

# Returns to scale in the generation map of repeated games

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November 15, 2011

## Abstract

This paper identifies a notion of decreasing returns to scale in the generation map of repeated games,  $B$ . Many continuous stage games, including standard oligopoly models, satisfy the condition at least for some classes of equilibria. We deduce two key implications for the set of equilibrium payoffs. First, in the infinitely repeated game, that set varies continuously in the discount factor. Second, in the finitely repeated game with any discount factor, equilibrium unraveling is not robust: a small perturbation of the long-but-finitely-repeated game is sufficient to restore nearly all equilibrium payoffs of the corresponding infinitely repeated game having the same discount factor. These properties have been previously sought, but sufficient conditions were unknown.

This paper identifies a notion of decreasing returns to scale in the generation map of repeated games,  $B$ . Introduced by Abreu, Pearce, and Stacchetti (1990) in the spirit of dynamic programming,  $B$  provides a recursive characterization of the set of equilibrium payoffs of a repeated game. If the set of equilibrium continuation payoffs beginning next period is  $W$ , then the set of equilibrium payoffs this period is  $B(W)$ . Here we focus on the case of perfect monitoring, in which case the map  $B$  is determined by the stage game, the discount factor, and the equilibrium solution concept. A number of results in the modern theory of dynamic

games employ the analytic framework of  $B$ .<sup>1</sup> The map  $B$  underlies methods for computing the set of equilibrium payoffs,  $\mathcal{E}$ . In addition, one can deduce properties of  $\mathcal{E}$  from conditions on  $B$ . For example, Abreu, Pearce, and Stacchetti show that the set of equilibrium payoffs  $\mathcal{E}$  is monotone increasing in the discount factor given a condition on  $B$  (that it is convex-valued). This paper is in that vein. We present a new condition on  $B$ , strictly decreasing returns to scale, and from that condition we deduce multiple important properties of the set of equilibrium payoffs of the repeated game. These properties have been previously sought, but sufficient conditions were unknown.

Our main results require that  $B$  exhibits a form of strictly decreasing returns to scale, a condition which we abbreviate as SDRS. We show that many continuous games, like standard oligopoly models, satisfy the condition at least in regards to certain classes of equilibria. In contrast, games with finite action spaces do not satisfy the condition except in degenerate cases. We return to this after describing two key implications of the SDRS condition.

First, we show that SDRS implies that the set of equilibrium payoffs of the infinitely repeated game,  $\mathcal{E}_\delta$ , varies continuously in the discount factor  $\delta$ . This means that if  $\delta$  and  $\delta'$  are near enough, then  $\mathcal{E}_{\delta'}$  and  $\mathcal{E}_\delta$  are of similar size. Absent SDRS, it is known that  $\mathcal{E}_\delta$  might be discontinuous. For example, in the prisoner's dilemma there is some minimum value  $\delta^*$  at which players may cooperate in equilibrium. For all smaller values,  $\mathcal{E}_\delta$  includes only the joint defection payoff. At the value  $\delta^*$ ,  $\mathcal{E}_\delta$  explodes.<sup>2</sup> For continuous stage games,  $\mathcal{E}_\delta$  might instead vary continuously in  $\delta$ . For example, consider a continuous oligopoly stage game. Again there is some minimum discount factor  $\delta^*$  such that the firms can achieve the monopoly profit in equilibrium. For  $\delta$  just smaller than  $\delta^*$ , one might anticipate that

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<sup>1</sup>A selection of results based on  $B$  and its extensions follows. Fudenberg, Levine, and Maskin (1994) prove a folk theorem for repeated games of imperfect public monitoring. Judd, Yeltekin, and Conklin (2003) and Abreu and Sannikov (2011) provide refined algorithms for computing the set of equilibrium payoffs. Athey and Bagwell (2008) characterize the set of equilibrium in a dynamic oligopoly model with varying private information. Cherry and Smith (2010) describe an upper bound on the set of equilibrium payoffs attainable in a repeated game with any private monitoring structure.

<sup>2</sup>Without SDRS,  $\mathcal{E}_\delta$  is upper hemicontinuous but not in general lower hemicontinuous. Stahl (1991) describes the graph of  $\mathcal{E}_\delta$  for the prisoner's dilemma, including this lower hemidiscontinuity. Though they are not as simple to describe, lower hemidiscontinuities of  $\mathcal{E}_\delta$  may occur even for continuous stage games.

the firms can achieve almost the monopoly profit in equilibrium. At first glance it might appear that this will be so simply because the stage game is continuous, however that is not sufficient but SDRS is. We are not aware of any previous result establishing conditions under which  $\mathcal{E}_\delta$  varies continuously. Continuity of  $\mathcal{E}_\delta$  is a basic technical property; it may also be directly important in interpreting oligopoly behavior.<sup>3</sup>

Second, we show that SDRS implies that equilibrium unraveling in the finitely repeated game is not robust for all discount factors. Recall that if the stage game has a unique equilibrium payoff, which we normalize to zero, there is complete unraveling:  $\mathcal{E}_{T,\delta} = 0$  for all finite  $T$ . Such unraveling has been viewed with concern at least since Luce and Raiffa (1957). More generally, even for large  $T$ , the set  $\mathcal{E}_{T,\delta}$  of equilibrium payoffs of the  $T$ -times repeated game may be much smaller than the set  $\mathcal{E}_{\infty,\delta}$  of equilibrium payoffs of the corresponding infinitely repeated game. In response, a number of finite-horizon folk theorems establish conditions under which such unraveling is absent in the limit as  $\delta$  approaches 1. We show that if SDRS is satisfied in addition to conditions similar to those of the finite-horizon folk theorems, then such unraveling will be absent for all values of  $\delta$ . For example, suppose that the stage game has two equilibrium payoffs  $v$  and  $v'$ , of which all players strictly prefer  $v$  to  $v'$ . In that case the finite-horizon folk theorem of Benoit and Krishna (1985, 2000) implies that if  $T$  and  $\delta$  are both large enough, then  $\mathcal{E}_{T,\delta}$  is nearly as large  $\mathcal{E}_{\infty,\delta}$ ; we say that there is complete un-unraveling asymptotically as  $\delta$  approaches 1. However, for any fixed value of  $\delta \in [0, 1)$ ,  $\mathcal{E}_{T,\delta}$  might still be much smaller than  $\mathcal{E}_{\infty,\delta}$  for all finite  $T$ . We show that given SDRS, this will generally not be so, instead there will be complete un-unraveling for all  $\delta$ .

Bernheim and Dasgupta (1995) present a folk theorem for asymptotically finite-horizon games. Again they find complete un-unraveling asymptotically as  $\delta$  approaches 1. They do ask whether there might also be complete un-unraveling for all  $\delta$ . In general, the answer may

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<sup>3</sup>Suppose that we observe a discontinuous drop in the average price level in some industry. Absent a discontinuous change in some other parameter, such as production cost or  $\delta$ , one might infer that the price drop is due to a shift from a collusive phase to a “price war” phase. However, one cannot be confident in that inference absent an understanding that the collusive outcome itself would vary continuously in the parameters. Otherwise it could be that a tiny decrease in  $\delta$  results in a large drop in the largest price achievable in collusive equilibrium.

be negative, and Bernheim and Dasgupta construct a game where that is so. We answer in the affirmative given our SDRS condition. In their conclusion, Bernheim and Dasgupta conjecture that the answer would be affirmative given an alternative condition on  $B$ .<sup>4</sup> This paper is partly in the spirit of that conjecture, though we find it to be mistaken. Apart from that conjecture, we are not aware that others have considered whether  $B$  might satisfy similar conditions related to SDRS.

That the issue of returns to scale matters is familiar from other areas of economics. However, regarding the map  $B$  it is not clear what decreasing returns to scale (abbreviated to DRS) ought to mean or in which circumstances it would hold. Here in the introduction we assume that the stage game has a unique equilibrium payoff, which we normalize to zero. In that case we say that  $B$  exhibits DRS if it is subhomogeneous, that is  $\theta B(W) \subset B(\theta W)$  for all  $W \subset \mathbf{R}^n$  and  $\theta \in (0, 1]$ . The results mentioned above require that  $B$  exhibits strictly decreasing returns to scale, not for all sets  $W$  but particularly for the set of equilibrium payoffs of the infinitely repeated game,  $\mathcal{E}_\infty$ .<sup>5</sup>

Whether or not  $B$  exhibits (strictly) decreasing returns to scale depends on how fast  $B(W)$  grows as  $W$  grows. Both  $W$  and  $B(W)$  are sets of points in  $\mathbf{R}^n$ , which complicates the matter. Given that the stage game has a unique equilibrium payoff, normalized to zero, we have  $B(0) = 0$ . For the set of equilibrium payoffs of the infinitely repeated game  $\mathcal{E}$  we also have  $0 \in \mathcal{E}$  and  $B(\mathcal{E}) = \mathcal{E}$ . For  $\theta \in [0, 1]$ , consider the scalar multiple  $\theta\mathcal{E} = \{\theta v : v \in \mathcal{E}\}$ . For example, if  $\mathcal{E}$  is the disk of radius 1 centered at the origin, then  $\mathcal{E}/2$  is the disk of radius  $1/2$  centered at the origin. We compare  $B(\theta\mathcal{E})$  to  $\theta\mathcal{E}$ . We saw that those two are equal at both endpoints:  $\theta = 0$  and  $\theta = 1$ . At intermediate values, say  $\theta = 1/2$ , it could be that  $B(\mathcal{E}/2)$

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<sup>4</sup>Bernheim and Dasgupta propose that an un-unraveling result like ours would hold provided that  $B$  has no intermediate fixed points, that is none apart from  $\mathcal{E}$  and perhaps the set of stage game equilibrium payoffs. They also conjecture that  $B$  would satisfy that condition of no intermediate fixed points provided that the payoff function of the stage game is continuous with a globally concave payoff function. We find that both conjectures are mistaken, but related to SDRS. If  $B$  satisfies SDRS then it also satisfies their fixed-point condition, which is important but insufficient for the desired conclusion.

<sup>5</sup>We say that  $B$  exhibits strictly decreasing returns to scale along  $\mathcal{E}$  if for each  $\theta \in (0, 1)$  there exists  $\theta_+ > \theta$  such that  $\theta_+ B(\mathcal{E}) \subset B(\theta\mathcal{E})$ . Here “ $\subset$ ” denotes weak inclusion, which some instead write as “ $\subseteq$ ”.

contains  $\mathcal{E}/2$ , the latter contains the former, or that neither contains the other. Decreasing returns to scale implies the first option:  $\mathcal{E}/2 \subset B(\mathcal{E}/2)$ . Strictly decreasing returns to scale implies that  $(1 + \epsilon)\mathcal{E}/2 \subset B(\mathcal{E}/2)$  for some  $\epsilon > 0$ . Whether or not  $B$  exhibits (S)DRS depends on the stage game but not the discount factor or the horizon.

If the stage game  $G$  has a finite action space, then  $B$  exhibits a step-function-like property which precludes DRS. Consider the prisoner's dilemma and suppose that  $\delta$  is large enough so that  $\mathcal{E}$  includes the joint cooperation payoff. As  $\theta$  increases from zero toward one, we will come to some critical value such that  $\theta\mathcal{E}$  is just large enough to enforce joint cooperation. At that point,  $B(\theta\mathcal{E})$  will explode, which is incompatible with DRS. Stage games that exhibit DRS must have uncountably infinite action spaces, and are usually continuous, by which we mean that they have convex action spaces and continuous payoff functions.

Different solution concepts yield different versions of  $B$ . For example, if we restrict attention to pure strategy equilibrium then we get a different  $B$  from that if we allow mixed strategies. The two main results that we described before are agnostic about the class of equilibria to be considered. For example, if the pure-strategy version of  $B$  exhibits SDRS, then the set of pure-strategy equilibrium payoffs of the infinitely repeated game varies continuously in  $\delta$ . If the mixed-strategy version of  $B$  exhibits SDRS, then the set of mixed-strategy equilibrium payoffs varies continuously, and so on for a number of standard solution concepts. Here in the introduction we explore why SDRS may hold in a very simple case.

Consider pure, symmetric trigger strategies in a Cournot duopoly game. Given a discount factor  $\delta$ , let  $v^*(\delta)$  be the greatest collusive profit achievable in such an equilibrium. Suppose that each firm chooses production quantities out of some interval, not a discrete set. In this case, one might anticipate that  $v^*(\delta)$  varies continuously in  $\delta$ , so  $v^*(\delta_-)$  is not much smaller than  $v^*(\delta)$  given  $\delta_-$  just smaller than  $\delta$ . That would be so for the following reason. Let  $q^*$  be the collusive equilibrium output level that achieves the profit  $v^*(\delta)$ , and let  $q^c$  be the competitive level. For  $q$  strictly between  $q^*$  and  $q^c$ , one expects that  $q$  will be strictly easier to enforce than  $q^*$  in many cases. That is often so not only in absolute terms, but in the

following relative sense. Let  $\pi(q, q)$  be the profit that results when both firms produce the quantity  $q$  for one period; suppose it is quasi-concave in  $q$ . Let  $l(q, q)$  be each firm's potential gain in that period from deviation, that is  $l(q, q) = \max_{q'} \pi(q', q) - \pi(q, q)$ . Let  $(q^c, q^c)$  be the unique equilibrium of the stage game, so  $l(q^c, q^c) = 0$ . We find that in many Cournot games  $l(q, q) / (\pi(q, q) - \pi(q^c, q^c))$  is strictly increasing as  $q$  moves away from the competitive level  $q^c$  toward the lower monopoly level  $q^m$ . That is, quantities nearer the competitive level are relatively easier to enforce. We show this implies that the simply version of  $B$  for pure, symmetric trigger strategies exhibits SDRS, so  $v^*(\delta)$  does vary continuously by the first result described above. We generalize this argument to other stage games and to general pure strategies with public randomization.

## 1 Implications of decreasing returns to scale in $B$

In this section we describe what it means for  $B$  to exhibit strictly decreasing returns to scale and we derive two key implications of that property. Our results here rely on a definition of  $B$  comprising only the seven general properties further below, (A1)-(A7). However, we have a more specific situation in mind, which follows.

There is an  $n$ -player stage game  $G$ . Player  $i$  has action space  $A_i$ , and payoff function  $u_i : A \rightarrow \mathbf{R}$  where  $A = \times_i A_i$  is the space of action profiles. There is a corresponding repeated game  $G_\delta^T$ , where  $T$  is the finite or infinite number of repetitions and  $\delta \in (0, 1)$  is the discount factor. Let  $\mathcal{E}_{T,\delta} \subset \mathbf{R}^n$  be  $(1 - \delta)$  times the set of total discounted equilibrium payoff profiles of the repeated game. So  $\mathcal{E}_{\infty,\delta}$  is the set of average discounted equilibrium payoffs of the infinitely repeated game. In place of  $\mathcal{E}_{T,\delta}$ , we often write  $\mathcal{E}_T$ ,  $\mathcal{E}_\delta$  or simply  $\mathcal{E}$  when it is not necessary to make explicit the values of  $T$  or  $\delta$ .

Given a set of hypothetical continuation payoffs  $W \subset \mathbf{R}^n$ , consider a continuation payoff function  $w : A \rightarrow W$ , and consider the one-shot game with payoffs  $(1 - \delta)u(a) + \delta w(a)$ . If  $a$  is an equilibrium of some such game then we say it is *enforced* by  $W$ , and if  $v$  is an equilibrium

payoff profile of such a game we say it is *generated* by  $W$ . The output of the generation map,  $B_\delta(W)$ , is the set of payoffs generated by  $W$ . We often drop the subscript  $\delta$  on  $B$ . The set of equilibrium payoffs of the  $(T + 1)$ -times repeated game is generated by the set of equilibrium payoffs of the  $T$ -times repeated game, that is,  $\mathcal{E}_{T+1} = B(\mathcal{E}_T)$ . Further, for the infinitely repeated game,  $\mathcal{E}_\infty$  is the largest, bounded self-generating set: if  $W$  is bounded and  $W \subset B(W)$  then  $W \subset \mathcal{E}_\infty$ . It is also a fixed point:  $\mathcal{E}_\infty = B(\mathcal{E}_\infty)$ . These last two properties are central results of Abreu, Pearce, and Stacchetti (1990).

Provided some mild regularity conditions on  $G$ ,  $\mathcal{E}_\infty$  is compact and  $B$  preserves compactness. For example, we may assume that either  $A$  is finite and  $u$  is bounded, or  $A$  is compact and  $u$  is continuous, as in Mailath and Samuelson (2006). We assume that the set of equilibrium payoffs of the stage game,  $\mathcal{N}$ , is non-empty. Lastly, without loss of generality, we shift the payoff function  $u$  so that zero is a stage-equilibrium payoff, that is  $0 \in \mathcal{N}$ . Notice that  $B(0) = (1 - \delta)\mathcal{N}$ .

Our results here abstract from the description of  $B$  above. Instead we define  $B$  in terms of the seven general properties below. This perhaps offers some clarity on the mechanism of our results, but more importantly it allows us to remain agnostic about the class of strategies to be considered. For example, we can define a version of  $B$  for mixed strategies and a different version for pure strategies, and both of these versions of  $B$  will satisfy the following properties. In the next section we will focus on pure strategies with public randomization, and on strongly symmetric strategies.

**Definition.** For each  $\delta \in (0, 1)$ , the map  $B_\delta$  and the set  $\mathcal{E}_{\infty, \delta}$  jointly satisfy the following properties.

$$(A1) \ B_\delta : \{W \subset \mathbf{R}^n\} \rightarrow \{W \subset \mathbf{R}^n\}$$

$$(A2) \ B_\delta \text{ is monotone increasing in its argument: if } W' \subset W, \text{ then } B_\delta(W') \subset B_\delta(W)$$

$$(A3) \ \text{If } W \text{ is bounded and } W \subset B_\delta(W) \text{ then } W \subset \mathcal{E}_{\infty, \delta}$$

$$(A4) \ \mathcal{E}_{\infty, \delta} = B_\delta(\mathcal{E}_{\infty, \delta})$$

$$(A5) \ B_\delta \text{ preserves compactness, and } \mathcal{E}_{\infty, \delta} \text{ is compact}$$

$$(A6) \ 0 \subset B_\delta(0)$$

$$(A7) \ B_{\delta'}(W) = \frac{1-\delta'}{1-\delta} B_\delta \left( \frac{\delta'}{1-\delta'} \frac{1-\delta}{\delta} W \right)$$

The first two properties are basic. The third and fourth are the key properties of  $B$  found by Abreu, Pearce, and Stacchetti. The fifth requires weak regularity conditions on  $G$  as we mentioned before. Central to our presentation, the property  $0 \subset B(0)$  holds having normalized zero to be an equilibrium payoff of the stage game. The last property describes the relationship between  $B$  for different discount factors.

Our results here rely on a notion of strictly decreasing returns to scale in  $B$ , which we abbreviate as SDRS. In the next section, we will describe various conditions under which this SDRS condition is satisfied.

Recalling a bit of producer theory, let  $x \geq 0$  be a vector of production input quantities and  $f(x) \geq 0$  the quantity of the single output. We would say that the production technology  $f$  exhibits decreasing returns to scale if  $\theta f(x) \leq f(\theta x)$  for all  $\theta \in (0, 1)$  and all  $x \geq 0$ . That corresponds to  $f$  being subhomogeneous. An equivalent condition is that  $f(\theta x)/\theta$  is non-increasing in  $\theta > 0$ . Given that  $f(0) = 0$ , it would suffice that  $f$  is concave in  $x$ , or, more weakly, that  $f(\theta x)$  is concave in  $\theta$ . (What we call decreasing returns to scale, some might call weakly decreasing returns to scale or non-increasing returns to scale.)

Where the production function  $f$  takes points in  $\mathbf{R}_+^n$  and returns points in  $\mathbf{R}_+$ , the generation map  $B$  takes sets in  $\mathbf{R}^n$  and returns the same. Still, we can define returns to scale similarly. We say that  $B$  exhibits *decreasing returns to scale* (DRS) if  $\theta B(W) \subset B(\theta W)$  for all  $\theta \in (0, 1]$  and all sets  $W \subset \mathbf{R}^n$ . (By “ $\subset$ ”, we mean weak set inclusion, which some write “ $\subsetneq$ ”. By  $\theta B$  and  $\theta W$  we mean the product,  $\theta W = \{\theta w : w \in W\}$ .) An equivalent condition is that  $B(\theta W)/\theta$  is non-increasing in  $\theta$ . Setting  $\theta = 0$  in the definition of DRS implies  $0 \subset B(0)$ , which is the normalization we have mentioned previously. Absent that normalization, the condition corresponding to DRS would become more complicated.<sup>6</sup>

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<sup>6</sup>Suppose instead that the set of stage equilibrium payoffs,  $\mathcal{N}$ , is a singleton, but we do not normalize it to zero. Then we would say that  $B$  exhibits DRS if  $\theta B(W) + (1 - \theta)\mathcal{N} \subset B(\theta W + (1 - \theta)\mathcal{N})$ . That condition

Our results here do not require that  $B$  exhibits decreasing returns to scale globally as we have just described, but only along certain paths:

**Definition 1** (DRS).  $B$  exhibits *decreasing returns to scale along*  $W \subset \mathbf{R}^n$  if for all  $\theta \in [0, 1]$ ,  $\theta B(W) \subset B(\theta W)$ .

Our results will require that  $B$  exhibits DRS particularly along the set of equilibrium payoffs of the infinitely repeated game,  $\mathcal{E}$ . The following lemma is immediate.

**Lemma 1.** *If  $B$  exhibits DRS along  $\mathcal{E}$ , then  $\mathcal{E}$  is radially convex, that is  $\theta\mathcal{E} \subset \mathcal{E}$  for all  $\theta \in [0, 1]$ .*

(Instead of “radially convex” some might say “star-shaped about 0”. A convex set containing the origin meets this condition, but not the converse.)

*Proof.* Let  $\theta \in [0, 1]$ . The set  $\mathcal{E}$  has the property  $\mathcal{E} = B(\mathcal{E})$ . DRS then implies that  $\theta\mathcal{E}$  is self-generating:  $\theta\mathcal{E} = \theta B(\mathcal{E}) \subset B(\theta\mathcal{E})$ .

Because  $\mathcal{E}$  is the largest, bounded self-generating set (A3), we have  $\theta\mathcal{E} \in \mathcal{E}$ . That this holds for all  $\theta \in [0, 1]$  is to say that  $\mathcal{E}$  is radially convex as desired.  $\square$

Our results will require that  $B$  exhibits a strict form of decreasing returns to scale.<sup>7</sup>

**Definition 2.**  $B$  exhibits *strictly decreasing returns to scale along*  $W \subset \mathbf{R}^n$  if for all  $\theta \in (0, 1)$ ,  $\theta_+ B(W) \subset B(\theta W)$  for some  $\theta_+ > \theta$ .

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is less easily recognized as decreasing returns to scale. If  $\mathcal{N} = 0$ , that condition reduces to the definition of DRS above,  $\theta B(W) \subset B(\theta W)$ . If  $\mathcal{N}$  is not a singleton but  $0 \in \mathcal{N}$ , that condition implies the definition of DRS above but not the converse.

If  $\mathcal{N}$  is not a singleton, there are multiple ways to normalize  $0 \in \mathcal{N}$ . We require only that the (S)DRS condition is met for some such normalization.

<sup>7</sup>One would say that the production technology  $f$  exhibits strictly decreasing returns to scale if  $\theta f(x) < f(\theta x)$ . There we have simply replaced weak with strict inequality. We might similarly strengthen DRS in  $B$  by replacing weak set inclusion “ $\subset$ ” with strict set inclusion “ $\subsetneq$ ” above, but that would not be sufficient for our purposes. We instead require that  $B(\theta W)$  is strictly greater than  $\theta B(W)$  in the sense that the former contains  $(1 + \epsilon)$  times the later for some  $\epsilon > 0$ . If  $B(W)$  is radially convex, then this is stronger than strict set inclusion.

## 1.1 Continuity of $\mathcal{E}$ in $\delta$

It is known that  $\mathcal{E}_\delta$ , the set of equilibrium payoffs of the infinitely repeated game with discount factor  $\delta$ , varies upper hemicontinuously in the discount factor.<sup>8</sup> However, it may not be lower hemicontinuous. As we mentioned in the introduction regarding the prisoner's dilemma,  $\mathcal{E}_{\delta'}$  could remain much smaller than  $\mathcal{E}_\delta$  even as  $\delta'$  converges to  $\delta$  from below. Here we first show that DRS implies that the set of total discounted equilibrium payoffs,  $\mathcal{E}_\delta/(1-\delta)$ , is monotone non-decreasing, which trivially implies lower hemicontinuity from the right. We then show that SDRS additionally implies lower hemicontinuity from the left. Lastly, we show that the converse is not true, but continuity together with monotonicity of  $\mathcal{E}_\delta/(1-\delta)$  do require that  $B$  exhibits a more general form of decreasing returns to scale.

We saw in the previous lemma that DRS of  $B$  along  $\mathcal{E}$  implies that  $\mathcal{E}$  is radially convex, from which we derive the following.

**Proposition 1.** *If  $\mathcal{E}_\delta$  is radially convex for all  $\delta$ , then  $\mathcal{E}_\delta/(1-\delta)$  is monotone non-decreasing in  $\delta$ .*

*Proof.* Let  $\delta_- < \delta$  and let  $\mathcal{E}_{\delta_-}$  be radially convex.

We have

$$\begin{aligned} \mathcal{E}_{\delta_-} &= B_{\delta_-}(\mathcal{E}_{\delta_-}) \\ &= \frac{1-\delta_-}{1-\delta} B_\delta \left( \frac{(1-\delta)}{(1-\delta_-)} \frac{\delta_-}{\delta} \mathcal{E}_{\delta_-} \right) \\ &\subset \frac{1-\delta_-}{1-\delta} B_\delta \left( \frac{(1-\delta)}{(1-\delta_-)} \mathcal{E}_{\delta_-} \right) \\ &\Rightarrow \frac{(1-\delta)}{(1-\delta_-)} \mathcal{E}_{\delta_-} \subset \mathcal{E}_\delta. \end{aligned}$$

The second step is simply the relationship between  $B_{\delta_-}$  and  $B_\delta$ . The third step follows from the monotonicity of  $B$ , noticing that  $\frac{(1-\delta)}{(1-\delta_-)} \frac{\delta_-}{\delta} \mathcal{E}_{\delta_-} \subset \frac{(1-\delta)}{(1-\delta_-)} \mathcal{E}_{\delta_-}$  due to its radial convexity. Rearranging the inequality of the third line shows that  $(1-\delta)\mathcal{E}_{\delta_-}/(1-\delta_-)$  is self-generating under  $B_\delta$ , which implies that it is contained in  $\mathcal{E}_\delta$  as desired.  $\square$

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<sup>8</sup>See for example Fudenberg and Maskin (1986), Theorem D.

If  $B$  is convex-valued, which is guaranteed if we allow public randomization, then a stronger result is known:  $\mathcal{E}_\delta$  is itself non-decreasing in  $\delta$ . (See Abreu, Pearce, and Stacchetti (1990, Theorem 6).) The weaker monotonicity result of the proposition above suffices to guarantee that  $\mathcal{E}$  is lower hemicontinuous from the right given DRS. The stronger condition, SDRS, implies that  $\mathcal{E}$  is additionally lower hemicontinuous from the left. DRS is not sufficient in place of SDRS for that result.<sup>9</sup>

**Theorem 1.** *If, for all  $\delta$ ,  $B_\delta$  exhibits SDRS along  $\mathcal{E}_\delta$ , then  $\mathcal{E}_\delta$  is lower hemicontinuous in  $\delta \in (0, 1)$ .*

*Proof.* Let  $\theta < 1$  and  $\delta \in (0, 1)$ . We will show that  $\theta\mathcal{E}_\delta \subset \mathcal{E}_{\delta'}$  for all  $\delta'$  in a neighborhood of  $\delta$ .

Consider any  $\delta_+ \in (\delta, 1)$ . We saw by the previous lemma and proposition  $\frac{1-\delta_+}{1-\delta}\mathcal{E}_\delta \subset \mathcal{E}_{\delta_+}$ .

Consider instead  $\delta_- < \delta$ . We have

$$\begin{aligned} B_{\delta_-}(\theta\mathcal{E}_\delta) &= \frac{1-\delta_-}{1-\delta}B_\delta\left(\frac{\delta_-}{1-\delta_-}\frac{1-\delta}{\delta}\theta\mathcal{E}_\delta\right) \\ &\supset \frac{\delta_-}{\delta}B_\delta(\theta\mathcal{E}_\delta) \\ &\supset \frac{\delta_-}{\delta}\theta_+B_\delta(\mathcal{E}_\delta) \\ &= \frac{\delta_-}{\delta}\theta_+\mathcal{E}_\delta \end{aligned}$$

where the first step is the relationship between  $B_{\delta_-}$  and  $B_\delta$ . The second follows from DRS. The third follows from SDRS for some  $\theta_+ > \theta$  that does not depend on  $\delta_-$ .

That implies  $\theta\mathcal{E}_\delta \subset B_{\delta_-}(\theta\mathcal{E}_\delta)$  provided  $\frac{\delta_-}{\delta}\theta_+ \geq \theta$ , that is  $\delta_- \in [\delta\theta/\theta_+, \delta)$ . In turn this implies  $\theta\mathcal{E}_\delta \subset \mathcal{E}_{\delta_-}$ .

We have seen that so long as  $\delta'$  is contained in a certain neighborhood of  $\delta$ , then  $\theta\mathcal{E}_\delta \subset \mathcal{E}_{\delta'}$ . We now show that this implies lower hemicontinuity of  $\mathcal{E}_\delta$  via the usual sequential characterization. Consider a sequence  $\delta_n \rightarrow \delta \in (0, 1)$ , where  $\delta_n \in (0, 1)$  for all  $n$ . Let  $v \in \mathcal{E}_\delta$ . We construct a sequence  $v_n \rightarrow v$  such that  $v_n \in \mathcal{E}_{\delta_n}$  for all  $n$  greater than some  $N$ . Recall  $\mathcal{E}$  is closed. Let

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<sup>9</sup>If  $B$  were to exhibit constant returns to scale, in the sense that  $B(\theta W) = \theta W$  for all  $W$ , then  $\mathcal{E}_\delta$  would be discontinuous. The Bertrand stage game with perfect competition and constant production costs yields a related example where  $B$  exhibits constant returns to scale up to a point, and  $\mathcal{E}_\delta$  varies discontinuously.

$\theta_n = \max\{\theta \leq 1 : \theta v \in \mathcal{E}_{\delta_n}\}$ , so  $\theta_n v \in \mathcal{E}_{\delta_n}$ . Because  $\delta_n \rightarrow \delta$ , for every neighborhood of  $\delta$  there exists an  $N$  such that for  $n \geq N$ ,  $\delta_n$  is in the specified neighborhood. Thus for each  $\theta < 1$ , there exists an  $N$  such that for  $n \geq N$ ,  $\theta \mathcal{E}_\delta \subset \mathcal{E}_{\delta_n}$ , so  $\theta_n \geq \theta$ . Thus  $\theta_n \rightarrow 1$ .  $\square$

We saw that SDRS implies that  $\mathcal{E}_\delta$  is continuous and  $\mathcal{E}_\delta/(1 - \delta)$  is monotone increasing in  $\delta$ , and radially convex; now we consider the converse. Public randomization suffices to guarantee that  $\mathcal{E}_\delta$  is monotone and (radially) convex. We can then examine whether additionally assuming continuity of  $\mathcal{E}_\delta$  implies some notion of decreasing returns to scale. That question is complicated by the fact that we only have a partial order on input sets  $W$  and identically the output sets  $B(W)$ . We cannot in general say how much larger one set is than another.

It is convenient to consider an operator related to  $B$  in place of  $B$  itself. Recall  $B(W)$  is the set of equilibrium payoffs of one-shot games with payoffs  $(1 - \delta)u(a) + \delta w(a)$  where  $w : A \rightarrow W$ . Let  $F(W)$  be the set of equilibrium payoffs of one-shot games with payoffs  $u(a) + w(a)$ . So  $F$  does not depend on  $\delta$ . Let  $V_\delta$  be the set of total discounted equilibrium payoffs, so  $V_\delta = \mathcal{E}_\delta/(1 - \delta)$ . Corresponding to the relationship  $B_\delta(\mathcal{E}_\delta) = \mathcal{E}_\delta$  we have  $F(\delta V_\delta) = V_\delta$ . It can be shown that  $B_\delta$  exhibits (S)DRS along  $\mathcal{E}_\delta$  if and only if  $F$  exhibits (S)DRS along  $\delta V_\delta$ .

Consider the relationship

$$F(\delta V_\delta) = V_\delta.$$

So long as  $V_\delta$  is monotone increasing in  $\delta$  that already implies a weak notion of decreasing returns to scale across certain points: While  $\delta V_\delta$  generates  $1/\delta$  times itself, there is a strictly smaller set that generates relatively more. For example,  $\frac{\delta}{2} V_{(\delta/2)}$  generates  $2/\delta$  times itself. Given radial convexity, if  $V_{\delta/2} \neq 0$  then it is in fact larger than  $\frac{\delta}{2} V_{(\delta/2)}$  in a satisfying sense. However, if  $V$  is not continuous in  $\delta$  it may be that there is no smaller set near  $\delta V_\delta$  that generates as much  $1/\delta$  times itself. In that case it would seem that  $F$  does not exhibit any notion of decreasing returns to scale locally at  $\delta V_\delta$ .

Suppose that  $V_\delta$  is not only monotone and radially convex, but also continuous in  $\delta$ . As  $\delta$  varies between zero and one,  $\delta V_\delta$  traces out a monotone, continuous path. Further, given an element on this path above zero, and any neighborhood of that element, there is a smaller element on the path which is in that neighborhood, and that smaller element generates relatively more. That is, we can pick  $\delta_-$  just below  $\delta$  so that  $\delta_- V_{\delta_-}$  is arbitrarily near to, but smaller than,  $\delta V_\delta$ , and while  $\delta V_\delta$  generates  $1/\delta$  times itself,  $\delta_- V_{\delta_-}$  generates  $1/\delta_- > 1/\delta$  times itself. Thus  $F$  exhibits a form of strictly decreasing returns to scale along that continuous path. This more general form is not a familiar one.

We view the question of whether  $\mathcal{E}_\delta$  varies continuously to be a basic one. Thus the question of whether  $F$  exhibits the general form of strictly decreasing returns to scale above is basic. Given their relationship —  $B_\delta(W) = (1 - \delta)F(\delta W/(1 - \delta))$  —  $B$  would exhibit a similar form of decreasing returns to scale as  $F$ . However, we are not aware of any method for verifying that  $B$  exhibits such a general form of strictly decreasing returns to scale apart from verifying that  $B$  exhibits the more specific form of SDRS as we defined it before. On the other hand, it is easy to construct games where this general form of decreasing returns to scale is not satisfied. For example, any nondegenerate game with a finite action space.

## 1.2 Un-unraveling

We find that SDRS implies that equilibrium unraveling is not robust in the long-but-finitely repeated game with any fixed discount factor  $\delta \in (0, 1)$ .

For some finite  $T$  and  $\delta \in (0, 1)$ , let  $\mathcal{E}_T$  be the set of equilibrium payoffs of the  $T$ -times repeated game,  $G_\delta^T$ . Though we do not assume it for our results here,  $B$  has the property that  $\mathcal{E}_T = B(\mathcal{E}_{T-1}) = B^T(0)$ . That property does motivate the following result. Suppose that the set of equilibrium payoffs of the stage game,  $\mathcal{N}$ , is a singleton, which we normalize to zero. Then for all finite  $T$ ,  $\mathcal{E}_T = B^T(0) = 0$ . That is complete equilibrium unraveling. Even if  $\mathcal{N}$  is larger,  $\mathcal{E}_T$  may be small relative to  $\mathcal{E}_\infty$  for all  $T$ . SDRS implies that such unraveling is not robust in the following general sense.

**Theorem 2.** *If  $B_\delta$  exhibits SDRS along  $\mathcal{E}_{\infty,\delta}$  and  $W_0$  contains a neighborhood of zero, then for any  $\theta \in (0, 1)$  there exists finite  $T$  such that  $\theta\mathcal{E}_{\infty,\delta} \subset B_\delta^T(W_0)$  for all  $t \geq T$ .*

We first describe three stronger corollaries based on general properties of the generation map  $B$  that are not included in and do not follow from our abstract definition of  $B$ . (1) As stated the theorem concludes only that  $\liminf_{T \rightarrow \infty} B^T(W_0) \supset \mathcal{E}$ , but that may be sharpened to  $\lim_{T \rightarrow \infty} B^T(W_0) = \mathcal{E}$  so long as  $W_0$  is bounded.<sup>10</sup> (As usual, we mean convergence in terms of the Hausdorff distance.) (2) The assumption that  $W_0$  contains a neighborhood of zero can be weakened to require only that  $W_0$  contains a neighborhood of some point.<sup>11</sup> (3) The assumption that  $W_0$  contains some entire neighborhood can often be weakened to require only that  $W_0$  contains two strictly Pareto-ranked points, or more generally that it contains a set of at least two distinct points  $\{\bar{w}, \underline{w}(1), \dots, \underline{w}(n)\}$  such that  $\underline{w}_i(i) < \bar{w}_i$  for all players  $i$ .<sup>12</sup>

The theorem and its corollaries have the following general application. Suppose we take the  $T$ -times repeated game  $G_\delta^T$  and to that we append a second game that has equilibrium payoff set  $W_0$ . The equilibrium payoff set of the combined game is  $B^T(W_0)$ , which is the object of the theorem. Our theorem bears comparison to the main result of Benoit and Krishna (2000), which instead relates to  $\lim_{\delta \rightarrow 1} B_\delta^T(W_0)$ , and does not assume SDRS. Benoit and Krishna's theorem provides a foundation for several finite-horizon folk theorems, which establish complete un-unraveling as  $\delta \rightarrow 1$ .<sup>13</sup> The finite-horizon folk theorems assume condi-

<sup>10</sup>In general  $B$  has the additional property that for any bounded set  $W_0$ ,  $\limsup_{T \rightarrow \infty} B^T(W_0) \subset \mathcal{E}$ . Abreu, Pearce, and Stacchetti (1990, Theorem 5) showed that if  $W'$  is compact and  $\mathcal{E} \subset B(W') \subset W'$  then  $\lim_{T \rightarrow \infty} B^T(W) = \mathcal{E}$ . If for any set  $W$  there is a larger compact set  $W' \supset W$  such that  $B(W') \subset W'$ , then it follows that  $\limsup_{T \rightarrow \infty} B^T(W) \subset \mathcal{E}$ . In general there must be such a larger set  $W'$ , in particular given SDRS.

<sup>11</sup>Because the generation map  $B$  has the property that  $B(W+k) = B(W) + \delta k$  for any fixed point  $k \in \mathbf{R}^n$ . So for example, if  $\lim_{T \rightarrow \infty} B^T(W_0)$  exists, it is equal to  $\lim_{T \rightarrow \infty} B^T(W_0)$ .

<sup>12</sup>Suppose the set  $W_0$  is framed by the two points  $\underline{w}$  and  $\bar{w}$  in the sense that  $\underline{w} \leq w \leq \bar{w}$  for all  $w \in W_0$ . Under the restriction to pure strategies,  $B$  has the property that if  $\lim_{T \rightarrow \infty} B^T(W_0)$  and  $\lim_{T \rightarrow \infty} B^T(\{\underline{w}, \bar{w}\})$  both exist, the latter contains the former.

Alternatively, for both mixed or pure strategies it is often the case that if  $\underline{w}$  and  $\bar{w}$  are strictly Pareto ranked, then  $B(\{\underline{w}, \bar{w}\})$  contains some ball. This is true for the class of smooth games with interior equilibria considered by Bernheim and Dasgupta (1995). Games that exhibit SDRS often belong to that class.

These properties also hold substituting the richer set  $\{\bar{w}, \underline{w}(1), \dots, \underline{w}(n)\}$  in place of  $\{\underline{w}, \bar{w}\}$ .

<sup>13</sup>We review the finite-horizon folk theorems partly following the unified presentation of Benoit and Kr-

tions which yield a terminal equilibrium payoff set  $W_0$  larger than a singleton, of one of the forms described above. Given such a terminal set and SDRS, our theorem implies complete un-unraveling for all  $\delta$ .

Benoit and Krishna (1985, 2000) present a finite-horizon folk theorem for stage games with multiple equilibrium payoffs. Suppose  $\mathcal{N}$  contains two strictly Pareto ranked points. Then  $\lim_{\delta \rightarrow 1, T \rightarrow \infty} \mathcal{E}_{T,\delta} = \lim_{\delta \rightarrow 1} \mathcal{E}_{\infty,\delta}$ . Here we have shown that, at least for pure strategies, if the stage game additionally satisfies SDRS, then the limit  $\delta \rightarrow 1$  can be dropped, that is we conclude  $\lim_{T \rightarrow \infty} \mathcal{E}_{T,\delta} = \mathcal{E}_{\infty,\delta}$  for all  $\delta \in (0, 1)$ . (Beyond pure strategies, this is true under the conditions of footnote 12.)

Following the famous literature on reputation in dynamic games, Fudenberg and Maskin (1986) present a folk theorem for finitely-repeated games with a small bit of imperfect information. In that literature, there is a small chance that the other players are crazy types that play arbitrary strategies. That is enough to establish complete un-unraveling as  $\delta \rightarrow 1$ . Suppose, more simply, that there is a small probability that each player goes crazy only in the final period. Under weak conditions on the stage game, this provides a terminal payoff set  $\{\bar{w}, \underline{w}(1), \dots, \underline{w}(n)\}$  such that  $\underline{w}_i(i) < \bar{w}_i$  for all players  $i$ , as we discussed before. Again, given that terminal payoff set and SDRS, the third corollary to our theorem above implies complete un-unraveling for all  $\delta \in (0, 1)$ .

Bernheim and Dasgupta (1995) present a finite-horizon folk theorem for repeated games with asymptotically finite horizons, wherein a stage game is repeated a long-but-finite number of times with discount factor  $\delta$  and is then infinitely repeated with a discount factor that declines to zero not too quickly over time. They establish complete un-unraveling as  $\delta \rightarrow 1$ . Under their conditions, for all  $\delta$ , the set of equilibrium payoffs of the dynamic game includes some neighborhood. Again, given that payoff set and SDRS, the second corollary to our theorem above implies complete un-unraveling for all  $\delta$ .

*Proof of Theorem 2.* Because  $W_0$  contains a neighborhood of 0 and  $\mathcal{E}$  is bounded,

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ishna. In the following discussion we assume the stage game has non-equivalent utilities (see Abreu, Dutta and Smith (1994)).

$W_0$  contains  $\theta_0\mathcal{E}$  for some  $\theta_0 \in (0, 1)$ . So  $B^T(W_0) \supset B^T(\theta_0\mathcal{E})$  for all  $T$ , due to the monotonicity of  $B$ . We will see that the later sequence converges up to  $\mathcal{E}$ , given SDRS.

DRS implies that  $\theta_0\mathcal{E}$  is self-generating, so monotonicity of  $B$  implies that the sequence is monotone non-decreasing.

Each element  $B^T(\theta_0\mathcal{E})$  is compact, because  $\mathcal{E}$  is compact and  $B$  preserves compactness. The sequence must be bounded above by  $\mathcal{E}$ , because each element is compact and self-generating and  $\mathcal{E}$  is the largest such set.

Let  $\theta_T = \max\{\theta : \theta\mathcal{E} \subset B^T(\theta_0\mathcal{E})\}$ , and let  $\theta_\infty = \lim_{T \rightarrow \infty} \theta_T$ . We must establish that  $\theta_\infty = 1$ . Suppose instead  $\theta_\infty = \theta \in (0, 1)$ . SDRS implies, for any  $\theta \in (0, 1)$ , there exists  $\theta^+ > \theta$  such that  $\theta^+\mathcal{E} \subset B(\theta\mathcal{E})$ . DRS then implies that for a strictly smaller value,  $\theta^- = \frac{\theta}{\theta^+}\theta < \theta$ , we have  $\theta\mathcal{E} \subset B(\theta^-\mathcal{E})$ . Thus we cannot have  $\theta_\infty < 1$ . For if that were so there would be some  $\theta_T \in (\theta^-, \theta)$  in which case  $\theta_{T+1} \geq \theta$  and  $\theta_{T+2} \geq \theta^+ > \theta_\infty$  — a contradiction.  $\square$

## 2 Conditions for decreasing returns to scale in $B$

We present a new representation of  $B$  for pure strategies and perfect monitoring, which illuminates decreasing returns to scale. Examining this representation reveals that the question of DRS in  $B$  is related to the question of whether the sets of pure-strategy,  $\epsilon$ -equilibrium payoffs of the game exhibit decreasing returns to scale in the value  $\epsilon \geq 0$ . We identify related conditions sufficient for SDRS in  $B$ .

Let  $G$  be an  $n$ -player stage game. Let  $A_i$  be player  $i$ 's action space;  $A = \times_i A_i$ , the space of action profiles;  $u_i : A \rightarrow \mathbf{R}$ , player  $i$ 's payoff function; and  $u = (u_1, \dots, u_n)$ , the payoff profile. As usual, let  $u(a'_i, a_{-i})$  denote the payoffs that result when player  $i$  plays  $a'_i$  while all other players  $j \neq i$  play  $a_j$ .

Consider the following measure of how far an action profile,  $a \in A$ , is from equilibrium in  $G$ ,

$$\ell_i(a) = \sup_{a'_i \in A_i} u_i(a'_i, a_{-i}) - u_i(a).$$

When player  $i$  has a best response to  $a_{-i}$ ,  $\ell_i(a)$  is the amount that she could increase her

current stage payoff by deviating from  $a_i$  to that best response. We call this quantity player  $i$ 's *temptation* at  $a$ . As usual, we extend this to a temptation profile,  $\ell(a) = (\ell_1(a), \dots, \ell_n(a))$ . Notice that action profile  $a$  is enforceable on the set of two points  $\{0, -x\} \subset \mathbf{R}^n$  if and only if  $\ell(a) \leq \frac{\delta}{1-\delta}x$ .

Given a non-negative vector  $x \in \mathbf{R}_+^n$ , consider the set of payoffs achievable at temptation not more than  $x$ ,

$$C(x) = \{u(a) : a \in A, \ell(a) \leq x\}.$$

For example,  $C(0) = \mathcal{N}$ , the set of pure-strategy equilibrium payoffs of the stage game, and for  $\epsilon \in \mathbf{R}_+$ ,  $C((\epsilon, \dots, \epsilon))$  is the set of pure-strategy  $\epsilon$ -equilibrium payoffs of  $G$ .

Let  $W \subset \mathbf{R}^n$  be a compact payoff set. Let  $\underline{w}_i = \min_{w \in W} w_i$ , that is player  $i$ 's minimum payoff in  $W$ . For  $w \in W$ , define

$$p_i(w, W) = \frac{\delta}{1-\delta} (w_i - \underline{w}_i).$$

If the promised continuation payoff is  $w$ , then  $p_i(w, W)$  is the size of the stick with which we can dissuade player  $i$  from deviating in this period. As usual, we extend this to the profile  $p = (p_1, \dots, p_n)$ .

For  $W$  closed, we can now represent  $B$  as follows,

$$B(W) = \bigcup_{w \in W} ((1-\delta)C(p(w, W)) + \delta w).$$

That representation of  $B$  holds for pure strategies without public randomization.

It will be useful to guarantee that  $C$  and  $B$  are convex-valued. To do so we can allow public randomization. In that case  $C$  becomes the convex hull of what was previously described:

$$C(x) = \text{co} \{u(a) : a \in A, \ell(a) \leq x\},$$

and  $B$  becomes,

$$B(W) = \text{co} \bigcup_{w \in \text{co} W} ((1 - \delta)C(p(w, W)) + \delta w).$$

That representation of  $B$  holds for pure strategies with public randomization.

We define similar notions of DRS and SDRS for  $C$  as for  $B$ :

**Definition 3.**  $C$  exhibits *decreasing returns to scale* if for all  $x \in \mathbf{R}_+^n$  and all  $\theta \in (0, 1)$ ,  $\theta C(x) \subset C(\theta x)$ .

$C$  exhibits *strongly decreasing returns to scale* if for all  $x \in \mathbf{R}_+^n$  and all  $\theta \in (0, 1)$ ,  $\theta_+ C(x) \subset C(\theta x)$  for some  $\theta^+ > \theta$ .

If  $C$  exhibits DRS, then  $\theta$  times the set of pure-strategy  $\epsilon$ -equilibrium payoffs of the stage game is contained within the set of pure-strategy  $(\theta\epsilon)$ -equilibrium payoffs. DRS in  $C$  generalizes that condition to asymmetric bounds on the players' temptations.

We show that if  $C$  exhibits DRS then so does  $B$ , and the same is almost true for SDRS:

**Theorem 3.** *Regarding  $B$  for pure strategies with or without public randomization,*

(a) *Let  $W \subset \mathbf{R}^n$  be closed. If  $C$  exhibits DRS then  $B$  exhibits DRS along  $W$ .*

(b) *Let  $W \subset \mathbf{R}^n$  be closed, contain a neighborhood of zero and be self-generating. If  $C$  and  $B$  are convex-valued, and  $C$  exhibits SDRS, then  $B$  exhibits SDRS along  $W$ .*

We apply part (b) with  $W = \mathcal{E}$  to establish the SDRS conditions of the previous Theorems 1 and 2. For a substantial class of continuous stage games, if  $\mathcal{E}$  is larger than zero it contains some neighborhood of 0, given our normalization  $0 \in \mathcal{N}$ .<sup>14</sup> In this case the theorem implies that SDRS in  $C$  is sufficient for SDRS in  $B$  along  $\mathcal{E}$ , given public randomization. So we will seek to establish SDRS in  $C$ .

*Proof.* (a) We want to show that  $\theta B(W) \subset B(\theta W)$  for all  $\theta \in [0, 1]$ .

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<sup>14</sup>These stage games include the class considered by Bernheim and Dasgupta, which include Cournot and differentiated-product Bertrand oligopoly.

$C$  is subhomogeneous, that is for all  $\theta \in [0, 1]$  and all  $x \in [0, \infty)^n$ , we have that  $\theta C(x) \subset C(\theta x)$ . Also notice that  $p$  is homogeneous of degree one:  $\theta p(w, W) = p(\theta w, \theta W)$ . Consequently

$$\theta C(p(w, W)) \subset C(p(\theta w, \theta W)).$$

We use that fact to establish the inclusion on the third line below.

$$\begin{aligned} \frac{B(\theta W)}{\theta} &= \bigcup_{v \in \theta W} \left( \frac{1-\delta}{\theta} C(p(v, \theta W)) + \frac{\delta}{\theta} v \right) \\ &= \bigcup_{w \in W} \left( \frac{1-\delta}{\theta} C(p(\theta w, \theta W)) + \frac{\delta}{\theta} \theta w \right) \\ &\subset \bigcup_{w \in W} ((1-\delta)C(p(w, W)) + \delta w) = B(W). \end{aligned}$$

(This argument can be extended to non-closed  $W$ , but is less elegant in that case.)

(b) We want to show that for all  $\theta \in (0, 1)$ ,  $\theta_+ B(W) \subset B(\theta W)$  for some  $\theta_+ > \theta$ .

For  $N \in \mathbf{R}^n$  with length  $\|N\| = 1$ , let  $h(N, B(W)) = \max_{v \in B(W)} Nv$ . Both  $B(W)$  and  $B(\theta W)$  are convex-valued, by assumption, so it suffices to show that  $\theta_+ h(B(W), N) \leq h(B(\theta W), N)$  for all  $N$ . Because  $h$  is continuous in  $N$ , which takes values in a compact set, it suffices to show that  $\theta h(B(W), N) < h(B(\theta W), N)$  for all  $N$ .

Because  $C$  exhibits SDRS and is convex-valued, we have that either  $h(C(x), N) = 0 = h(C(\theta x), N)$  or  $\theta h(C(x), N) < h(C(\theta x), N)$ . We will establish that the latter is true where necessary for  $B$  to exhibit SDRS.

Recalling the representation of  $B$  in terms of  $C$ ,

$$h(B(W), N) = \max_{v \in B(W)} Nv = (1-\delta)h(C(p(w^*, W)), N) + \delta Nw^*$$

for some optimal  $w^*$ . Because  $W$  is self generating and contains a neighborhood of zero,  $h(B(W), N) \geq h(W, N) > 0$ . Additionally  $\delta Nw^* < h(W, N)$  because  $\delta < 1$ . So we must have  $h(C(p(w^*, W)), N) > 0$ . SDRS in  $C$  then implies  $\theta h(C(p(w^*, W)), N) < h(C(\theta p(w^*, W)), N)$ , so

$$\theta h(B(W), N) < (1-\delta)h(C(\theta p(w^*, W)), N) + \delta \theta Nw^* \leq h(B(\theta W), N)$$

as desired.

□

As discussed in Plan (2011),  $C$  exhibits DRS but not SDRS for example in the Bertrand oligopoly model with undifferentiated products and linear production cost. For that stage game, if  $v$  is a feasible payoff that can be achieved at some temptation  $t$ , then  $\theta v$  can be achieved at exactly  $\theta t$  for all  $\theta \in (0, 1)$ . That is, temptation is linear in payoffs, and  $C$  exhibits constant returns to scale, up to a point.

We now seek conditions on the primitives of the stage game sufficient for SDRS in  $C$ .

## 2.1 Games with separable payoffs

Consider the following prisoner's dilemma:

	$C$	$D$
$C$	1, 1	-1, 2
$D$	2, -1	0, 0

We know that stage game does not exhibit DRS because the action space is finite, but it could be extended to be continuous. We first examine which of those extensions exhibit (S)DRS. We will restrict attention to symmetric games, so we can focus on player one.

The prisoner's dilemma above can instead be represented by the action space  $A_1 = \{0, 1\}$  and linear payoff function  $u_1 = 2a_2 - a_1$ . Notice that payoffs are additively separable. Here action 1 corresponds to cooperate and 0 to defect. We can extend the action space to the entire unit interval,  $A_1 = [0, 1]$ . The resulting game can be interpreted as the mixed extension of the original game, in which case  $a_i \in [0, 1]$  is the probability with which player one cooperates. Notice that temptation is  $l_1(a_1) = a_1$ , so both payoffs and temptation are linear in  $a$ . Because of that linearity, this game exhibits constant returns to scale up to a point: if  $t \leq (1, 1)$  and  $\theta \in (0, 1)$  then  $C(\theta t) = \theta C(t)$ . More generally, the extended game exhibits DRS but not SDRS.

A notable feature of the above linear extension of the prisoner's dilemma is that the

player's first order conditions are not satisfied in equilibrium:  $l_1'(0)$  equals one not zero. An alternative extension is given by the payoff function  $u_1 = 2a_2 - a_1^2$ . Here cooperative effort is initially free but becomes increasingly costly:  $l_1 = a_1^2$  so  $l_1'(0) = 0$ . For this game,  $B$  does exhibit SDRS along  $\mathcal{E}$  as desired.

More generally consider a symmetric, two-player game with additively separable payoff function  $u_1(a_1, a_2) = \bar{u}_1(a_2) - l_1(a_1)$  and action space  $A_1 = [0, 1]$ . Like in examples above, we assume that both  $\bar{u}_1$  and  $l_1$  are continuous and strictly increasing in their arguments, and that both  $\bar{u}_1(0) = 0 = l_1(0)$ . If  $\bar{u}_1(1) = 2$  and  $l_1(1) = 1$ , then we can view such a game as a continuous extension of the prisoner's dilemma above. We consider pure strategies with public randomization. We find the following conditions for (S)DRS,

**Proposition 2.**  *$B$  exhibits (S)DRS along  $\mathcal{E}$  if  $\bar{u}_1(a_2)/l_2(a_2)$  is (strictly) decreasing in  $a_2 > 0$ .*

For  $\bar{u}_1(a_2)/l_2(a_2)$  to be decreasing, it suffices that the numerator is concave while the denominator is convex, which is true if the total payoff  $u_1(a)$  is concave in both players' actions. If in addition payoffs are strictly concave in the player's own action, that is  $l_2$  is strictly convex, then  $\bar{u}_1/l_2$  is strictly decreasing. Those conditions are common. We postpone the proof of the proposition until the end of this section.

A notable feature of the prisoner's dilemma is that the unique stage equilibrium coincides with the minmax point. That remains so in the continuous extensions we just considered. We now enlarge the action space to allow worse punishments. For example, let  $A_1 = [-1, 1]$  and  $u_1 = 2a_2 - |a_1|$ . This extends the game with linear payoffs that we considered above. In this new game both  $C$  and  $B$  exhibit DRS. A second example is  $A_1 = [-1, 1]$  and  $u_1 = 2a_2 - a_1^2$ . Here  $C$  exhibits SDRS for all  $t \gg 0$  and  $B$  exhibits SDRS along  $\mathcal{E}$  as desired.

More generally, we maintain the previous assumptions about the payoff function but now enlarge the action space to  $A_1 = [\underline{a}_1, \bar{a}_1]$ , which contains zero in its interior. We assume that  $l_1$  is not globally increasing but rather it is strictly quasi-concave and minimized at  $l_1(0) = 0$ . We assume that the action space is symmetric about the origin in the sense that

$l_1(\underline{a}_1) = l_1(\bar{a}_1)$ .<sup>15</sup> Unlike in the previous case where  $A_1 = [0, \bar{a}_1]$ , here we are able to make strong statements about (S)DRS in  $C$ ,

**Proposition 3.** (a)  $C$  exhibits DRS if and only if  $\bar{u}_1(a_2)/l_2(a_2)$  is decreasing at all  $a_2 \neq 0$ .  
 (b)  $C$  exhibits SDRS for all  $t \gg 0$  if  $\bar{u}_1(a_2)/l_2(a_2)$  is strictly decreasing at all  $a_2 \neq 0$ . If in addition payoffs are differentiable, then  $B$  exhibits SDRS along  $\mathcal{E}$ .

As we mentioned before, for example the payoff function  $u_1 = 2a_2 - a_1^2$  satisfies the conditions of part (b).

Bernheim and Dasgupta (1995) conjectured that  $B$  has no intermediate fixed points if  $u$  is jointly concave in both players actions. We find a counter example inspired by the conditions of this proposition.<sup>16</sup>

*Proof.* The conditions imply  $\max_{a_2: l_1(a_2) \leq t_2} \bar{u}_1(a_2)$  is (strictly) subhomogeneous in  $t_2$ , and similarly regarding  $-\bar{u}_1$ .

(a) Sufficiency of the stated condition for DRS: Another way to state the condition is

$$\frac{|\bar{u}_1(\rho a_2)|}{l_2(\rho a_2)} \geq \frac{|\bar{u}_1(a_2)|}{l_2(a_2)}$$

for all  $0 \neq a_2 \in A_2$  and  $\rho \in (0, 1)$ . We want to show that  $\theta \max_{a: l(a) \leq t} Nu(a) \leq \max_{a: l(a) \leq \theta t} Nu(a)$ . Suppose that action profile  $a$  is optimal for the first maximization. Choose  $a'$  between zero and  $a$  so that  $l(a') = \theta l(a)$ , which is possible due to the strict quasi-concavity and continuity of  $l$ . We want

$$\theta Nu(a) \leq Nu(a'),$$

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<sup>15</sup>A natural way for the symmetry condition to be satisfied is if  $A_1 = [-\infty, \infty]$  and  $l_1(-\infty) = \infty = l_1(\infty)$ . Alternatively, we can restrict attention to strategies where each player's actions lie in a truncation of the action space that satisfies the condition. Fix a discount factor  $\delta$ , and suppose that the action space  $A_1$  is very large in the sense that  $l_1(\underline{a}_1)$  and  $l_1(\bar{a}_1)$  are large. Then all equilibrium strategies will indeed lie in some subset of the action space  $[\underline{a}'_1, \bar{a}'_1]$  such that  $l_1(\underline{a}'_1) = l_1(\bar{a}'_1)$  as desired. In this way, the symmetry condition can be satisfied without restriction for fixed  $\delta$  if  $A_1$  is large enough.

<sup>16</sup>For the condition in part (a) to hold, it suffices that  $|u_1|$  and  $-l_2$  are both concave. Concavity of  $u_1$  helps when  $u_1$  is positive, but hurts when it is negative. That inspires the following counter example. Suppose  $l_2 = a_2^2$  and  $\bar{u}_1(a_2) = a_2$ . For  $\delta$  small enough, all equilibrium actions lie in the space  $(-1, 1)^2$  and the set of equilibrium payoffs lies inside  $(-2, 2)^2$ . Now suppose that for  $a_2 < -1$ ,  $\bar{u}_1(a_2)$  instead equals  $-1 + \gamma(a_2 + 1)$  for some  $\gamma > 0$ . We can smooth out  $\bar{u}_1$  at  $-1$  so that payoffs remain differentiable. For  $\gamma > 1$ , this is consistent with payoffs being concave. The old set of equilibrium payoffs remains a fixed point of  $B_\delta$ . As  $\gamma$  gets larger, it gets easier to generate large punishments, which in turn may enforce cooperation. For  $\gamma$  large enough, the new set of equilibria expands beyond  $(-1, 1)^2$ , so  $B_\delta$  now has a second larger fixed point.

which means

$$\theta N\bar{u}(a) - \theta Nl(a) \leq N\bar{u}(a') - Nl(a').$$

The second terms on both sides are equal, so we need

$$\theta N\bar{u}(a) \leq N\bar{u}(a').$$

We are guaranteed that  $N_1$  shares the same sign as  $a_2$ , and thus of  $\bar{u}_1$ . Otherwise we could have increased  $Nu(a)$  by choosing  $\tilde{a}_2$  of the opposite sign such that  $l_2(a_2) = l_2(\tilde{a}_2)$ . (This relies on the assumption that  $l_2(\underline{a}_2) = l_2(\bar{a}_2)$ .) So it suffices to show that

$$\theta N_1 \bar{u}_1(a_2) \leq N_1 \bar{u}_1(a'_2),$$

that is

$$\frac{|\bar{u}_1(a'_2)|}{|\bar{u}_1(a_2)|} \geq \theta = \frac{l_2(a'_2)}{l_2(a_2)},$$

equivalently

$$\frac{|\bar{u}_1(a'_2)|}{l_2(a'_2)} \geq \frac{|\bar{u}_1(a_2)|}{l_2(a_2)}.$$

Noting that  $a'_2$  is between zero and  $a_2$ , this follows from the restated condition of the proposition above.

(a) Necessity of the stated condition for DRS: Choose  $N = (1, 0)$ . Fix  $a_1 = 0$ ,  $a_2 > 0$  and  $a'_2 = \rho a_2$ . Pick  $\theta = l(a'_2)/l(a_2)$ . We must have  $\bar{u}_1(a'_2) \geq \theta \bar{u}_1(a_2) = \bar{u}_1(a_2)l(a'_2)/l(a_2)$ . Noting  $\bar{u}_1 > 0$ , rearranging gives the previously stated condition.

(b) Sufficiency of the stated condition for SDRS: First notice that if payoffs are differentiable,  $u'_1(0)$  must be strictly positive while  $l'_2(0) = 0$ . Thus  $\mathcal{E}_\delta$  contains a neighborhood of zero for all  $\delta$ , so we can apply Theorem 3.

Provided that  $t_2 > 0$  so the optimal  $a_2 \neq 0$ , we can substitute strict inequalities in place of weak inequalities in the argument for DRS above. If  $t_2 = 0$  but  $N_2 \neq 0$  then we can make the same argument regarding  $\bar{u}_2(a_1)$  instead. If  $t_2 = 0$ ,  $N_2 = 0$  and  $N_1 = 1$ , then  $\max_{a:l(a) \leq t} Nu = 0$ , so SDRS is satisfied trivially. However, if  $t_2 = 0$ ,  $N_2 = 0$  and  $N_1 = -1$ , then  $\max_{a:l(a) \leq t} = t_1$ . This is linear in  $t$ , exhibiting DRS but not SDRS. However, we can see that SDRS of  $C$  for this specific  $t$  and  $N$  is unnecessary to have SDRS of  $B$  along  $\mathcal{E}$ . Returning to the proof of Theorem 3, notice it cannot be that  $\max_{v \in B(\mathcal{E})} Nv > 0$  is achieved for  $N = (-1, 0)$  with a continuation payoff

$w$  such that  $p_2(w, \mathcal{E}) = 0$ . At such a continuation payoff we would have  $\max_{v \in B(\mathcal{E})} Nv = \delta Nw$ , contradicting the fact that  $\mathcal{E}$  is self-generating.  $\square$

We now prove the former proposition, which requires a repetition of some of the arguments in Theorem 3.

*Proof of Proposition 2.* Consider,

$$h(B(\mathcal{E}); N) = \max_{w \in \mathcal{E}} \left( \left( \max_{a: l(a) \leq w} Nu(a) \right) + \delta Nw \right)$$

Let  $a^*$  and  $w^*$  be optimal. It must be that  $Nu(a^*) > 0$  if  $h(\mathcal{E}; N) > 0$  because  $\mathcal{E}$  is self-generating.

We will show the following inequality

$$\theta Nu(a^*) = \theta \max_{a: l(a) \leq w^*} Nu(a) < \max_{a: l(a) \leq \theta w^*} Nu(a)$$

for all  $N$  such that  $h(\mathcal{E}; N) > 0$  and  $\theta \in (0, 1)$ . Because the minmax payoff is zero,  $\mathcal{E}$  is restricted to the first quadrant, so if  $h(\mathcal{E}; N) > 0$  it must be that  $N_1 > 0$  or  $N_2 > 0$ .

Choose  $a'$  such that  $l(a') = \theta l(a^*)$ . We want to show

$$\theta Nu(a^*) < Nu(a'),$$

that is

$$\theta N\bar{u}(a^*) - \theta Nl(a^*) < N\bar{u}(a') - Nl(a').$$

Canceling the second terms on both sides gives

$$\theta N\bar{u}(a^*) < N\bar{u}(a').$$

Claim: At least one of the components  $N_1\bar{u}_1(a_2^*)$  and  $N_2\bar{u}_2(a_1^*)$  is strictly positive, while the other is non-negative. If they are both zero, then  $N_1a_2^* + N_2a_1^* = 0$ . If  $N \gg 0$ , that would imply  $Nl(a^*) = 0 = Nu(a^*)$ , contradicting the condition that  $h(\mathcal{E}; N) > 0$ . If instead  $N = (1, 0)$  and both of the previous components are zero, then  $a_2^* = 0$  again implying  $Nu(a^*) = 0$ . If one component is negative, suppose it is the first, then  $a^*$  is not optimal, it would be better to choose  $a_2^* = 0$ .

Without loss of generality we can assume it is the first component,  $N_1 \bar{u}_1(a_2^*)$ , that is strictly positive. It then suffices to show

$$\theta \bar{u}_1(a_2^*) < \bar{u}(a'),$$

which is true because  $u_1(a_1)/l(a_1)$  is strictly decreasing.

Following a similar argument as in the proof of the previous theorem, we have shown that for all  $N$  either  $\theta h(B(\mathcal{E}); N) < h(B(\theta\mathcal{E}); N)$  or both sides equals zero. Because of the latter possibility, we have not yet established that  $\theta^+ B(\mathcal{E}) \subset B(\theta\mathcal{E})$  for some  $\theta^+ > \theta$ . It suffices to show that if  $\mathcal{E}$  is greater than zero, then it intersects the  $x$  and  $y$ -axes at a point larger than zero. Recall  $\mathcal{E}$  is restricted to the first quadrant and contains the origin. If  $\mathcal{E}$  is greater than zero, then by symmetry it contains a point on the 45 degree line strictly greater than zero; call that point  $(w_0; w_0)$ . Choose  $a_1 = 0$  and  $a_2$  such that  $l_2(a_2) = w_0 \delta / (1 - \delta)$ . In this way we generate the point  $((1 - \delta) \bar{u}_1(a_2) + \delta w_0, 0)$  as claimed.  $\square$

## 2.2 Strongly-symmetric equilibria

The generation map  $B$  is substantially simplified under the restriction to strongly-symmetric, pure strategies.<sup>17</sup> We establish conditions under which this version of  $B$  exhibits SDRS. These conditions are satisfied in many oligopoly stage games.

Suppose that the stage game  $G$  is symmetric. As all players are the same, we will focus on player one. For an action profile where player one plays action  $a_1$  while all others play action  $a_2$ , instead of writing “ $(a_1, a_2, \dots, a_2)$ ”, here we will simply write “ $(a_1, a_2)$ ”, as if this were a two-player game. If all play the same action  $a_0$ , we will write “ $(a_0, a_0)$ ”.

We now examine the set

$$C(\epsilon) = \{u_1(a_0, a_0) : a_0 \in A_1, \ell_1(a_0, a_0) \leq \epsilon\},$$

---

<sup>17</sup>For such strategies, Cronshaw and Luenberger (1994) show that if convex-valued,  $B$  can be reduced to a map from  $\mathbf{R}$  to  $\mathbf{R}$ . We do not employ that simplification here, in order to remain closer to the analysis for all pure strategies.

which is the set of payoffs of symmetric, pure-strategy  $\epsilon$ -equilibrium of the stage game, where  $\epsilon \in \mathbf{R}_+$ . Following the proof of Theorem 3 it can be shown that if  $W$  is closed, contains zero, and is self-generating, and this version of  $C$  exhibits SDRS then  $B$  exhibits SDRS along  $W$  for strongly-symmetric, pure strategies with public randomization. So we seek conditions under which this  $C$  exhibits SDRS.

We imagine that  $G$  might be a Cournot or differentiated-product Bertrand stage game. We assume the following about  $G$ . Each player's action space,  $A_1$ , is a compact interval in  $\mathbf{R}$ . The payoff function,  $u_1$ , is continuous. There is a symmetric, pure-strategy equilibrium which is unique. We normalize that equilibrium to be  $(0, 0)$  with payoffs  $u_1(0, 0) = 0$ . Each player's payoff is strictly concave in her own action, so given that all others play  $a_0$ , player one has a unique best response,  $r_1(a_0)$ . The best response  $r_1(a_0)$  always lies in the interior of  $A_1$ . We assume that  $u_1$  is monotone increasing in  $a_2$ . (If this is a Cournot game, the action  $a$  could represent the negative of the quantity.)

We further assume that along symmetric action profiles, the payoff  $u_1(a_0, a_0)$  is monotone increasing in  $a_0$ . It would be more natural to assume that  $u_1(a_0, a_0)$ , is quasi-concave in  $a_0$ , for which it would suffice that payoffs are jointly concave in  $a$ . We can get the former assumption from the latter assumption by truncating the action space in a way that will not reduce the greatest equilibrium payoff.<sup>18</sup>

**Lemma 2.** *Suppose that payoffs are twice differentiable and*

$$\frac{d^2 u_1(a_1, a_2)}{da_1^2} + \frac{d^2 u_1(a_1, a_2)}{da_1 da_2} < 0.$$

*Then (a)  $\ell_1(a_0, a_0)$  is strictly quasi-convex in  $a_0$ ; and (b)  $(a_0 - r_1(a_0))a_0 > 0$  for all  $a_0 \neq 0$ .*

For the condition of the lemma it is sufficient that the Hessian of the payoff function is

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<sup>18</sup>Let  $a_0^M = \arg \max_{a_0} u_1(a_0, a_0)$ , so if this is a Bertrand stage game,  $a_0^M$  is the monopoly price level. If  $a_0^M \geq 0$ , truncate the action space to exclude all actions strictly greater than  $a_0^M$ . In instead  $a_0^M < 0$ , exclude all actions strictly less than  $a_0^M$ . Because  $u_1(a_0, a_0)$  was quasi-concave on the original action space, it is monotone on the newly truncated action space. Further, this truncation does not reduce the set of equilibrium payoffs in the class of games that we will identify as exhibiting SDRS.

strictly diagonally dominant, which is common in differentiated-product Bertrand oligopoly. It would also be sufficient that actions are strategic substitutes, which is common in Cournot oligopoly. The lemma implies that  $C$  is convex-valued, so public randomization is unnecessary.

*Proof.* (b) Because best responses are interior, the FOC for  $r_1$  holds,

$$\frac{du_1(r_1(a_0), a_0)}{da_1} = 0.$$

Implicitly differentiating with respect to  $a_0$  yields,

$$r_1'(a_0) = \frac{d^2u_1(r_1, a_0)}{da_1 da_2} / \left( -\frac{d^2u_1(r_1, a_0)}{da_1^2} \right)$$

which is strictly less than one by the diagonal dominance and the strict concavity in own payoffs. Thus  $a_0 - r_1(a_0) > 0$  for  $a_0 > 0$  and  $r_1(a_0) - a_0 > 0$  for  $a_0 < 0$ .

(a) Recall  $\ell_1(0, 0) = 0$  so for quasi-convexity we want to establish  $a_0 \frac{d\ell_1(a_0, a_0)}{da_0} > 0$ .

$$\begin{aligned} \frac{d\ell_1(a_0, a_0)}{da_0} &= \frac{d}{da_0} (u_1(r_1, a_0) - u_1(a_0, a_0)) \\ &= \left( \frac{du_1(r_1, a_0)}{da_1} + \frac{du_1(r_1, a_0)}{da_2} \right) - \left( \frac{du_1(a_0, a_0)}{da_1} + \frac{du_1(a_0, a_0)}{da_2} \right) \\ &= \int_{a_0}^{r_1} \frac{d}{da_1} \left( \frac{du_1(a_1, a_0)}{da_1} + \frac{du_1(a_1, a_0)}{da_2} \right) da_1 \\ &= \int_{a_0}^{r_1} \left( \frac{d^2u_1(a_1, a_0)}{da_1^2} + \frac{d^2u_1(a_1, a_0)}{da_1 da_2} \right) da_1 \end{aligned}$$

(In moving from the first to second line we use the fact that FOC holds for  $r_1$ .) The integrand on the last line is strictly negative by the diagonal dominance and strict concavity with respect to own action. And we have seen  $a_0(r_1 - a_0)$  is strictly negative. So  $a_0 \frac{d\ell_1}{da_0} > 0$  as desired.  $\square$

**Proposition 4.** *Suppose that the conclusions of the previous lemma are satisfied and in addition*

$$\frac{u_1(r_1(a_0), a_0)}{u_1(a_0, a_0)}$$

is strictly increasing in  $a_0 \neq 0$ . Then  $C$  and  $B$  exhibit SDRS for strongly-symmetric pure strategies.

If payoffs are differentiable then it suffices for the condition of the proposition that

$$\left( \frac{d^2 \log |u_1(a_1, a_2)|}{da_1^2} + \frac{d^2 \log |u_1(a_1, a_2)|}{da_1 da_2} \right) u_1(a_1, a_2) < 0.$$

for all  $a_2$  and all  $a_1$  between  $a_2$  and the best response  $r_1(a_2)$ . That condition is guaranteed if payoffs are strategic complements in the sense of log-submodularity, as would be common in Cournot games. The condition requires that payoffs cannot be log-supermodular to a degree that would violate log-diagonal-dominance. Such a diagonal-dominance condition is common in Bertrand games with differentiated products. (See Vives (2001).) Imagining that this is a Bertrand stage game and that payoffs are currently positive, consider the growth in  $u_1$  as all prices increase by a penny. The condition above requires that this is decreasing in player one's price while her price is above her best response. That seems natural to the extent that she finds increasing her own price by a penny to be more painful the higher her price already is. We have verified the condition of the proposition directly for Bertrand stage games with constant production costs and linear or logit demand.

*Proof.* The condition of the proposition implies

$$\left( \frac{u_1(r_1(a_0), a_0)}{u_1(r_1(a'_0), a'_0)} - \frac{u_1(a_0, a_0)}{u_1(a'_0, a'_0)} \right) u_1(a_0, a_0) > 0,$$

for all  $a_0 \in A_1$ ,  $a_0 \neq 0$  and  $a'_0$  strictly between 0 and  $a_0$ .

We want to establish that the following is strictly subhomogeneous in  $x_0$ ,

$$C(x_0) = \{u_1(a_0, a_0) : a_0 \in A_1, \ell_1(a_0, a_0) \leq x_0\},$$

where  $\ell_1(a_0, a_0)$  is player one's temptation when all play  $a_0$ . We have established that it is convex-valued, because payoffs are quasi-concave while temptation is quasi-convex in  $a_0$ . So it suffices to establish that for every  $a_0 \in A_1$

and  $\theta \in [0, 1]$ , there exists  $a'_0 \in A_1$  such that  $\ell_1(a'_0, a'_0) = \theta \ell_1(a_0, a_0)$  while  $u(a'_0, a'_0) > \theta u(a_0, a_0)$ .

So we want for  $a'_0$  between 0 and  $a_0$ ,

$$\frac{u_1(r_1(a_0), a_0) - u_1(a_0, a_0)}{u_1(r_1(a'_0), a'_0) - u_1(a'_0, a'_0)} > \frac{u_1(a_0, a_0)}{u_1(a'_0, a'_0)}.$$

We will write simply  $r$  in place of  $r(a_0)$  and  $r'$  in place of  $r(a'_0)$ . The denominator (and the numerator) on the LHS are positive, so we can multiply out and cancel terms,

$$\begin{aligned} u_1(r, a_0) - u_1(a_0, a_0) &> \frac{u_1(a_0, a_0) (u_1(r', a'_0) - u_1(a'_0, a'_0))}{u_1(a'_0, a'_0)} \\ &= \frac{u_1(a_0, a_0)}{u_1(a'_0, a'_0)} u_1(r', a'_0) - u_1(a_0, a_0) \\ u_1(r, a_0) &> \frac{u_1(a_0, a_0)}{u_1(a'_0, a'_0)} u_1(r', a'_0). \end{aligned}$$

Case 1:  $u_1(a_0, a_0) > 0$ . Then all the terms in that inequality are positive, because  $u_1(a_0, a_0)$  is strictly quasi-concave and  $u_1(r_1(a_0), a_0) > u_1(a_0, a_0)$ . So we want

$$\frac{u_1(r, a_0)}{u_1(r', a'_0)} > \frac{u_1(a_0, a_0)}{u_1(a'_0, a'_0)}.$$

Taking logs yields

$$\log u_1(r, a_0) - \log u_1(r', a'_0) > \log u_1(a_0, a_0) - \log u_1(a'_0, a'_0)$$

Because  $r$  maximizes  $u_1$  given  $a_0$ , it suffices that

$$\log u_1(r' + (a_0 - a'_0), a_0) - \log u_1(r', a'_0) > \log u_1(a_0, a_0) - \log u_1(a'_0, a'_0)$$

Without loss of generality, suppose  $a_0$  is positive, then  $r' < a'_0$  and  $a'_0 < a_0$ . So it suffices that

$$\log u_1(x + \Delta, y + \Delta) - \log u_1(x, y)$$

is strictly decreasing in  $x$  for  $\Delta > 0$ .

It suffices that

$$\frac{d^2 \log u_1}{da_1^2} + \frac{d^2 \log u_1}{da_1 da_2} < 0.$$

(This means that  $\log u_1$  cannot be so supermodular as to be not diagonal dominant.)

Case 2:  $u_1(a_0, a_0) < 0$ . If instead some of the terms are negative, then they are all negative (due to the monotone externality), and we want

$$\frac{u_1(r, a_0)}{u_1(r', a'_0)} < \frac{u_1(a_0, a_0)}{u_1(a'_0, a'_0)}$$

which means

$$\frac{|u_1(r, a_0)|}{|u_1(r', a'_0)|} < \frac{|u_1(a_0, a_0)|}{|u_1(a'_0, a'_0)|}.$$

$$\log |u_1(r, a_0)| - \log |u_1(r', a'_0)| < \log |u_1(a_0, a_0)| - \log |u_1(a'_0, a'_0)|$$

Because  $r$  is optimal at  $a_0$ ,  $u_1(r, a_0) > u_1(r' + (a_0 - a'_0), a_0)$  so  $|u_1(r, a_0)| < |u_1(r' + (a_0 - a'_0), a_0)|$  so  $\log |u_1(r, a_0)| < \log |u_1(r' + (a_0 - a'_0), a_0)|$ . So it suffices that

$$\log |u_1(r' + (a_0 - a'_0), a_0)| - \log |u_1(r', a'_0)| < \log |u_1(a_0, a_0)| - \log |u_1(a'_0, a'_0)|$$

□

## 2.3 Games with quadratic payoffs

Consider an  $n$ -player game  $G$  with quadratic payoffs,

$$u_i(a) = a^T Q_i a + q_i a,$$

where for each player  $i \in \{1, \dots, n\}$ ,  $Q_i$  is a symmetric  $n \times n$  matrix,  $q_i$  is a  $1 \times n$  row vector, and  $a$  is an  $n \times 1$  column vector, all of which are real-valued. Apart from an interest in DRS, not all values of  $Q$  and  $q$  will yield a well-behaved game. Suppose that each player's payoff is strictly concave in her own actions,  $Q_{i,ii} < 0$ , which is true in most quadratic games

of economic interest.<sup>19</sup> Suppose that  $G$  has a pure-action equilibrium. If, in addition, each player's action space is the entire real line, then  $G$  satisfies DRS for pure actions with public randomization. However, action spaces are usually smaller than the entire real line. If they are finite, then DRS cannot hold. If they are not finite, they are often intervals, in which case we identify some additional sufficient conditions for DRS.

**Proposition 5.** *Let  $G$  be an  $n$ -player game with quadratic payoffs where  $Q_{i,ii} < 0$ . Restrict attention to pure actions with public randomization. Let  $0$  be a pure action equilibrium of  $G$ .*

*DRS holds if  $A_i$  is convex,  $A_i = -A_i$ , and  $\sum_{j \neq i} Q_{i,ij} a_j / (-Q_{i,ii}) \in A_i$  for all  $a_{-i} \in A_{-i}$ .*

*SDRS in  $B$ , and in  $C(t)$  for  $t \gg 0$ , holds if in addition the  $q_i$  are linearly independent,  $Q_{i,ij} \neq 0$  for  $j \neq i$ , and  $A$  is small enough.*

The condition regarding  $\sum_{j \neq i} Q_{i,ij} a_j / (-Q_{i,ii})$  is just barely weaker than the condition that pure best responses are always interior. This is satisfied if  $A_i$  equals the whole real line. Given that  $A$  is origin symmetric, it is also satisfied given row diagonal dominance ( $|Q_{i,ii}| \geq \sum_{j \neq i} Q_{i,ij}$ ).

*Proof.* We want to establish that  $C$  is subhomogeneous, that is  $C(\theta t) / \theta$  is non-increasing in  $\theta > 0$  for all positive vectors  $t \in \mathbf{R}^n$ ,  $t_i \geq 0$ . Considering pure actions with public randomization yields  $C(t) = \text{conv}\{u(a) : a \in A, \ell(a) \leq t\}$ , where  $\ell_i(a) = \max_{a'_i \in A_i} u_i(a'_i, a_{-i}) - u_i(a)$ .

Because we have allowed public randomization,  $C$  is convex. Recall that if two subsets of  $\mathbf{R}^n$ ,  $X$  and  $Y$ , are convex then  $X \subset Y$  if and only if  $\max_{x \in X} N x \leq \max_{y \in Y} N y$  for all row vectors  $N^T \in \mathbf{R}^n$ . So for our purposes it suffices to establish that  $h(\theta; x, N) = \max_{v \in C(\theta x)} N v = \max_{a: \ell(a) \leq \theta x} N u(a)$  is subhomogeneous.

We first consider the case where each player's action space is the entire real line. Each player's best response is linear in the others' actions:

$$a_i^*(a_{-i}) = \frac{q_i + \sum_{j \neq i} Q_{i,ij} a_j}{-Q_{i,ii}}$$

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<sup>19</sup>If there is a pure-strategy equilibrium in the interior of the action space, then it must be that  $Q_{i,ii} \leq 0$  for all players  $i$ . If  $Q_{i,ii} = 0$  and there is such an interior equilibrium  $a^*$ , it must be that player  $i$ 's payoffs are invariant in  $a_i$  at  $a_{-i}^*$ .

Because we have assumed that best responses are always interior, the above holds even if the action space is smaller.

Without loss of generality, we can shift the action spaces so that the assumed equilibrium corresponds to zero, thus  $q_i = 0$ . Then we have  $a_i - a_i^*(a_{-i}) = a_i - \sum_{j \neq i} Q_{i,ij} a_j$ , which is linear in  $a$ . Temptation is quadratic in that difference:

$$\ell_i(a) = -Q_{i,ii} (a_i - a_i^*(a_{-i}))^2 = -Q_{i,ii} \left( a_i - \frac{\sum_{j \neq i} Q_{i,ij} a_j}{-Q_{i,ii}} \right)^2 = \frac{1}{-Q_{i,ii}} \left( \sum_{j=1}^n Q_{i,ij} a_j \right)^2.$$

Notice that  $\ell(a)$  is homogeneous of degree two, so the constraint  $\ell(a) \leq \theta t$  is equivalent to the constraint  $\ell(\sqrt{\theta} a) \leq t$ . Also notice that  $\ell(a) = \ell(-a)$ .

Notice that  $Nu$  is itself a quadratic form in  $a$ ,

$$Nu = a^T \underbrace{\left( \sum_i N_i Q_i \right)}_{=R} a + \underbrace{\left( \sum_i N_i q_i \right)}_{=r} a.$$

So we have

$$h(\theta) = \max_{\ell(\sqrt{\theta} a) \leq t, a \in A} a^T R a + r a.$$

Suppose that  $a^*$  is optimal, then it must be that  $r a^* \geq 0$ . If not then choosing  $-a^*$  would increase the objective while continuing to satisfy the constraints. (This is the only point at which we use the assumption that  $A = -A$ .)

Let  $\theta \in (0, 1)$  and suppose that  $a^*$  is optimal for  $h(1)$ . We have

$$h(1) = a^{*T} R a^* + r a^* \leq a^{*T} R a^* + \frac{1}{\sqrt{\theta}} r a^* \leq \frac{h(\theta)}{\theta}$$

The inequality at right is because  $(\sqrt{\theta} a^*)$  satisfies the constraints for  $h(\theta)$ . The inequality in the middle is because  $r a^* \geq 0$  and  $1/\sqrt{\theta} \geq 1$ .

To establish SDRS in  $C$  we must have that one of the inequalities directly above is a strict inequality, so we require one of the following:  $r a^* > 0$  or  $\sqrt{\theta} a^*$  is no longer optimal for  $h(\theta)$ . The former is true given that  $t \gg 0$ , the  $q_i$  are linearly independent and  $A$  is small enough. The linear independence implies that  $r = \sum_i N_i q_i \neq 0$ . So if  $r a^* = 0$ , it must be that  $r_i a_i^* < 0$  for some  $i$ . If  $A_{-i}$  is small enough then it must be that instead choosing  $-a_i^*$  would increase the objective. If on the other hand  $t \geq 0$  but not  $t \gg 0$  it may be that  $r a^* = 0$ . However, in this case, returning to the proof of Theorem 3 it is apparent that

$\theta w^*$  would no longer be optimal at  $\theta W$ .

□

### 3 Conclusion

Our results apply only to a special class of stage games, those where  $B$  exhibits SDRS — that condition is not entirely a limitation of this paper. That the answer is not “for all stage games” does not diminish the importance of the question, “For which stage games does  $\mathcal{E}$  vary continuously in  $\delta$ ?” The folk theorem, which does hold for all stage games, describes an important way in which all repeated games are the same. In contrast, this paper describes two important ways in which two substantial classes of repeated games are different. In the first class are those stage games that exhibit SDRS. We have shown that this class includes many continuous stage games, like standard oligopoly models, at least under the restriction to strongly-symmetric strategies. For this class of games, our theorem 1 implies that the set of equilibrium payoffs of the infinitely repeated game,  $\mathcal{E}$ , varies continuously in the discount factor, and our theorem 2 implies that unraveling in the finitely repeated game is not robust for all discount factors. In the second class are those stage games with finite action spaces, like the prisoner’s dilemma.<sup>20</sup> Except in degenerate examples, no finite stage game exhibits SDRS, so our theorems 1 and 2 do not apply, but instead the opposite conclusions are known to hold:  $\mathcal{E}$  is not continuous in  $\delta$  and unraveling is robust opposite to the sense of theorem 2.

For the case of strongly-symmetric pure-strategy equilibria, we provide general conditions on the primitives of the stage game sufficient for SDRS. For the broader class of pure strategy equilibria, we provide a higher level condition on the stage game sufficient for SDRS, and we find sufficient primitive conditions in two special cases. In many other cases, we have been unable to determine whether or not SDRS holds — that is a limitation of this paper.

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<sup>20</sup>The second class also includes stage games where  $B$  exhibits increasing returns to scale along  $\mathcal{E}$ , but such games are unusual.

However, the class of strongly symmetric strategies is large enough to be of interest on its own. (Previous important papers in game theory have restricted attention to those strategies, for example, Abreu (1986).) Furthermore, our higher level condition for SDRS is itself valuable, because it is a condition only on the stage game and from it we deduce properties of the repeated game. We continue to examine the question of broader primitive conditions for all pure strategies. The structure of  $B$  remains an active area of research more generally, see for example Abreu and Sannikov (2011).

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