 0$ , and signals a high valuation, the object is sold at the price 1 to bidder two. This yields the profit  $-0.7b + (H_2^t(b) - 0.7)(1 - b) = 0$ . Hence there is no profit from such signaling strategy. Bidder one has no incentive to signal high valuation either. Thus the announcement of all the bids by the auctioneer does not destroy the respective equilibrium.

In general, however, when all the bids are announced, the signalling strategy can destroy the increasing equilibrium. Take the following example  $v_1(t) = t, v_2(t) = 2t$ . We have the symmetric bidding strategy for the maximum common value without bid disclosure

$$b_{\max}(t) = \frac{1}{t} \int_0^t 2r dr = t.$$

If bids are announced, bidder two with valuation  $2t$  can lower the bid to  $b < t$ , and buys the object from bidder one at the lower price  $2b$  when the auction is lost. The profit becomes

$$\phi_1(b)(2t - b) + (1 - \phi_1(b))(2t - 2b) = \phi_1(b)b - 2b + 2t = b^2 - 2b + 2t.$$

The derivative of the profit function is  $2b - 2 < 0$ . This means that when the bid  $b$  is lowered, the profit will increase, thus destroying the above equilibrium strategy when bids are announced, and signalling is effective. Similarly, the symmetric equilibrium for the minimum common value case with no bid disclosure is

$$b_{\min}(t) = \frac{1}{t} \int_0^t r dr = \frac{t}{2}.$$

When bids are announced, bidder one with valuation  $t$  can raise the bid  $b > \frac{t}{2}$ , and sells the object to bidder two at the price  $2b$ . The profit from this deviation is

$$\phi_2(b)(2b - b) = 2b^2,$$

which is an increasing function of  $b$ . The signalling strategy destroys the no disclosure equilibrium.

As we shall see in a minute, if we assume that only the winning bid is disclosed (which is normally the case in the real world auctions), the Coase Theorems (Gul, Sonnenschein, and Wilson (1986)) will allow us to obtain similar equilibrium outcome in auctions with resale in the limit. For convenience, we shall refer to the maximum case as the case of seller's market. This means that the resale market is a monopoly market in which the seller captures all the surplus from trade. Similarly, the minimum case is referred to as the buyer's market in which the buyer captures all the trade surplus.

Gupta and Lebrun (1999) have shown that there is a reversal of ranking between the first-price auction with resale and the second-price auction with

resale in this case. We now show a similar result for the comparison of revenues in auctions with and without resale. Let  $R^r$  denote the revenue of the first-price auction with resale, and  $R^o$  denote the revenue of the first-price auction without resale.

**Theorem 7** *Assume that  $v_1(t) \leq v_2(t)$ , for all  $t$  and  $v_1(t) \neq v_2(t)$  for a subset of  $[0, 1]$  of non-zero measure. If  $w(t_1, t_2) = \max\{v_1(t_1), v_2(t_2)\}$ , then  $R^r > R^o$ .*

*If  $w(t_1, t_2) = \min\{v_1(t_1), v_2(t_2)\}$ , then we have  $R^o > R^r$ .*

**Proof.** Let

$$F(x) = \min\{F_1(x), F_2(x)\}.$$

and  $v(t) = F^{-1}(t)$ . Then we have  $v(t) = \max\{v_1(t), v_2(t)\}$ . Then  $R^r$  is the revenue of a symmetric auction with the valuation distribution  $F(x)$  for each bidder. Let  $H$  be equilibrium bid distribution of each bidder in this symmetric auction. Let  $H_1, H_2$  be the equilibrium bid distribution of bidder one and two respectively in the asymmetric auction without resale. According to Proposition of Maskin and Riley (2000),  $H$  first-order stochastically dominate  $H_2$  which also first-order stochastically dominates  $H_1$ . Hence we have  $H^2$  first-order stochastically dominates  $H_1H_2$ , and we have  $R^r > R^o$ . The result for the case of  $w(t_1, t_2) = \min\{v_1(t_1), v_2(t_2)\}$  is completely similar. ■

**Remark 8** *There is a difference between full extraction of surplus and monopoly or monopsony. For the example in section 2, when ties are broken in favor of bidder two, the following is an equilibrium in the auction with the monopsony resale market,*

$$H_1(b) = \frac{0.55}{1-b} = H_2(b), b \in [0, 0.45].$$

*This is not a full extraction equilibrium, as bidder two fails to extract all the surplus of a bidder one with valuation 0. The revenue of this equilibrium is*

$$\int_0^{0.45} \left(1 - \frac{0.55^2}{(1-b)^2}\right) db = 0.2025.$$

*The bid-your-valuation equilibrium in the second-price auction with resale yields the revenue 0.18. Hence the equilibrium in the first-price auction with monopsony resale has higher revenue than the second-price auction with resale. The full-extraction monopsony equilibrium for the same example has the revenue*

$$\int_0^{0.3} \left(1 - \frac{0.7^2}{(1-b)^2}\right) db = 0.09,$$

which is lower than the second-price auction with resale. The revenue of the first-price auction without resale is

$$\int_0^{0.6} \left(1 - \frac{1.4 * 0.4}{(1-b)(2-b)}\right) db = 0.28662.$$

Hence this is also an example in which the first-price auction without resale may yield a higher revenue than the auction with a monopsony resale without full extraction of surplus.

We now look at the implications of the Coase Theorem. Assume that the object for sale is a durable good, and both the seller and the buyer are patient (with a discount factor close to 1) or when the offers are made in increasingly short intervals. The Coase (1972) conjecture in fact says that the monopolist may lose all bargaining power if the buyer anticipates lower prices in future offers. This has been formalized in Gul, Sonnenschein and Wilson (1986)<sup>16</sup>. In their model, the monopolist makes a sequence of offers until the offer is accepted by the buyer. Assume that the seller's valuation is common knowledge (as in the case when the winning bid is announced). Consider the set of equilibria with the stationary property that the state of the market, after any price that is lower than all preceding prices, is independent of the earlier price history in the market. Such equilibria are said to be stationary, since the acceptance and rejection decisions and future offers depend only on the current price. They show that for such equilibria, all prices including the first offer goes to the marginal cost of the monopolist (Theorem 3).

Applying this result to our model, the winner of the auction is the seller, and the winning bidder's valuation is the marginal cost of the seller. When the winning bid is announced, the seller's valuation becomes common knowledge. Since the first offer price is accepted by the buyer with the highest valuation, this price is the common value. When the first offer converges to the marginal cost, the common value function then converges to  $\min\{v_1(t), v_2(t)\}$ <sup>17</sup>. If only the losing bid is announced, while the loser makes offers, similar arguments lead us to the common value function  $\max\{v_1(t), v_2(t)\}$  in the limit. Thus we can see these two cases as extreme cases of auctions with resale.

In the literature on Coase conjecture, the seller's cost is usually fixed, and equal to 0. In our resale model, the seller's cost can be any number within the

<sup>16</sup>Fudenberg, Levine, and Tirole (1985) have a Coase Theorem in the "gap" case in an infinite horizon model of bargaining when the discount rate is close to 1. Our model does not allow the "gap" case.

<sup>17</sup>When alternating offers are allowed, Ausubel and Deneckere (1992, Theorem 3.2) show that a version of the Coase conjecture also holds. Assume that the equilibrium satisfies stationarity, monotonicity, pure strategies, and no free screening. As the time interval between successive periods is made sufficiently short, the initial serious offer by the seller or buyer in an alternating-offer bargaining game must be close to the marginal cost of the monopolist for an entire family of distribution functions. Thus the buyer's market could also arise with alternating offers. In private communication, Ausubel has indicated that this result does not imply the authors's belief in the generality of the Coase conjecture for bargaining with alternating offers.

range  $[0, a_1]$ . To show how the Coase Theorem can be adapted for any cost of the seller with heterogeneous beliefs due to updating, we illustrate with the finite horizon model of Sobel and Takahashi (1983). We show that for any given discount factor  $\delta_1 < 1$  of the seller, the Coase conjecture holds as  $\delta_2 \rightarrow 1$ , and the number of periods goes to infinity. We focus on the linear case of Sobel and Takahashi (1983).

Assume that bidder one and two have uniform IPV distributions over the intervals  $[0, a_1], [0, a_2]$  respectively and  $a_1 < a_2$ . After the first-price auction in stage one, the winning bid is announced. In stage two, the winner of the auction makes no commitment offers (except the last one which is a take-it-or-leave-it offer) to the loser for  $n$  periods. In this case, only bidder one will make offers after winning the auction. First we derive the unique perfect Bayesian equilibrium of this finite-offer game and show that the revenue ranking is reversed. Let the seller has the valuation  $x$  and in equilibrium she believes that the buyer's valuation is uniformly distributed over  $[0, y], y = \frac{a_2}{a_1}x$ . We denote this bargaining game by  $L_n(x, y)$ . The proof of the following result is given in the appendix.

**Proposition 9** *The first period offer of the bargaining game  $L_n(x, y)$  in the resale stage with  $n$  periods of offers is given by*

$$p = c_n y + (1 - c_n)x$$

where  $c_n$  is defined recursively by

$$c_1 = \frac{1}{2}, c_k = \frac{(1 - \delta_2 + \delta_2 c_{k-1})^2}{2(1 - \delta_2 + \delta_2 c_{k-1}) - \delta_1 c_{k-1}}.$$

Fix  $\delta_1 < 1$ , and let  $\delta_2 \rightarrow 1$ , we have

$$c_k \rightarrow \frac{c_{k-1}}{2 - \delta_1} \text{ for all } k.$$

Since  $c_1 = \frac{1}{2}$ , we have  $c_n = \frac{1}{2(2 - \delta_1)^{n-1}} \rightarrow 0$ , as  $n \rightarrow \infty$ . Therefore the first period offer  $p$  converges to  $x = \min\{x, y\}$  as  $n \rightarrow \infty$ . By Theorem ??, the revenue ranking is reversed if  $\delta_1 < 1$  is fixed,  $\delta_2$  is close to 1, and the number of offer periods  $n$  is sufficiently large. In this example, Coase Theorem holds as long as the buyer is sufficiently patient, and the number of bargaining period is sufficiently large.

## 6 Effects of Bargaining Power

The main purpose of this section is to illustrate the applications of the bid-equivalence result and explore the effects of bargaining power on many important questions regarding auctions with resale. The bid equivalence result

makes it possible to analyze the revenue and efficiency questions in the auction with resale when the resale market is more complicated than the monopoly or monopsony market in Hafalir and Krishna (2008).

We shall first offer a simple way of formulating the idea that the revenue of the auction with resale is an increasing function of the bargaining power of the weak bidder. Let bidder one and two be a pair of weak-strong bidders. Take the two polar cases of section 5. The common value function for the monopoly market with full extraction of surplus is given by  $w_p(t_1, t_2) = \max(v_1(t_1), v_2(t_2)) = v_2(t_2)$ . Similarly, let the common value function of the monopsony market with full extraction of surplus be denoted by  $w_s(t_1, t_2) = v_1(t_1)$ . Assume that bidders do not know before the auction who will be making offers, and the offer-maker is determined by a stochastic mechanism before the resale. Let  $q \in [0, 1]$  be the probability that the seller will be making offers. In this formulation, the offer-maker has the bargaining power, and the weak bidder has more bargaining power, the more likely he is the offer-maker. Consider the common value function

$$w(t_1, t_2) = (1 - q)w_s(t_1, t_2) + qw_p(t_1, t_2) \quad (13)$$

where  $q$  represents the bargaining power of bidder one. The equilibrium of the common-value model with this common-value function is the same as the auction with resale in which the offer-maker is stochastically determined. The higher  $q$  is, the higher the transaction price in the resale stage. Gupta and Lebrun (1999) have noted that as  $q$  changes from 0 to 1, there is a reversal between the ranking of first-price auction and second price auction with resale. There is  $q_0 \in (0, 1)$  such that the two auctions yield the same revenue. This implies that  $R^F > (<)R^S$  if and only if  $q > (<)q_0$ .

There is a similar result for the revenue comparisons between the auction with resale and auction without resale.

**Theorem 10** *For a weak-strong pair, the auctioneer's revenue is an increasing function of the bargaining power  $q$  of the weak bidder. There exists  $q_1 \in (0, 1)$  such that  $R^r > (<)R^o$  if and only if  $q > (<)q_1$ .*

Lebrun (2007) shows that the revenue for the monopoly market is higher than that for the monopsony market. This accords with our intuition of the bargaining power effect. Let the auction with monopoly resale be represented by the common-value function  $w_p$ , and the auction with monopsony resale be represented by the common-value  $w_s$ . Then the auction with resale in which bidder one is the offer-maker with probability  $q$ . Then we also know that the revenue is an increasing function of  $q$ .

In the following subsections, we examine the resale markets in more details and endogenize the determination of the bargaining power by many factors in the bargaining process.

## 6.1 Bargaining power and delay costs

In a bilateral trade mechanism, If we allow repeated offers with no commitment, it is well-known (Sobel and Takahashi (1983), Fudenberg and Tirole (1983)) that high delay costs weaken the bargaining power of the monopolist. The weakened bargaining power may lead to low trade prices when the auction winner makes offers to the loser. We show by an example that the auction with resale may yield lower revenue than the the auction without resale when the bargaining power is substantially reduced in bargaining with repeated offers.

The resale market is one in which the winner of the auction makes a first offer to the buyer. If the offer is accepted, the game ends. If the first offer is rejected, a second final offer is made to the buyer. The bargaining problem with repeated offers from one-side to the other with delay costs is similar to that of Sobel and Takahashi (1983). However, there is a main difference: the seller may have different non-zero costs (or valuations) and different valuations imply different beliefs about the buyer's valuations due to the outcome of the first-stage auction. The delay costs are expressed by discount factors  $\delta_1, \delta_2$  for bidder one, two respectively. Our example assumes that bidder one has low  $\delta_1$  (close to 0), and bidder two has high  $\delta_2$  (close to 1). Note that when there are delay costs in repeated offers, the common-value is the first offer price and later offers are not involved in the equilibrium revenue.

Consider the weak-strong pair of bidders  $v_1(t) = t, v_2(t) = 1.5t$  over  $[0, 1]$ . There are only two rounds of offers. For the example, we adopt the notations  $x, y$  for  $x_i, x_j$  respectively. We have  $F_1(x) = x, F_2(y) = \frac{2}{3}y$ . In equilibrium, bidder one with valuation  $x$  believes that bidder two valuation distribution is  $F_2|_{1.5x}$ , after she wins the auction. We let  $y = 1.5x$ . Given the first price offer  $p_1$ , bidder two has a threshold of acceptance  $z$ . The offer will be accepted if and only if bidder two's valuation is higher than  $z$ . When bidder two rejects the offer, the equilibrium period two offer is given by  $p_2(x, z) = \frac{x+z}{2}$ . The following equation determines the equilibrium  $z$

$$z - p_1 = \delta_2(z - \frac{z+x}{2}),$$

and we have

$$z = \frac{p_1 - 0.5\delta_2x}{1 - 0.5\delta_2}.$$

The optimal first offer  $p_1$  maximizes the profit function

$$\begin{aligned} \frac{2}{3}(y-z)(p_1-x) + \frac{2}{3}\delta_1(z-p_2)(p_2-x) &= \frac{2}{3}(y-z)(p_1-x) + \frac{2}{3}\frac{\delta_1}{4}(z-x)^2 \\ &= \frac{2}{3}(y - \frac{p_1 - 0.5\delta_2x}{1 - 0.5\delta_2})(p_1-x) + \frac{2}{3}\frac{\delta_1}{4}(\frac{p_1-x}{1 - 0.5\delta_2})^2. \end{aligned}$$

The first order condition for  $p_1$  is

$$y - \frac{2p_1 - (1 + 0.5\delta_2)x}{1 - 0.5\delta_2} + \frac{\delta_1}{2(1 - 0.5\delta_2)^2}(p_1 - x) = 0,$$

and we get the optimal first period offer

$$p_1(x, y, \delta_1, \delta_2) = \frac{(1 - 0.5\delta_2)^2}{2 - \delta_2 - 0.5\delta_1}y + \frac{1 - 0.5\delta_1 - 0.25\delta_2^2}{2 - \delta_2 - 0.5\delta_1}x.$$

where  $y = 1.5x$ .

Since the first price auction revenue  $R^F$  with resale is increasing in  $p_1$ , and  $p_1$  is increasing in  $\delta_1$ , and decreasing in  $\delta_2$ , we know that  $R^F$  is increasing in  $\delta_1$  and decreasing in  $\delta_2$ . Therefore we know that a higher delay cost (or lower bargaining power) for bidder one hurts the revenue in the first price auction, while the opposite is true for bidder two. When  $\delta_1 = 0, \delta_2 = 1$ , we have the lowest revenue in the first price auction. In this case, we have  $p(x, y) = \frac{1}{4}y + \frac{3}{4}x = 1.125x$ , hence

$$R^F = \int_0^1 2(1-t)1.125tdt = 0.375.$$

We now compute the revenue without resale. By Plum (1992), the equilibrium bidding strategies of the two bidders are given by

$$b_w(t) = \frac{9}{5} \frac{1 - \sqrt{1 - \frac{5}{9}t^2}}{t}, b_s(t) = \frac{6}{5} \frac{\sqrt{1 + \frac{5}{4}t^2} - 1}{t}.$$

with the maximum bid  $\frac{3}{5}$ . The inverse bidding functions can be easily computed as

$$\varphi_w(b) = \frac{18b}{9 + 5b^2}, \varphi_s(b) = \frac{12b}{9 - 5b^2}, b \in [0, \frac{3}{5}]$$

From this, we can compute the revenue of the auction without resale:

$$R = \int_0^{\frac{3}{5}} bd\left(\frac{18b}{9 + 5b^2} \frac{12b}{9 - 5b^2}\right) = \frac{3}{5} - \int_0^{\frac{3}{5}} \left(\frac{18b}{9 + 5b^2} \frac{12b}{9 - 5b^2}\right)db = 0.40462$$

which is higher than the revenue in the auction with resale.

## 6.2 Bargaining power in k-double auctions

The k-double auction with linear bidding strategies has been studied in Chatterjee and Samuelson (1983). In the bargaining after the auction, we have noted a difference from their model: the heterogeneous beliefs of the seller and the buyer. Because of this difference, the equilibrium strategy is different, but is still linear. In the standard bargaining with homogeneous beliefs, Williams (1987) has shown that such linear equilibrium satisfies efficient properties, and in experimental studies (Radner and Schotter (1989)), it seems to be the relevant one in practice. Williams (1987) also shows that the parameter  $k$  can be a proxy of the utility payoffs of the bargainer. Thus in this case,  $k$  can be a

proxy of the bargaining power as well. As  $k$  takes the extreme values 0, 1, we get the monopoly or monopsony market. Although other non-linear equilibria exist, the linear equilibrium case deserves special attention. With heterogeneous beliefs, it is desirable that the efficiency properties of Williams (1987) continue to hold, so that we can argue that the linear equilibrium is the one to study, as the bargainers will tend to adopt an efficient bargaining mechanism. We can not explore this question here, but will take it with a grain of faith that such equilibrium is important to study. It is also the only tractable one so far to analyze.

As another application of the bid equivalence, we consider the  $k$ -double resale market in which both the seller and the buyer make simultaneous offers. Let  $p_s, p_b$  be the seller and buyer offers respectively. The transaction takes place if and only if  $p_s \leq p_b$  at the price  $(1-k)p_s + kp_b$ . When  $k$  increases, the bargaining power shifts from the seller to the buyer. This may appear to be counter-intuitive as a higher  $k$  means a higher price for the seller, and the seller benefits if there is no change in the offer strategies. Chatterjee and Samuelson ((1983), example one and two) have shown that the seller profit is actually decreasing in  $k$  as the offer strategies do change when  $k$  changes. This result is for the case of homogeneous beliefs. We will show that with heterogeneous beliefs, the conclusion is the same. Notice however that the beliefs of the players are not homogeneous, and hence the solution we get is different from theirs. With the uniform distribution of the valuations, there exist a unique (piecewise) linear equilibrium in our model as in Chatterjee and Samuelson (1983).

In this framework, we show that the auctioneer's revenue is an increasing function of the bargaining power  $1 - k$  of the weak bidder. When  $k = 0, 1$  respectively, the equilibrium in the resale market is the same as the monopoly and monopsony market respectively. However, there is a trade-off between the revenue and efficiency. The efficiency, measured by the realized trade surplus of the auction with resale, is a decreasing function of the bargaining power  $1 - k$  of the weak bidder.

We first note the result of Lebrun (2007) that the auctioneer's revenue is higher in an auction with a monopoly resale market than with a monopsony resale market.

Assume that the signals are independent and the valuations are given by<sup>18</sup>  $v_1(t) = t, v_2(t) = 2t$  so that  $F_1(x) = x, F_2(x) = \frac{x}{2}$ . The first stage is a first-price auction. We have the following result. Let  $R^F(k)$  be the revenue of the first-price auction with the  $k$ -double auction resale market. Let the  $E(k)$  be the efficiency of the auction measured by the realized surplus of the auction with resale. We also have exploit solution of the equilibrium bidding strategy of the auction with resale.

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<sup>18</sup>More general parameters can be allowed without affecting the results as long as the distributions are uniform.

**Proposition 11** *The revenue  $R^F(k)$  is a decreasing function of  $k$ , while the efficiency  $E(k)$  is increasing in  $k$ . The equilibrium bidding strategy is given by*

$$\begin{aligned} b_1(t) = b_2(t) &= \frac{3}{4}t \text{ for } t \leq \frac{3-k}{4-k}, \\ &= \frac{3-k}{2} \left[ k + \frac{1-k}{2}t - \frac{k(3-k)}{2(4-k)t} \right] \text{ for } t \geq \frac{3-k}{4-k}. \end{aligned} \quad (14)$$

**Proof.** In the resale game, let  $p_s, p_b$  be the offer price by the seller and buyer respectively. The transaction takes place if and only if  $p_s \leq p_b$ , and the transaction price is given by

$$p = (1-k)p_s + kp_b.$$

■

Let the inverse bidding strategy in the first-price auction with resale be  $\phi_1, \phi_2$  and in equilibrium we have  $\phi_2(b) = 2\phi_1(b)$  by the symmetry property. To find an equilibrium with linear strategies in the resale game, let  $p_s(v_1) = c_1v_1 + d_1, p_b(v_2) = c_2v_2 + d_2$  be the equilibrium strategies as functions of valuations. Bidder one with valuation  $v_1$  chooses  $p \leq 2c_2v_1 + d_2$  to maximize

$$\int_{\frac{p-d_2}{c_2}}^{2v_1} [(1-k)p + k(c_2v_2 + d_2) - v_1] dv_2.$$

The derivative of the payoff with respect to  $p$  is given by

$$\begin{aligned} &-\frac{p-v_1}{c_2} + (1-k) \int_{\frac{p-d_2}{c_2}}^{2v_1} dv_2 \\ &= \frac{1}{c_2} [-(2-k)p + (1+2(1-k)c_2)v_1 + (1-k)d_2], \end{aligned}$$

which is decreasing in  $p$ . Therefore the payoff function is concave. The first-order condition of optimality gives us

$$p_s(v_1) = \frac{1+2(1-k)c_2}{2-k}v_1 + \frac{1-k}{2-k}d_2$$

For the bidder two with valuation  $v_2$ , the price offer  $p \geq \frac{v_2}{2}c_1 + d_1$  maximizes

$$\int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} [v_2 - (1-k)(c_1v_1 + d_1) - kp] dv_1.$$

The first-order condition for the optimal offer is

$$\frac{v_2-p}{c_1} - k \int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} dv_1 = 0$$

or

$$\frac{1}{c_1} \left( \left(1 + \frac{kc_1}{2}\right)v_2 - (1+k)p + kd_1 \right) = 0,$$

and we have the optimal offer of the buyer

$$p_b(v_2) = \frac{1 + \frac{kc_1}{2}}{1+k}v_2 + \frac{k}{1+k}d_1.$$

To be an equilibrium, we must have

$$\begin{aligned} d_1 &= \frac{1-k}{2-k}d_2, d_2 = \frac{k}{1+k}d_1, \\ c_1 &= \frac{1+2(1-k)c_2}{2-k}, c_2 = \frac{1+\frac{kc_1}{2}}{1+k}. \end{aligned}$$

Solving the equations, we have

$$d_1 = d_2 = 0, c_1 = \frac{3-k}{2}, c_2 = \frac{4-k}{4}.$$

The (piecewise) linear equilibrium in the resale game is then given by

$$\begin{aligned} p_s(v_1) &= \frac{3-k}{2}v_1, v_1 \in [0, 1], \\ p_b(v_2) &= \frac{4-k}{4}v_2 \text{ for } v_2 \leq \frac{6-2k}{4-k}, \\ &= \frac{3-k}{2} \text{ for } v_2 > \frac{6-2k}{4-k}. \end{aligned}$$

Note that when  $k = 0$ , the strong bidder bids the true valuation when  $v_2 \leq \frac{6-2k}{4-k}$ , and the transaction takes place if and only if  $2v_1 \geq v_2 \geq p_s(v_1) = \frac{3}{2}v_1$ . This is equivalent to the monopoly case in which the seller offers the optimal monopoly price and the buyer accepts if and only if the buyer valuation is above the price. Similarly, when  $k = 1$ , the weak bidder bids the true valuation, and the transaction takes place if and only if  $\frac{1}{2}v_2 \leq v_1 \leq p_b(v_2) = \frac{3}{4}v_2$  when  $v_2 \leq \frac{6-2k}{4-k}$ . This is equivalent to the monopsony case in which the buyer offers the optimal monopsony price and the seller accepts if and only if the seller valuation is below the price.

The transaction price is given by

$$\begin{aligned} w(t_1, t_2) &= \frac{(1-k)(3-k)}{2}v_1(t_1) + \frac{k(4-k)}{4}v_2(t_2) = \frac{(1-k)(3-k)}{2}t_1 + \frac{k(4-k)}{2}t_2 \text{ if } v_2(t_2) \leq \frac{6-2k}{4-k}, \\ &= \frac{3-k}{2}((1-k)t_1 + k) \text{ if } v_2(t_2) > \frac{6-2k}{4-k}. \end{aligned}$$

Here  $Q = \{(t_1, t_2) : t_1 \geq t_2, \min(\frac{4-k}{4}v_2(t_2), \frac{3-k}{2}) \geq \frac{3-k}{2}v_1(t_1)\}$ , or  $Q = \{(t_1, t_2) : t_1 \geq t_2 \geq \frac{3-k}{4-k}t_1\}$ . Trade occurs with probability one if and only if  $(t_1, t_2) \in Q$ , and there is no trade outside  $Q$ . This means that the object is in the hand of

the strong bidder if and only if  $v_2 \geq \frac{6-2k}{4-k}v_1$ . The slope  $\frac{6-2k}{4-k} = 2(1 - \frac{1}{4-k})$  is decreasing in  $k$ . This implies that the efficiency of the auction increases as  $k$  becomes larger with the highest efficiency for the monopsony market.

We have

$$\begin{aligned} w(t, t) &= [(1-k)(3-k) + k(4-k)] \frac{t}{2} = \frac{3}{2}t \text{ for } t \leq \frac{3-k}{4-k} \\ &= \frac{3-k}{2}((1-k)t + k) \text{ for } t \geq \frac{3-k}{4-k}. \end{aligned}$$

We want to show that  $w(t, t)$  is decreasing in  $k$  when  $t \geq \frac{3-k}{4-k}$ . The derivative with respect to  $k$  is

$$\begin{aligned} \frac{3-k}{2}(1-t) - \frac{(1-k)t+k}{2} &= \frac{3-2k}{2} - (2-k)t \leq \frac{3-2k}{2} - \frac{(2-k)(3-k)}{4-k} \\ &= \frac{-k}{2(4-k)} < 0, \end{aligned}$$

hence  $p(t, t)$  is decreasing on the range  $t \geq \frac{3-k}{4-k}$ . Since the revenue of the first-price auction with resale is

$$2 \int_0^1 (1-t)p(t, t)dt,$$

the revenue is a decreasing function of  $k$ .

We have the equilibrium bidding strategies:

$$b_1(t_1) = \frac{1}{t_1} \int_0^{t_1} w(t, t)dt = \frac{1}{t_1} \int_0^{t_1} \frac{3}{2}t dt = \frac{3}{4}t_1, \text{ for } t_1 \leq \frac{3-k}{4-k}.$$

For  $t_1 \geq \frac{3-k}{4-k}$ , we have

$$\begin{aligned} b_1(t_1) &= \frac{3(3-k)^2}{4(4-k)^2 t_1} + \frac{1}{t_1} \int_{\frac{3-k}{4-k}}^{t_1} \frac{3-k}{2}((1-k)t + k)dt \\ &= \frac{3(3-k)^2}{4(4-k)^2 t_1} + \frac{3-k}{2t_1} \left[ \frac{1-k}{2}t_1^2 + kt_1 - \frac{1-k}{2} \left( \frac{3-k}{4-k} \right)^2 - \frac{k(3-k)}{4-k} \right] \\ &= \frac{3-k}{2} \left[ k + \frac{1-k}{2}t_1 - \frac{k(3-k)}{2(4-k)t_1} \right]. \end{aligned}$$

Next we show that Theorem ?? can be applied to this  $k$ -double auction resale market.

Theorem ?? can be applied to the  $k$ -double auction model. For instance, in the  $\frac{1}{2}$ -double auction example, for  $F_1(x_1) = F_2(x_2), x_2 \leq 1$ , we have

$$p(x_1, x_2) = \frac{5}{8}x_1 + \frac{5}{16}x_2 = \frac{5}{4}x_1 = \frac{p(x_1, x_1) + p(x_2, x_2)}{2},$$

and condition (C') holds for this example. In fact, it holds for all  $k$ -double auctions for this example.

In a  $k$ -double auction, if the winning bid is announced, the valuation of the winner is known in the resale stage. As a result, the equilibrium in the resale game is the same as the equilibrium in the monopoly market. This implies that the auctioneer revenue is higher when the winning bid is announced. Similarly, if the losing bid is announced, then the revenue is lower compared to the  $k$ -double auction.

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Appendix:

Proof of Proposition 9:

Let the number of periods remaining be  $k$ , and denote the optimal offer by  $p_k$ . The updated belief of the highest valuation  $z_k$  of the buyer is the threshold of acceptance in the period before. By backward induction,  $p_k$  depends only on  $x, z_k$ , and we use the notation  $p_k(x, z_k)$ . Let  $\pi_k(x, z_k)$  be the expected profit function when  $k$  periods are remaining. Again by backward induction,  $z_k$  depends only on  $x$  and  $z_{k+1}$ . Given  $p_k, p_{k-1}$ , bidder two has a threshold level of acceptance  $z_{k-1}$ . Bidder two will accept the offer  $p_k$  whenever his or her valuation is above  $z_{k-1}$ . Given  $p_k, p_{k-1}$ , we can determine  $z_{k-1}$  by the condition

$$z_{k-1} - p_k = \delta_2(z_{k-1} - p_{k-1})$$

Thus we have the equation

$$(1 - \delta_2)z_{k-1} + \delta_2 p_{k-1} = p_k \tag{15}$$

If the offer  $p_k$  is rejected, the bidder  $i$  updates his belief of the valuation of bidder  $j$ , and the new highest (lowest) valuation of the buyer (seller) is now  $z_{k-1}$ . Let  $p_{k-1}(x_i, z_{k-1})$  be the optimal offer with  $k-1$  periods remaining with the updated  $z_{k-1}$ . We can rewrite (15) as

$$(1 - \delta_2)z_{k-1} + \delta_2 p_{k-1}(x, z_{k-1}) = p_k \quad (16)$$

If the optimal offer  $p_{k-1}$  with  $k-1$  periods remaining has been determined by backward induction and is increasing in  $z_{k-1}$ . The left-hand side of (16) is increasing in  $z_{k-1}$ , and there is a unique solution denoted by  $z_{k-1}(x_i, p_{k-1})$ . Thus we know how  $z_{k-1}$  is determined once  $p_k$  is chosen.

The choice of  $p_k$  is determined by the maximization of the profit function of the seller given by

$$[F_2(z_k) - F_2(z_{k-1}(x, p_k))](p_k - x) + \delta_1 \pi_{k-1}(x, z_{k-1}) \quad (17)$$

The first order condition for  $p_k$  is

$$F_2(z_k) - F_2(z_{k-1}) - f_2(z_{k-1})(p_k - x) \frac{\partial z_{k-1}}{\partial p_k} + \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}} \frac{\partial z_{k-1}}{\partial p_k} = 0.$$

Take the implicit derivative of (15) with respect to  $p_k$ , we have

$$(1 - \delta_2) \frac{\partial z_{k-1}}{\partial p_k} + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}} \frac{\partial z_{k-1}}{\partial p_k} = 1,$$

or

$$\frac{\partial z_{k-1}}{\partial p_k} = \frac{1}{(1 - \delta_2) + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}}. \quad (18)$$

Substitute (18) into the first order condition, we have

$$F_2(z_k) - F_2(z_{k-1}) - \frac{f_2(z_{k-1})(p_k - x) - \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}}}{(1 - \delta_2) + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}} = 0.$$

For uniform distributions, we have  $f_2 = 1$ . Hence we have the first order condition

$$z_k - z_{k-1} - \frac{p_k - x - \delta_1 \frac{\partial \pi_{k-1}}{\partial z_{k-1}}}{(1 - \delta_2) + \delta_2 \frac{\partial p_{k-1}}{\partial z_{k-1}}} = 0 \quad (19)$$

When  $k = 1$ , we have

$$p_1(x, y) = \frac{x + y}{2}, \pi_1(x, y) = \left(\frac{y - x}{4}\right)^2$$

and  $p_1(x, z_1) = \frac{x + z_1}{2}, \pi_1(x, z_1) = \left(\frac{z_1 - x}{2}\right)^2$ . Hence

$$\frac{\partial p_1}{\partial z_1} = \frac{1}{2}, \frac{\partial \pi_1}{\partial z_1} = \frac{z_1 - x}{2}.$$

The theorem holds for  $k = 1$  with  $c_1 = \frac{1}{2}$ . More generally, by mathematical induction, assume that the theorem holds for  $k - 1$ , and we have

$$p_{k-1} = c_{k-1}z_{k-1} + (1 - c_{k-1})x, \pi_{k-1} = 0.5c_{k-1}(z_{k-1} - x)^2$$

$$\frac{\partial p_{k-1}}{\partial z_{k-1}} = c_{k-1}, \frac{\partial \pi_{k-1}}{\partial z_{k-1}} = c_{k-1}(z_{k-1} - x).$$

The first order condition (19) for  $z_{k-1}, p_k$  is

$$y - z_{k-1} = \frac{(1 - \delta_2)z_{k-1} + \delta_2(c_{k-1}z_{k-1} + (1 - c_{k-1})x) - x - \delta_1c_{k-1}(z_{k-1} - x)}{1 - \delta_2 + \delta_2c_{k-1}}$$

or

$$y - z_{k-1} = \frac{(1 - \delta_2 + \delta_2c_{k-1} - \delta_1c_{k-1})(z_{k-1} - x)}{1 - \delta_2 + \delta_2c_{k-1}}$$

or

$$\frac{2(1 - \delta_2 + \delta_2c_{k-1}) - \delta_1c_{k-1}}{1 - \delta_2 + \delta_2c_{k-1}}z_{k-1} = y + \frac{1 - \delta_2 + \delta_2c_{k-1} - \delta_1c_{k-1}}{1 - \delta_2 + \delta_2c_{k-1}}x$$

and we have

$$z_{k-1} = \frac{1 - \delta_2 + \delta_2c_{k-1}}{2(1 - \delta_2 + \delta_2c_{k-1}) - \delta_1c_{k-1}}y + \frac{1 - \delta_2 + \delta_2c_{k-1} - \delta_1c_{k-1}}{2(1 - \delta_2 + \delta_2c_{k-1}) - \delta_1c_{k-1}}x.$$

Let

$$d_{k-1} = \frac{1 - \delta_2 + \delta_2c_{k-1}}{2(1 - \delta_2 + \delta_2c_{k-1}) - \delta_1c_{k-1}},$$

then

$$z_{k-1} = d_{k-1}y + (1 - d_{k-1})x.$$

We have

$$p_k = (1 - \delta_2)z_{k-1} + \delta_2p_{k-1} = (1 - \delta_2)z_{k-1} + \delta_2(c_{k-1}z_{k-1} + (1 - c_{k-1})x)$$

$$= (1 - \delta_2 + \delta_2c_{k-1})z_{k-1} + \delta_2(1 - c_{k-1})x = c_k y + (1 - c_k)x$$

where

$$c_k = \frac{(1 - \delta_2 + \delta_2c_{k-1})^2}{2(1 - \delta_2 + \delta_2c_{k-1}) - \delta_1c_{k-1}} = (1 - \delta_2 + \delta_2c_{k-1})d_{k-1}.$$

The expected profit can be written as

$$\begin{aligned} \pi_k &= (y - z_{k-1})(p_k - x) + \delta_1\pi_{k-1} \\ &= c_k(1 - d_{k-1})(y - x)^2 + 0.5\delta_1c_{k-1}(z_{k-1} - x)^2 \\ &= (y - x)^2(c_k - c_kd_{k-1} + 0.5\delta_1c_{k-1}d_{k-1}^2) \\ &= (y - x)^2(c_k - (1 - \delta_2 + \delta_2c_{k-1})d_{k-1}^2 + 0.5\delta_1c_{k-1}d_{k-1}^2) \\ &= (y - x)^2(c_k - 0.5d_{k-1}^2(2(1 - \delta_2 + \delta_2c_{k-1}) - \delta_1c_{k-1})) \\ &= (y - x)^2(c_k - 0.5\frac{(1 - \delta_2 + \delta_2c_{k-1})^2}{2(1 - \delta_2 + \delta_2c_{k-1}) - \delta_1c_{k-1}}) \\ &= (y - x)^2(c_k - 0.5c_k) = 0.5c_k(y - x)^2. \end{aligned}$$

By mathematical induction, the proof is complete.